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Robust $\mathcal{H}_\infty$ Control Design for Uncertain Fuzzy Systems with Markovian Jumps: An LMI Approach

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***Abstract*— This paper investigates the problem of designing a robust output feedback controller for a class of uncertain Markovian jump nonlinear systems that guarantees the $\mathcal{L}_2$-gain from an exogenous input to a regulated output is less than or equal to a prescribed value. First, we approximate this class of uncertain Markovian jump nonlinear systems by a class of uncertain Takagi-Sugeno fuzzy models with Markovian jumps. Then, based on an LMI approach, LMI-based sufficient conditions for the uncertain Markovian jump nonlinear systems to have an $\mathcal{H}_\infty$ performance are derived. An illustrative example is used to illustrate the effectiveness of the proposed design techniques.**

I. Introduction

Many physical systems may experience abrupt changes in their structure and parameters, caused by phenomena such as component and interconnection failures, parameters shifting, tracking, and the time required to measure some of the variables at different stages. Such system can be modelled by a hybrid system with two components in the state vector. The first one which varies continuously is referred to be the continuous state of the system and the second one which varies discretely is referred to be the mode of the system. There has been an increasing interest in these types of systems during the last decade, mostly due to the growing use of computers in the control of physical plants but also as a result of the hybrid nature of physical processes. A special class of hybrid systems known as Markovian jump systems has been widely used to model manufacturing systems [1] and communication systems [2]. Although linear Markovian jump systems have been extensively studied [3]-[13], to the best of our knowledge, the control design of nonlinear Markovian jump dynamical systems remains as an open research area. Recently, there has been some attempt in this area. In [14], Hamilton-Jacobi-equation-based sufficient conditions for nonlinear Markovian jump systems to have an $\mathcal{H}_\infty$ performance have been derived. However, until now, it is still very difficult to find a global solution to the HJE either analytically or numerically.

Over the past two decades, there has been rapidly growing interest in application of fuzzy logic to control problem. Researches have been focused on its application to industrial processes and a number of successful results have been reported in the literature. In spite of these successes, there are many basic issues remain to be addressed. One of them is how to achieve a systematic design that guarantees closed-loop stability and performance. Recently, a great amount of effort has been devoted to describing a nonlinear system using a Takagi-Sugeno fuzzy model (see [15]-[30]). The Takagi-sugeno fuzzy model represents a nonlinear system by a family of local linear models which smoothly blended together through fuzzy membership functions. Unlike conventional modelling techniques which uses a single model to describe the global behavior of a nonlinear system, fuzzy modelling is essentially a multi-model approach in which simple sub-models (typically linear models) are fuzzily combined to described the global behavior of a nonlinear system. Based on this fuzzy model, a number of systematic model-based fuzzy control design methodologies have been developed.

The aim of this paper is to study the problem of designing a robust $\mathcal{H}_\infty$ fuzzy output feedback controller for a class of uncertain nonlinear systems with Markovian jumps. First, we approximate this class of uncertain nonlinear systems with Markovian jumps by a Takagi-Sugeno fuzzy model with Markovian jumps. Then based on an LMI approach, we develop a technique for designing a robust $\mathcal{H}_\infty$ fuzzy output feedback controller such that the $\mathcal{L}_2$-gain of the mapping from the exogenous input noise to the regulated output is less than a prescribed value.

This paper is organized as follows. In Section II, system descriptions and definition are presented. In Section III, based on an LMI approach, we develop a technique for designing a robust $\mathcal{H}_\infty$ fuzzy output feedback controller such that the $\mathcal{L}_2$-gain of the mapping from the exogenous input noise to the regulated output is less than a prescribed value for the system described in Section II. The validity of this approach is demonstrated by an example from a literature in Section IV. Finally, conclusions are given in Section V.

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II. System Descriptions and Definitions

The class of uncertain nonlinear system with Markovian jumps under consideration is described by the following TS

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fuzzy model with Markovian jumps:

$$\begin{array}{rcl}\dot{x}(t) & = & \sum_{i=1}^{r}\mu_i(\nu(t))\Big[[A_i(\eta(t))+\Delta A_i(\eta(t))]x(t)\\ & & +[B_{1_i}(\eta(t))+\Delta B_{1_i}(\eta(t))]w(t)\\ & & +[B_{2_i}(\eta(t))+\Delta B_{2_i}(\eta(t))]u(t)\Big],\ \ x(0)=0,\\ z(t) & = & \sum_{i=1}^{r}\mu_i(\nu(t))\Big[[C_{1_i}(\eta(t))+\Delta C_{1_i}(\eta(t))]x(t)\\ & & +[D_{12_i}(\eta(t))+\Delta D_{12_i}(\eta(t))]u(t)\Big]\\ y(t) & = & \sum_{i=1}^{r}\mu_i(\nu(t))\Big[[C_{2_i}(\eta(t))+\Delta C_{2_i}(\eta(t))]x(t)\\ & & +[D_{21_i}(\eta(t))+\Delta D_{21_i}(\eta(t))]w(t)\Big]\end{array} \tag{1}$$

