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State Feedback Control of Piecewise-Affine Systems with Norm Bounded Noise

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Abstract— **Being able to design controllers that are robust to noisy measurements is of fundamental importance in practical applications. With this objective in mind, the original contribution of this paper is to propose a design technique for state feedback control of piecewise-affine systems that is robust to bounded noise in the measurements. More specifically, the paper gives conditions under which a piecewise-affine state feedback controller designed to stabilize a noise-free piecewise-affine system to a target point still stabilizes the system when the state is subject to norm bounded noisy measurements. It will be shown that controllers designed using a globally quadratic Lyapunov function are robust to noisy state measurements and that the state trajectories of the closed-loop system will still converge to a region around the equilibrium point in the presence of noise. The size of this region will be related to the norm bound on the noise.**

I. Introduction

Piecewise-affine systems are multi-model systems that offer a good modeling framework for complex dynamical systems involving nonlinear phenomena. In fact, many nonlinearities that appear frequently in engineering systems are either piecewise-affine (e.g., a saturated linear actuator characteristic) or can be approximated as piecewise-affine functions. Piecewise-affine systems are also a class of hybrid systems, i.e, they are systems with a continuous time-driven state and a discrete event-driven state. For piecewise-affine systems the discrete-event state is associated with discrete modes of operation. The continuous-time state is associated with the affine (linear with offset) dynamics valid within each discrete mode. Piecewise-affine systems pose challenging problems because of its switched structure. In fact, the analysis and control of even some simple piecewise-affine systems have been shown to be either an $\mathcal{NP}$ hard problem or undecidable [1].

State and output feedback control of continuous-time piecewise-affine systems has received increasing interest over the past decade [2], [3], [4], [5]. The state feedback approach presented in [3] relies on computing upper and lower bounds to the optimal cost of the controller obtained as the solution to the Hamilton-Jacobi-Bellman equation. The continuous-time controller resulting from the approach in [3] is a patched LQR that cannot be guaranteed to avoid sliding modes at the switching and, therefore, is not provably stabilizing. Reference [4] presents a formulation of both state and output feedback controller synthesis for piecewise-affine systems. The control design is based on a piecewise-quadratic control Lyapunov function and is formulated as an optimization problem subject to a bilinear matrix inequality. Because of this constraint, the problem is $\mathcal{NP}$-hard and suboptimal solutions must be sought for problems of considerable size, as the ones occuring frequently in applications. Three local solution algorithms were suggested in [4] to find a suboptimal solution. Several examples demonstrated the performance of the controllers. Inspired by the work of Hassibi and Boyd [2], a new formulation of the piecewise-affine state feedback synthesis problem was suggested in [5]. The control problem was formulated as a convex optimization program subject to an infinite number of LMI constraints analitically parameterized by a vector. This vector can then be sampled and a relaxation of the problem can be solved. Alternatively, as described in [5], a concave optimization problem can be formulated and in case there is a solution to the concave problem it was shown that it will also be a solution to the original state feedback problem. However, none of previous approaches [2], [3], [4], [5] consider the case when the state measurement is subject to noise, which is the case typically faced in a practical application.

Based on these considerations, the contribution of this paper is to propose a design technique for state feedback control of piecewise-affine systems that is robust to bounded noise in the measurements. More specifically, the paper presents conditions under which a state feedback controller designed for a noise-free piecewise-affine system still stabilizes the system when subject to norm bounded noise. The paper starts by stating the problem assumptions followed by a section on the design of piecewise-affine state feedback controllers for noise-free piecewise-affine systems. Then, it is shown under which conditions the trajectories of the closed-loop system converge to a region around the closed-loop equilibrium point in the presence of noise. The size of this region will be connected to the bound on the norm of the noise in the measurements.