where $\nu(t)=[\nu_1(t)\ \cdots\ \nu_\vartheta(t)]$ is the premise variable that may depend on states in many cases, $\mu_i(\nu(t))$ denote the normalized time-varying fuzzy weighting functions for each rule, ϑ is the number of fuzzy sets, $x(t)\in\Re^n$ is the state vector, $u(t)\in\Re^m$ is the input, $w(t)\in\Re^p$ is the disturbance which belongs to $\mathcal{L}_2[0,\infty)$, $y(t)\in\Re^\ell$ is the measurement, $z(t)\in\Re^s$ is the controlled output, the matrix functions $A_i(\eta(t))$, $B_{1_i}(\eta(t))$, $B_{2_i}(\eta(t))$, $C_{1_i}(\eta(t))$, $C_{2_i}(\eta(t))$, $D_{12_i}(\eta(t))$, $D_{21_i}(\eta(t))$, $\Delta A_i(\eta(t))$, $\Delta B_{1_i}(\eta(t))$, $\Delta B_{2_i}(\eta(t))$, $\Delta C_{1_i}(\eta(t))$, $\Delta C_{2_i}(\eta(t))$, $\Delta D_{12_i}(\eta(t))$ and $\Delta D_{21_i}(\eta(t))$ are of appropriate dimensions. $\{\eta(t))\}$ is a continuous-time discrete-state Markov process taking values in a finite set $\mathcal{S}=\{1,2,\cdots,s\}$ with transition probability matrix $Pr\overset{\Delta}{=}\{P_{\imath k}(t)\}$ given by

$$\begin{array}{rcl}P_{\imath k}(t) & = & Pr(\eta(t+\Delta)=k|\eta(t)=\imath)\\ & = & \left\{\begin{array}{ll}\lambda_{\imath k}\Delta+O(\Delta) & \text{if } \imath\neq k\\ 1+\lambda_{\imath\imath}\Delta+O(\Delta) & \text{if } \imath=k\end{array}\right.\end{array} \tag{2}$$

where $\Delta>0$, and $\lim_{\Delta\longrightarrow 0}\frac{O(\Delta)}{\Delta}=0$. Here $\lambda_{\imath k}\geq 0$ is the transition rate from mode $\imath$ (system operating mode) to mode k ($\imath\neq k$), and

$$\lambda_{\imath\imath}=-\sum_{k=1,k\neq\imath}^{s}\lambda_{\imath k}. \tag{3}$$

For the convenience of notations, we let $\mu_i\overset{\Delta}{=}\mu_i(\nu(t))$, $\eta=\eta(t)$, and any matrix $M(\mu,\imath)\overset{\Delta}{=}M(\mu,\eta=\imath)$. The matrix functions $\Delta A_i(\eta)$, $\Delta B_{1_i}(\eta)$, $\Delta B_{2_i}(\eta)$, $\Delta C_{1_i}(\eta)$, $\Delta C_{2_i}(\eta)$, $\Delta D_{12_i}(\eta)$ and $\Delta D_{21_i}(\eta)$ represent the time-varying uncertainties in the system and satisfy the following assumption.

Assumption 1:

$$\Delta A_i(\eta)=F(x(t),\eta,t)H_{1_i}(\eta),$$

$$\Delta B_{1_i}(\eta)=F(x(t),\eta,t)H_{2_i}(\eta),$$

$$\Delta B_{2_i}(\eta)=F(x(t),\eta,t)H_{3_i}(\eta),$$

$$\Delta C_{1_i}(\eta)=F(x(t),\eta,t)H_{4_i}(\eta),$$

$$\Delta C_{2_i}(\eta)=F(x(t),\eta,t)H_{5_i}(\eta),$$

$$\Delta D_{12_i}(\eta)=F(x(t),\eta,t)H_{6_i}(\eta),$$

$$\text{and }\ \Delta D_{21_i}(\eta)=F(x(t),\eta,t)H_{7_i}(\eta)$$

where $H_{j_i}(\eta)$, $j=1,2,\cdots,7$ are known matrices which characterize the structure of the uncertainties. Furthermore, there exists a positive function $\rho(\eta)$ such that the following inequality holds:

$$\|F(x(t),\eta,t)\|\leq\rho(\eta). \tag{4}$$

We recall the following definition.

Definition 1: Suppose γ is a given positive number. A system of the form (1) is said to have the $\mathcal{L}_2$-gain less than or equal to γ if

$$\mathbf{E}\left[\int_0^{T_f}\{z^T(t)z(t)-\gamma^2w^T(t)w(t)\}\,dt\right]\leq 0,\ x(0)=0 \tag{5}$$

where $\mathbf{E}[\cdot]$ stands for the mathematical expectation, for all T_f and all $w(t)\in\mathcal{L}_2[0,T_f]$.

Note that for the symmetric block matrices, we use $(*)$ as an ellipsis for terms that are induced by symmetry.

III. Robust $\mathcal{H}_\infty$ Fuzzy Output Feedback Control Design

In this section, we will present the robust $\mathcal{H}_\infty$ fuzzy output feedback control design. We first consider the following full order $\mathcal{H}_\infty$ fuzzy output feedback which is inferred as the weighted average of the local models of the form:

$$\begin{array}{rcl}\dot{\hat{x}}(t) & = & \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\Big[\hat{A}_{ij}(\imath)\hat{x}(t)+\hat{B}_i(\imath)y(t)\Big]\\ u(t) & = & \sum_{i=1}^{r}\mu_i\hat{C}_i(\imath)\hat{x}(t).\end{array} \tag{6}$$

where $\hat{A}_{ij}(\imath,\varepsilon)\in\Re^{n\times n}$. We describe the problem under our study as follows.

Problem Formulation: Given a prescribed $\mathcal{H}_\infty$ performance $\gamma>0$, design a robust output feedback controller of the form (6) that guarantees the inequality (5).

Before presenting our next result, the following lemma is needed.

Lemma 1: Consider the system (1). Given a prescribed $\mathcal{H}_\infty$ performance $\gamma>0$ and any positive constants $\delta(\imath)$, for $\imath=1,2,\cdots,s$, if there exists a matrix function $P(\imath)=P^T(\imath)$ satisfying the following linear matrix inequalities:

$$P(\imath)>0 \tag{7}$$

$$\left(\begin{array}{ccc}\left(\begin{array}{c}P(\imath)A_{cl}^{ij}(\imath)\\ +(A_{cl}^{ij}(\imath))^TP(\imath)\\ +\sum_{k=1}^{s}\lambda_{\imath k}P(k)\end{array}\right) & (*)^T & (*)^T\\ (P(\imath)B_{cl}^{ij}(\imath))^T & -\gamma^2I & (*)^T\\ C_{cl}^{ij}(\imath) & 0 & -I\end{array}\right)<0 \tag{8}$$

where $i,j=1,2,\cdots,r$,

$$A_{cl}^{ij}(\imath)=\left[\begin{array}{cc}A_i(\imath) & B_{2_i}(\imath)\hat{C}_j(\imath)\\ \hat{B}_i(\imath)C_{2_j}(\imath) & \hat{A}_{ij}(\imath)\end{array}\right],$$

$$B_{cl}^{ij}(\imath)=\left[\begin{array}{c}\tilde{B}_{1_i}(\imath)\\ \hat{B}_i(\imath)\tilde{D}_{21_j}(\imath)\end{array}\right]$$

$$\textit{and }\ C_{cl}^{ij}(\imath)=[\tilde{C}_{1_i}(\imath)\quad\tilde{D}_{12_i}(\imath)\hat{C}_j(\imath)]$$

with

$$
\begin{aligned}
\tilde{B}_{1_i}(\imath) &= [\delta(\imath)I \;\; I \;\; \delta(\imath)I \;\; 0 \;\; B_{1_i}(\imath) \;\; 0] \\
\tilde{C}_{1_i}(\imath) &= \Big[\frac{\gamma\rho(\imath)}{\delta(\imath)}H_{1_i}^T(\imath) \;\; 0 \;\; \frac{\gamma\rho(\imath)}{\delta(\imath)}H_{5_i}^T(\imath) \\
&\qquad \sqrt{2}\aleph(\imath)\rho(\imath)H_{4_i}^T(\imath) \;\; \sqrt{2}\aleph(\imath)C_{1_i}^T(\imath)\Big]^T \\
\tilde{D}_{12_i}(\imath) &= \Big[0 \;\; \frac{\gamma\rho(\imath)}{\delta(\imath)}H_{3_i}^T(\imath) \;\; 0 \\
&\qquad \sqrt{2}\aleph(\imath)\rho(\imath)H_{6_i}^T(\imath) \;\; \sqrt{2}\aleph(\imath)D_{12_i}^T(\imath)\Big]^T \\
\tilde{D}_{21_i}(\imath) &= [0 \;\; 0 \;\; 0 \;\; \delta(\imath)I \;\; D_{21_i}(\imath) \;\; I] \\
\aleph(\imath) &= \Big(1 + \rho^2(\imath)\sum_{i=1}^{r}\sum_{j=1}^{r}\Big[\|H_{2_i}^T(\imath)H_{2_j}(\imath)\| \\
&\qquad + \|H_{7_i}^T(\imath)H_{7_j}(\imath)\|\Big]\Big)^{\frac{1}{2}},
\end{aligned}
$$

then the inequality (5) is guaranteed.