II. Problem Assumptions

It is assumed that a PWA system and a corresponding partition of the state space with polytopic cells $\mathcal{R}_i$, $i \in \mathcal{I} = \{1, \ldots, M\}$ are given (see [6] for generating such a partition). Following [7], [8], [2], each cell is constructed as the intersection of a finite number (p_i) of half spaces

$$\mathcal{R}_i = \{x \,|\, H_i^T x - \tilde{g}_i < 0\}, \tag{1}$$

where $H_i = [h_{i1} \;\; h_{i2} \ldots h_{ip_i}]$, $\tilde{g}_i = [\tilde{g}_{i1} \;\; \tilde{g}_{i2} \ldots \tilde{g}_{ip_i}]^T$. Moreover, the sets $\mathcal{R}_i$ partition a subset of the state space $\mathcal{X} \subset I\!R^n$ such that $\cup_{i=1}^M \overline{\mathcal{R}}_i = \mathcal{X}$, $\mathcal{R}_i \cap \mathcal{R}_j = \emptyset$, $i \neq j$, where $\overline{\mathcal{R}}_i$ denotes the closure of $\mathcal{R}_i$. Within each cell the dynamics are affine of the form

$$\dot{x}(t) = A_i x(t) + \tilde{b}_i + B_i u(t), \tag{2}$$

where $x(t) \in I\!R^n$ and $u(t) \in I\!R^m$. For system (2), we adopt the following definition of trajectories or solutions presented in [9].

Definition 2.1: [9] Let $x(t) \in \mathcal{X}$ be an absolutely continuous function. Then $x(t)$ is a trajectory of the system (2) on $[t_0, t_f]$ if, for almost all $t \in [t_0, t_f]$ and Lebesgue measurable $u(t)$, the equation $\dot{x}(t) = A_i x(t) + \tilde{b}_i + B_i u(t)$ holds for $x(t) \in \overline{\mathcal{R}}_i$. □

Any two cells sharing a common facet will be called *level-1* neighboring cells. Let $\mathcal{N}_i = \{$*level*-1 neighboring cells of $\mathcal{R}_i\}$. It is also assumed that vectors $c_{ij} \in I\!R^n$ and scalars d_{ij} exist such that the facet boundary between cells $\mathcal{R}_i$ and $\mathcal{R}_j$ is contained in the hyperplane described by $\{x \in I\!R^n \,|\, c_{ij}^T x - d_{ij} = 0\}$, for $i = 1, \ldots, M$, $j \in \mathcal{N}_i$. A parametric description of the boundaries can then be obtained as [2]

$$\overline{\mathcal{R}_i} \cap \overline{\mathcal{R}_j} \subseteq \{x = \tilde{l}_{ij} + F_{ij} s \;|\; s \in I\!R^{n-1}\} \tag{3}$$

for $i = 1, \ldots, M$, $j \in \mathcal{N}_i$, where $F_{ij} \in I\!R^{n \times (n-1)}$ (full rank) is the matrix whose columns span the null space of c_{ij}^T, and $\tilde{l}_{ij} \in I\!R^n$ is given by $\tilde{l}_{ij} = c_{ij}\left(c_{ij}^T c_{ij}\right)^{-1} d_{ij}$. For systems whose polytopic cells are slabs, called *piecewise-affine slab systems*, each $\mathcal{R}_i$ can be outer approximated by a degenerate ellipsoid ε_i. This covering will be used to describe the regions instead of the polytopic description. The ellipsoidal description of piecewise-affine systems is useful because it often requires fewer parameters than the polytopic description and it enables to cast the synthesis problem as an optimization program involving a set of LMIs analitically parameterized by a vector. To describe the ellipsoidal covering, it is assumed that matrices E_i and $\tilde{f}_i$ exist such that