Proof: Due to limited pages, the detail of the proof is omitted for brevity. ∎

The left hand side of (8) can be re-expressed as follows:

$$
\begin{aligned}
&P(\imath)A_{cl}^{ij}(\imath) + (A_{cl}^{ij}(\imath))^T P(\imath) + \textstyle\sum_{k=1}^{s}\lambda_{\imath k}P(k) \\
&\gamma^{-2}P(\imath)B_{cl}^{ij}(\imath)(B_{cl}^{ij}(\imath))^T P(\imath) + (C_{cl}^{ij}(\imath))^T C_{cl}^{ij}(\imath).
\end{aligned} \tag{9}
$$

Before providing LMI-based sufficient conditions for the system (1) to have the $\mathcal{H}_\infty$ performance, let us partition the matrix $P(\imath)$ as follows:

$$
P(\imath) = \begin{bmatrix} X(\imath) & Y^{-1}(\imath) - X(\imath) \\ Y^{-1}(\imath) - X(\imath) & X(\imath) - Y^{-1}(\imath) \end{bmatrix} \tag{10}
$$

where $X(\imath) = X^T(\imath) \in \Re^{n\times n}$ and $Y(\imath) = Y^T(\imath) \in \Re^{n\times n}$. Utilizing the partition above, we define the new controller's matrices as

$$
\begin{aligned}
\mathcal{B}(\mu,\imath) &\triangleq \Big[Y^{-1}(\imath) - X(\imath)\Big]\hat{B}(\mu,\imath) \\
\mathcal{C}(\mu,\imath) &\triangleq \hat{C}(\mu,\imath)Y(\imath).
\end{aligned} \tag{11}
$$

The following theorem provides LMI-based sufficient conditions for the system (1) to have an $\mathcal{H}_\infty$ performance γ with μ_i being available for feedback.

Theorem 1: Consider the system (1). Given a prescribed $\mathcal{H}_\infty$ performance $\gamma > 0$ and any positive constants $\delta(\imath)$, for $\imath = 1, 2, \cdots, s$, if there exist matrices $X(\imath) = X^T(\imath)$, $Y(\imath) = Y^T(\imath)$, $\mathcal{B}_i(\imath)$ and $\mathcal{C}_i(\imath)$, $i = 1, 2, \cdots, r$, satisfying the following linear matrix inequalities:

$$
\begin{bmatrix} X(\imath) & I \\ I & Y(\imath) \end{bmatrix} > 0 \tag{12}
$$

$$
X(\imath) > 0 \tag{13}
$$

$$
Y(\imath) > 0 \tag{14}
$$

$$
\Psi_{11_{ii}}(\imath) < 0, \quad i = 1, 2, \cdots, r \tag{15}
$$

$$
\Psi_{22_{ii}}(\imath) < 0, \quad i = 1, 2, \cdots, r \tag{16}
$$

$$
\Psi_{11_{ij}}(\imath) + \Psi_{11_{ji}}(\imath) < 0, \quad i < j \le r \tag{17}
$$

$$
\Psi_{22_{ij}}(\imath) + \Psi_{22_{ji}}(\imath) < 0, \quad i < j \le r \tag{18}
$$

where

$$
\Psi_{11_{ij}}(\imath) = \begin{pmatrix}
\begin{pmatrix} A_i(\imath)Y(\imath) \\ +Y(\imath)A_i^T(\imath) \\ +\lambda_{\imath\imath}Y(\imath) \\ +\frac{\tilde{B}_{1_i}(\imath)\tilde{B}_{1_j}^T(\imath)}{\gamma^{-2}} \\ +B_{2_i}(\imath)\mathcal{C}_j(\imath) \\ +\mathcal{C}_i^T(\imath)B_{2_j}^T(\imath) \end{pmatrix} & (*)^T & (*)^T \\
\begin{pmatrix} \tilde{C}_{1_i}(\imath)Y(\imath) \\ +\tilde{D}_{12_i}(\imath)\mathcal{C}_j(\imath) \end{pmatrix} & -I & (*)^T \\
\mathcal{J}^T(\imath) & 0 & -\mathcal{Y}(\imath)
\end{pmatrix} \tag{19}
$$