$$\mathcal{R}_i \subseteq \varepsilon_i \tag{4}$$

where

$$\varepsilon_i = \{x | \, \|E_i x + \tilde{f}_i\| \leq 1\}. \tag{5}$$

This covering is especially useful in the case where $\mathcal{R}_i$ is a slab because in this case the matrices E_i and $\tilde{f}_i$ are guaranteed to exist and the covering (having one degenerate ellipsoid ε_i) is exact, ***i.e.***, $\varepsilon_i \subseteq R_i$ and $R_i \subseteq \varepsilon_i$. More precisely, if $\mathcal{R}_i = \{x \,|\, d_1 < c_i^T x < d_2\}$, then the degenerate ellipsoid is described by $E_i = 2c_i^T/(d_2 - d_1)$ and $\tilde{f}_i = -(d_2 + d_1)/(d_2 - d_1)$. Finally, it is assumed that the control objective is to stabilize the system to a given point x_{cl}. With the change of coordinates $z = x - x_{cl}$ the problem is transformed to the stabilization of the origin. In these coordinates, the system dynamics (2) are

$$\dot{z}(t) = A_i z(t) + b_i + B_i u(t), \tag{6}$$

where $b_i = \tilde{b}_i + A_i x_{cl}$. The parametric description of the boundaries (3) is written as

$$\overline{\mathcal{R}_i} \cap \overline{\mathcal{R}_j} \subseteq \{z = l_{ij} + F_{ij} s \,|\, s \in I\!R^{n-1}\} \tag{7}$$

where $l_{ij} = \tilde{l}_{ij} - x_{cl}$ for $i = 1, \ldots, M$, $j \in \mathcal{N}_i$. The description of the polytopic cells is

$$\mathcal{R}_i = \{z \,|\, H_i^T z - g_i < 0\}, \tag{8}$$

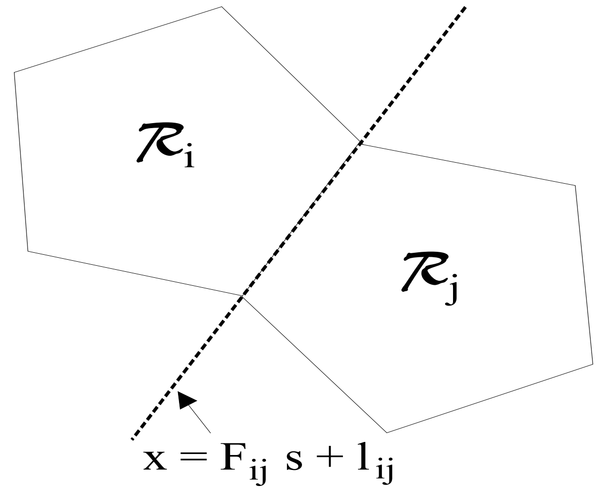

Fig. 1. Polytopic regions $\mathcal{R}_i$, $\mathcal{R}_j$ and boundary

where $g_i = \tilde{g}_i - H_i^T x_{cl}$, and the ellipsoidal covering is described by

$$\varepsilon_i = \{z \mid \|E_i z + f_i\| \leq 1\}, \tag{9}$$

where $f_i = \tilde{f}_i + E_i x_{cl}$.

III. Lyapunov-based Controller Synthesis

The piecewise-affine state feedback input signal is parameterized by K_i and m_i in the form

$$u = K_i z + m_i, \quad z \in \mathcal{R}_i \tag{10}$$

with $-l_0 \leq m_i \leq l_0$ where l_0 is a vector of upper bounds for the entries of $m_i,\ i = 1, \ldots, M$. The globally quadratic candidate control Lyapunov function is parameterized by $P = P^T$ as

$$V(z) = z^T P z. \tag{11}$$

The candidate control Lyapunov function (11) becomes a Lyapunov function with decay rate α if for fixed $\alpha \geq 0$, $V > 0$ and $\dot{V} < -\alpha V$. Using (6) and (10), sufficient conditions for exponential stability are $P = P^T > 0$ and

$$\begin{aligned} z \in \mathcal{R}_i \Rightarrow \left[(A_i + B_i K_i) z + (b_i + B_i m_i)\right]^T P z + \\ z^T P \left[(A_i + B_i K_i) z + (b_i + B_i m_i)\right] + \alpha z^T P z < 0. \end{aligned} \tag{12}$$

This expression can be recast as

$$z \in \mathcal{R}_i \Rightarrow \begin{bmatrix} z \\ 1 \end{bmatrix}^T \begin{bmatrix} \bar{A}_i^T P + P\bar{A}_i + \alpha P & P\bar{b}_i \\ (P\bar{b}_i)^T & 0 \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix} < 0, \tag{13}$$

where $\bar{A}_i = A_i + B_i K_i$ and $\bar{b}_i = b_i + B_i m_i$. If we relax the condition $z \in \mathcal{R}_i$ in (13) by $z \in \varepsilon_i$ and if we use expression (9) and the $\mathcal{S}-procedure$ [10] yields the following sufficient conditions for quadratic stabilization (see [5] for details)

$$\begin{gathered} P = P^T > 0, \quad \lambda_i < 0, \quad i = 1, \ldots, M, \\ \begin{bmatrix} \bar{A}_i^T P + P\bar{A}_i + \alpha P + \lambda_i E_i^T E_i & (\cdot) \\ (P\bar{b}_i + \lambda_i E_i^T f_i)^T & -\lambda_i \left(1 - f_i^T f_i\right) \end{bmatrix} < 0 \end{gathered} \tag{14}$$

These conditions are ***Bilinear Matrix Inequalities*** (BMIs) [11] because they involve terms with products of the unknowns P and K_i. However, for ***piecewise-affine slab systems***, the following procedure can be used to make the dependence on K_i and P be linear rather than bilinear. The procedure involves the change of variables $Q = P^{-1}$, $\mu_i = \lambda_i^{-1}$ and the algebraic manipulation presented in [5] to yield

$$\begin{gathered} Q = Q^T > 0, \quad \mu_i < 0, \quad i = 1, \ldots, M, \\ \begin{bmatrix} \bar{A}_i Q + Q\bar{A}_i^T + \alpha Q + \mu_i \bar{b}_i \bar{b}_i^T & \mu_i \bar{b}_i f_i^T + QE_i^T \\ (\mu_i \bar{b}_i f_i^T + QE_i^T)^T & -\mu_i \left(I - f_i f_i^T\right) \end{bmatrix} < 0 \end{gathered} \tag{15}$$

Performing now the substitution $\bar{A}_i = A_i + B_i K_i$ and introducing new variables $Y_i = K_i Q$ in (15) yields

$$\begin{gathered} Q = Q^T > 0, \quad \mu_i < 0, \quad i = 1, \ldots, M, \\ \begin{bmatrix} W_i + W_i^T + \alpha Q + \mu_i \bar{b}_i \bar{b}_i^T & \mu_i \bar{b}_i f_i^T + QE_i^T \\ (\mu_i \bar{b}_i f_i^T + QE_i^T)^T & -\mu_i \left(I - f_i f_i^T\right) \end{bmatrix} < 0, \end{gathered} \tag{16}$$

where $W_i = A_i Q + B_i Y_i$, $\bar{b}_i = b_i + B_i m_i$. Notice that the dependence on Q and Y_i is now linear at the cost of a bilinear dependence on m_i. However, m_i is typically of much lower dimension than Y_i (e.g, for single input systems Y_i has the same dimension of the state space, which can be quite large, and m_i is a scalar). The piecewise-affine state feedback stabilization problem is now formally defined.

Definition 3.1: The piecewise-affine state feedback problem is: for fixed $\alpha \geq 0,\ i = 1, \ldots, M$,

$$\begin{array}{ll} \text{find} & Q,\ Y_i,\ m_i,\ \mu_i \\ s.t. & Q = Q^T > 0, \quad \mu_i < 0, \quad (16) \\ & -l_1 \prec Y_i \prec l_1,\ -l_0 \prec m_i \prec l_0 \end{array}$$