$$
\Psi_{22_{ij}}(\imath) = \begin{pmatrix}
\begin{pmatrix} A_i^T(\imath)X(\imath) \\ +X(\imath)A_i(\imath) \\ +\mathcal{B}_i(\imath)C_{2_j}(\imath) \\ +C_{2_i}^T(\imath)\mathcal{B}_j^T(\imath) \\ +\tilde{C}_{1_i}^T(\imath)\tilde{C}_{1_j}(\imath) \\ +\sum_{k=1}^{s}\lambda_{\imath k}X(k) \end{pmatrix} & (*)^T \\
\begin{pmatrix} X(\imath)\tilde{B}_{1_i}(\imath) \\ +\mathcal{B}_i(\imath)\tilde{D}_{21_j}(\imath) \end{pmatrix}^T & -\gamma^2 I
\end{pmatrix} \tag{20}
$$

with

$$
\begin{aligned}
\mathcal{J}(\imath) &= \Big[\sqrt{\lambda_{1\imath}}Y(\imath) \;\; \cdots \;\; \sqrt{\lambda_{(i-1)\imath}}Y(\imath) \\
&\qquad \sqrt{\lambda_{(i+1)\imath}}Y(\imath) \;\; \cdots \;\; \sqrt{\lambda_{s\imath}}Y(\imath)\Big] \\
\mathcal{Y}(\imath) &= diag\Big\{Y(1), \;\; \cdots, \;\; Y(\imath-1), \\
&\qquad Y(\imath+1), \;\; \cdots \;\; ,Y(s)\Big\} \\
\tilde{B}_{1_i}(\imath) &= [\delta(\imath)I \;\; I \;\; \delta(\imath)I \;\; 0 \;\; B_{1_i}(\imath) \;\; 0] \\
\tilde{C}_{1_i}(\imath) &= \Big[\frac{\gamma\rho(\imath)}{\delta(\imath)}H_{1_i}^T(\imath) \;\; 0 \;\; \frac{\gamma\rho(\imath)}{\delta(\imath)}H_{5_i}^T(\imath) \\
&\qquad \sqrt{2}\aleph(\imath)\rho(\imath)H_{4_i}^T(\imath) \;\; \sqrt{2}\aleph(\imath)C_{1_i}^T(\imath)\Big]^T \\
\tilde{D}_{12_i}(\imath) &= \Big[0 \;\; \frac{\gamma\rho(\imath)}{\delta(\imath)}H_{3_i}^T(\imath) \;\; 0 \\
&\qquad \sqrt{2}\aleph(\imath)\rho(\imath)H_{6_i}^T(\imath) \;\; \sqrt{2}\aleph(\imath)D_{12_i}^T(\imath)\Big]^T \\
\tilde{D}_{21_i}(\imath) &= [0 \;\; 0 \;\; 0 \;\; \delta(\imath)I \;\; D_{21_i}(\imath) \;\; I] \\
\aleph(\imath) &= \Big(1 + \rho^2(\imath)\sum_{i=1}^{r}\sum_{j=1}^{r}\Big[\|H_{2_i}^T(\imath)H_{2_j}(\imath)\| \\
&\qquad + \|H_{7_i}^T(\imath)H_{7_j}(\imath)\|\Big]\Big)^{\frac{1}{2}},
\end{aligned}
$$

then the prescribed $\mathcal{H}_\infty$ performance $\gamma > 0$ is guaranteed. Furthermore, a suitable controller is of the form (6) with

$$
\begin{aligned}
\hat{A}_{ij}(\imath) &= \big[Y^{-1}(\imath) - X(\imath)\big]^{-1}\mathcal{M}_{ij}(\imath)Y^{-1}(\imath) \\
\hat{B}_i(\imath) &= \big[Y^{-1}(\imath) - X(\imath)\big]^{-1}\mathcal{B}_i(\imath) \\
\hat{C}_i(\imath) &= \mathcal{C}_i(\imath)Y^{-1}(\imath)
\end{aligned} \tag{21}
$$

where

$$\begin{aligned}\mathcal{M}_{ij}(\imath) \quad =& -A_i^T(\imath) - X(\imath)A_i(\imath)Y(\imath)\\ &-\big[Y^{-1}(\imath) - X(\imath)\big]\hat{B}_i(\imath)C_{2_j}(\imath)Y(\imath)\\ &-X(\imath)B_{2_i}(\imath)\hat{C}_j(\imath)Y(\imath)\\ &-\textstyle\sum_{k=1}^{s}\lambda_{\imath k}Y^{-1}(k)Y(\imath)\\ &-\tilde{C}_{1_i}^T(\imath)\Big[\tilde{C}_{1_j}(\imath)Y(\imath) + \tilde{D}_{12_j}(\imath)\hat{C}_j(\imath)Y(\imath)\Big]\\ &-\gamma^{-2}\Big\{X(\imath)\tilde{B}_{1_i}(\imath) + \big[Y^{-1}(\imath) - X(\imath)\big]\times\\ &\qquad \hat{B}_i(\imath)\tilde{D}_{21_i}(\imath)\Big\}\tilde{B}_{1_j}^T(\imath).\end{aligned} \tag{22}$$

Proof: Due to limited pages, the detail of the proof is omitted for brevity. ■

IV. Illustrative Example

Consider a modified Samuelson multiplier-accelerator economic model based on [34] which is governed by the following differential equations:

$$\begin{aligned}\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} &= \begin{bmatrix}1 & 1\\ -\alpha(\imath) & \begin{bmatrix}\alpha(\imath)+1\\ +\beta(\imath)x_2(t)\end{bmatrix}\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}\\ &\quad + \begin{bmatrix}0 & 0\\ 0.1 & 0\end{bmatrix}w(t) + \begin{bmatrix}0\\ 1\end{bmatrix}u(t)\\ &\quad + \begin{bmatrix}0 & 0\\ -\Delta\alpha(\imath) & \begin{bmatrix}\Delta\alpha(\imath)\\ +\Delta\beta(\imath)x_2(t)\end{bmatrix}\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}\\ z(t) &= \begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}\\ y(t) &= Jx(t) + [0 \;\; 0.1]w(t)\end{aligned} \tag{23}$$

where $x_1(t)$ and $x_2(t)$ are the state vectors, $u(t)$ is the controlled input which represents the deviation of the government expenditure from the desired government expenditure, $w(t)$ is the disturbance input which represents the unexpected behavior of the economy, $z(t)$ is the controlled output, $y(t)$ is the measured output, J is the sensor matrix, α is the accelerator coefficient, β is the marginal propensity to consume parameter and υ is the consume parameter ($\upsilon \geq 1$). $\Delta\alpha$ and $\Delta\beta$ are the uncertain accelerator coefficient and marginal propensity to consume parameter, respectively. We assume that $\upsilon = 2$, $|\Delta\alpha| \leq 0.1\alpha$ and $|\Delta\beta| \leq 0.1\beta$.

Based on [34], the general economic situations could be aggregated into three modes as shown in Table I:

TABLE I
Economic Terminology.

Mode $\imath$	Terminology	$\alpha(\imath) \pm \Delta\alpha(\imath)$	$\beta(\imath) \pm \Delta\beta(\imath)$
1	Normal	2.5 $\pm$10%	0.3 $\pm$10%
2	Boom	43.7 $\pm$10%	$-0.7 \pm$ 10%
3	Slump	$-5.3 \pm$ 10%	0.9 $\pm$10%

The transition probability matrix that relates the three operation modes is given as follows:

$$P_{\imath k} = \begin{bmatrix}0.67 & 0.17 & 0.16\\ 0.30 & 0.47 & 0.23\\ 0.26 & 0.10 & 0.64\end{bmatrix}.$$

The control objective is to control the state variable $x_2(t)$ for the range $x_2(t) \in [N_1 \; N_2]$. For the sake of simplicity, we will use as few rules as possible. Note that Figure 1 shows the plot of the membership functions represented by

$$M_1(x_2(t)) = \frac{-x_2(t) + N_2}{N_2 - N_1}$$

$$\text{and} \;\; M_2(x_2(t)) = \frac{x_2(t) - N_1}{N_2 - N_1}.$$

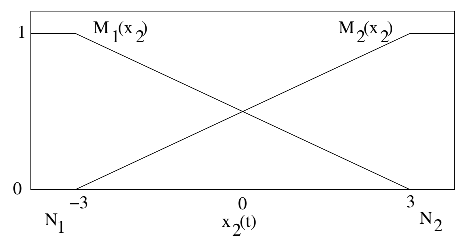

Fig. 1. Membership functions for the two fuzzy set.