Note that for fixed $m_i,\ i = 1, \ldots, M$, expression (16) is an LMI and the problem is convex. Therefore, although the problem formulated in (16) cannot be cast as one convex program, it is an infinite set of convex problems involving an LMI or, equivalently, an infinite number of LMIs analitically parameterized by the vector $\gamma = [m_1^T\ m_2^T\ \ldots\ m_M^T]^T$. Since each element $m_i,\ i = 1, \ldots, M$ has bounded components, γ belongs to an hypercube. Effective meshing techniques can then be used to sample the hypercube and solve a relaxation of problem 3.1 where now the LMI constraints form a finite set. The following algorithm is suggested to solve the state-feedback problem:

Algorithm # 1 – Sampling Method:

1) Define a grid for the domain of the vector γ to sample it at N points,
2) For fixed $\alpha \geq 0$, solve the corresponding feasibility problem 3.1 for each of the points in the grid until a feasible point is found.
3) If step 2 is successful or if the maximum number of iterations was reached, stop. Otherwise, increase the grid density and go back to Step 2.

The feasibility problem from definition 3.1 can be transformed into an optimization problem if the Q with minimum condition number is sought as follows:

Definition 3.2: The minimum condition number piecewise-affine state feedback problem is: for fixed $\alpha \geq 0,\ \epsilon > 0$

$$\begin{array}{ll} \min & k \\ s.t. & k > 0, \quad \epsilon I < Q < k\epsilon I \\ & Q = Q^T > 0, \quad \mu_i < 0, \quad (16) \\ & -l_1 \prec Y_i \prec l_1,\ -l_0 \prec m_i \prec l_0,\ i = 1, \ldots, M, \end{array}$$

where $\succ, \prec$ mean component-wise inequalities and $l_0,\ l_1$

are given vector bounds. □

Remark 1:* *Usually ϵ is selected to be unitary. Notice also that Algorithm # 1 can be changed to store for all grid points the one that yields the minimum value of k. For the same setting, the algorithm can be further improved if the derivative of the solution with respect to γ at each point is computed. In that case, for each selected sample point, the next sample point should be chosen in the direction opposite to the vector derivative. This will reduce the number of points from the grid that need to be used, thus reducing the computational burden of the algorithm. □

IV. Stability of the Closed-Loop System with Norm Bounded Noise

In this section a stability result is presented for the closed-loop system when the measurements are subject to norm bounded noise. It is assumed that a continuous-time state feedback controller has been designed (for example, by solving one of the optimization problems in definitions 3.1 or 3.2). It is also assumed that the state x of the system is measured with additive norm bounded noise η. In other words, the measurement is $y = x + \eta$ with $\|\eta\| < N$, for some positive constant N. The control input is then $u = K_j(y - x_{cl}) + m_j = K_j z_{meas} + m_j$, where it is possibly true that $z = x - x_{cl} \in \mathcal{R}_i$ and $z_{meas} \in \mathcal{R}_j,\ j \neq i$ because of the noise in the state measurement. The closed-loop system is then described by the differential equation

$$\dot{z} = \left(\bar{A}_i + B_i \Delta K_{ij}\right) z + \bar{b}_i + B_i \left(\Delta m_{ij} + K_j \eta\right), \quad (17)$$

for $z(t) \in \mathcal{R}_i,\ z_{meas}(t) \in \mathcal{R}_j$, with $\bar{A}_i,\ \bar{b}_i$ defined as before and $\Delta K_{ij} = K_j - K_i,\ \Delta m_{ij} = m_j - m_i$. The main result of this section can now be stated. It gives conditions under which the trajectories of the closed-loop system (17) converge to a region around the closed-loop equilibrium point. Furthermore, it relates the size of this region to the bound on the norm of the noise.