Knowing that $x_2(t) \in [N_1 \; N_2]$, the nonlinear system (23) can be approximated by the following TS fuzzy model:

$$\begin{aligned}\dot{x}(t) &= \sum_{i=1}^{r}\mu_i\Big[[A_i(\imath) + \Delta A_i(\imath)]x(t)\\ &\qquad + B_{1_i}(\imath)w(t) + B_{2_i}(\imath)u(t)\Big], \quad x(0) = 0,\\ z(t) &= \sum_{i=1}^{r}\mu_i C_{1_i}(\imath)x(t),\\ y(t) &= \sum_{i=1}^{r}\mu_i\Big[C_{2_i}(\imath)x(t) + D_{21_i}(\imath)w(t)\Big].\end{aligned}$$

where $x(t) = [x_1^T(t) \; x_2^T(t)]^T$,

$$A_i(1) = \begin{bmatrix}1 & 1\\ -2.5 & 3.5 + 0.7N_i\end{bmatrix},$$

$$A_i(2) = \begin{bmatrix}1 & 1\\ -43.7 & 44.5 + 1.7N_i\end{bmatrix},$$

$$A_i(3) = \begin{bmatrix}1 & 1\\ 5.3 & -4.3 + 0.1N_i\end{bmatrix},$$

$$B_{1_i}(\imath) = \begin{bmatrix}0 & 0\\ 0.1 & 0\end{bmatrix}, \; B_{2_i}(\imath) = \begin{bmatrix}0\\ 1\end{bmatrix},$$

$$C_{1_i}(\imath) = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \; C_{2_i}(\imath) = J,$$

$$D_{21_i}(\imath) = \begin{bmatrix}0 & 0.1\end{bmatrix},$$

$$\text{and } \Delta A_i(\imath) = F(x(t), \imath, t)H_{1_i}(\imath)$$

with $i = 1, 2$. Now, by assuming that in (23), $\|F(x(t), \imath, t)\| \leq \rho(\imath) = 1$, we have

$$H_{1_1}(\imath) = \begin{bmatrix}0 & 0\\ -0.1\alpha(\imath) & 0.1\alpha(\imath) + 0.1\beta(\imath)N_1\end{bmatrix}$$

and $H_{1_2}(\imath) = \begin{bmatrix} 0 & 0 \\ -0.1\alpha(\imath) & 0.1\alpha(\imath) + 0.1\beta(\imath)N_2 \end{bmatrix}$.

In this simulation, we select $N_1 = -3$ and $N_2 = 3$. Using the LMI optimization algorithm and Theorem 1 with $\gamma = 1$, $J = [0 \;\; 1]$, and $\delta(\imath) = 1$,we obtain

$$\hat{A}_{11}(1) = \begin{bmatrix} 10.0318 & 1.0079 \\ -123.0107 & -65.2591 \end{bmatrix},$$

$$\hat{A}_{12}(1) = \begin{bmatrix} 10.1591 & 0.9003 \\ -118.6456 & -68.7917 \end{bmatrix},$$

$$\hat{A}_{21}(1) = \begin{bmatrix} 10.0079 & 1.0933 \\ -122.9968 & -61.0529 \end{bmatrix},$$

$$\hat{A}_{22}(1) = \begin{bmatrix} 10.1352 & 0.9857 \\ -118.6317 & -64.5855 \end{bmatrix},$$

$$\hat{B}_1(1) = \begin{bmatrix} -31.6693 \\ 7.9142 \end{bmatrix}, \quad \hat{B}_2(1) = \begin{bmatrix} -31.5525 \\ 12.0990 \end{bmatrix},$$

$$\hat{C}_1(1) = \begin{bmatrix} -0.5184 & -141.9962 \end{bmatrix},$$

$$\hat{C}_2(1) = \begin{bmatrix} -5.8859 & -159.0397 \end{bmatrix},$$

$$\hat{A}_{11}(2) = 10^3 \times \begin{bmatrix} 0.1388 & 1.0140 \\ -1.3391 & -9.8999 \end{bmatrix},$$

$$\hat{A}_{12}(2) = 10^3 \times \begin{bmatrix} 0.0128 & 0.1141 \\ -0.1237 & -1.1135 \end{bmatrix},$$

$$\hat{A}_{21}(2) = 10^3 \times \begin{bmatrix} 0.0490 & 0.2208 \\ -0.4355 & -3.2840 \end{bmatrix},$$

$$\hat{A}_{22}(2) = 10^3 \times \begin{bmatrix} 0.0582 & 0.2511 \\ -0.5624 & -3.7084 \end{bmatrix},$$

$$\hat{B}_1(2) = \begin{bmatrix} -28.3321 \\ 173.4354 \end{bmatrix}, \quad \hat{B}_2(2) = \begin{bmatrix} -30.5197 \\ 204.8014 \end{bmatrix},$$

$$\hat{C}_1(2) = 10^3 \times \begin{bmatrix} -0.1107 & -1.2470 \end{bmatrix},$$

$$\hat{C}_2(2) = 10^3 \times \begin{bmatrix} -0.1532 & -1.3909 \end{bmatrix},$$

$$\hat{A}_{11}(3) = \begin{bmatrix} -44.6535 & -49.5945 \\ -198.0970 & -193.4395 \end{bmatrix},$$

$$\hat{A}_{12}(3) = \begin{bmatrix} -44.7323 & -49.9065 \\ -198.3023 & -194.2173 \end{bmatrix},$$

$$\hat{A}_{21}(3) = \begin{bmatrix} -44.5448 & -48.9764 \\ -198.0298 & -192.4832 \end{bmatrix},$$

$$\hat{A}_{22}(3) = \begin{bmatrix} -44.6237 & -49.2884 \\ -198.2351 & -193.2610 \end{bmatrix},$$

$$\hat{B}_1(3) = \begin{bmatrix} 57.1346 \\ 32.6070 \end{bmatrix}, \quad \hat{B}_2(3) = \begin{bmatrix} 57.4491 \\ 33.3825 \end{bmatrix},$$

$$\hat{C}_1(3) = \begin{bmatrix} -173.3989 & -161.6216 \end{bmatrix},$$

$$\hat{C}_2(3) = \begin{bmatrix} -173.5604 & -162.2239 \end{bmatrix}.$$

The resulting fuzzy controller is

$$\begin{aligned} \dot{\hat{x}}(t) &= \sum_{i=1}^{2}\sum_{j=1}^{2} \mu_i\mu_j \hat{A}_{ij}(\imath)\hat{x}(t) + \sum_{i=1}^{2} \mu_i \hat{B}_i(\imath) y(t) \\ u(t) &= \sum_{i=1}^{2} \mu_i \hat{C}_i(\imath)\hat{x}(t) \end{aligned}$$

where

$$\mu_1 = M_1(x_2(t)) \text{ and } \mu_2 = M_2(x_2(t)).$$

Remark 1: The robust fuzzy output feedback controller guarantees that the $\mathcal{L}_2$-gain, γ, is less than the prescribed value. Figure 2 shows the changing between modes during the simulation with the initial mode 1. The disturbance input signal, $w(t)$, which was used during simulation is the rectangular signal (magnitude 0.1 and frequency 10 Hz). The ratios of the regulated output energy to the disturbance input noise energy for both cases are depicted in Figure 3. After time = 3, the ratio of the regulated output energy to the disturbance input noise energy tends to a constant value which is about 0.22. Thus, for output feedback controller where $\gamma = \sqrt{0.22} = 0.469$, all are less than the prescribed value 1. □

V. Conclusion

This paper has proposed a technique for designing an $\mathcal{H}_\infty$ output feedback controller for a class of fuzzy Markovian jump dynamic systems that guarantees the $\mathcal{L}_2$-gain from an exogenous input to a regulated output is less or equal to a prescribed value. Based on an LMI approach, LMI-based sufficient conditions for the uncertain Markovian jump Takagi-Sugeno fuzzy model to have an $\mathcal{H}_\infty$ performance are established. The effectiveness of the proposed design methodology is demonstrated through a numerical simulation system.

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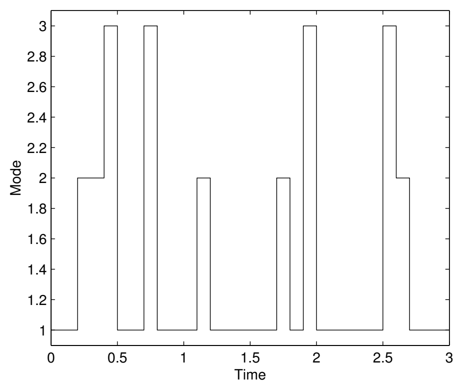

Fig. 2. The result of the changing between modes during the simulation with the initial mode 1.

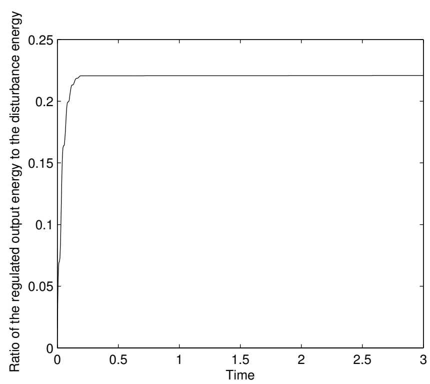

Fig. 3. The ratio of the regulated output energy to the disturbance noise energy, $\frac{\int_0^{T_f} z^T(t)z(t)dt}{\int_0^{T_f} w^T(t)w(t)dt}$.

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