Theorem 4.1: **Assume the Lyapunov function (11) is defined in $\mathcal{X} \subseteq I\!R^n$ and define the condition number $\chi(P) = \frac{\sigma_{\max}(P)}{\sigma_{\min}(P)}$. Assume there is a solution to the design problem from Definition 3.1 and assume that there exists $N > 0$ such that the noise term η from (17) satisfies $\|\eta\| < N$. Let $K = \max_{i=1,\ldots,M} \|K_i\|,\ B = \max_{i=1,\ldots,M} \|B_i\|,\ \Delta K = \max_{i=1,\ldots,M,\ j=1,\ldots,M} \|\Delta K_{ij}\|$ and $\Delta m = \max_{i=1,\ldots,M,\ j=1,\ldots,M} \|\Delta m_{ij}\|$. Define**

$$\mu_\theta = \frac{2\sigma_{\max}(P) B \left(KN + \Delta m\right)}{\alpha\theta\sigma_{\min}(P) - 2\sigma_{max}(P) B \Delta K},$$

and the region

$$\mathcal{S}_\theta = \{z \in \mathcal{X} \mid \|z\| \leq \mu_\theta\}$$

for any positive constant $\theta < 1$. Then, the trajectories of the closed-loop system (17) converge exponentially to the set

$$\Omega = \left\{z \in \mathcal{X} \mid V(z) \leq \sigma_{max}(P)\mu_\theta^2\right\}$$

provided

$$\Delta K < \frac{\chi^{-1}(P)\,\alpha\theta}{2B}$$

Proof: Using the dynamics (17), the derivative of the candidate Lyapunov function (11) along the trajectories of the system is

$$\frac{d}{dt}V(z) = \begin{bmatrix} z \\ 1 \end{bmatrix}^T \begin{bmatrix} \bar{A}_i^T P + P\bar{A}_i & P\bar{b}_i \\ (P\bar{b}_i)^T & 0 \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix} + 2z^T P B_i \left(\Delta K_{ij} z + \Delta m_{ij} + K_j \eta\right) \quad (18)$$

for the general case $z(t) \in \mathcal{R}_i,\ z_{meas}(t) \in \mathcal{R}_j$ where possibly $i \neq j$. However, note that by using the $\mathcal{S}$-procedure, expression (13) (or, equivalently, expression (16)) guarantees that for $z \in \mathcal{R}_i$

$$\begin{bmatrix} z \\ 1 \end{bmatrix}^T \begin{bmatrix} \bar{A}_i^T P + P\bar{A}_i & P\bar{b}_i \\ (P\bar{b}_i)^T & 0 \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix} < -\alpha z^T P z.$$

Therefore, for $z \in \mathcal{R}_i$ it follows that

$$\frac{d}{dt}V(z) < -\alpha z^T P z + 2z^T P B_i \left(\Delta K_{ij} z + \Delta m_{ij} + K_j \eta\right)$$

Noting that $V(z) = z^T P z$, taking norms and using the bounds $\|\eta\| < N,\ \|B_i\| \leq B,\ \|K_i\| \leq K,\ \|\Delta K_{ij}\| \leq \Delta K,\ \|\Delta m_{ij}\| \leq \Delta m,\ i,j = 1,\ldots,M$, yields

$$\frac{d}{dt}V(z) < -\alpha V(z) + 2\|z\|\sigma_{max}(P) B \left(\Delta K\|z\| + \Delta m + KN\right)$$

or, for any positive constant $\theta < 1$

$$\frac{d}{dt}V(z) < -(1-\theta)\alpha V(z) - \theta\alpha V(z) + 2\|z\|\sigma_{max}(P) B \left(\Delta K\|z\| + \Delta m + KN\right). \quad (19)$$

Therefore, given that $-V(z) = -z^T P z \leq -\sigma_{min}(P)\|z\|^2$, for $0 < \theta < 1$ we have

$$\frac{d}{dt}V(z) < -(1-\theta)\alpha V(z) \quad (20)$$

for $z \in I\!R^n \setminus \mathcal{S}_\theta$, i.e, for

$$\|z\| > \frac{2\sigma_{\max}(P) B \left(KN + \Delta m\right)}{\alpha\theta\sigma_{\min}(P) - 2\sigma_{max}(P) B \Delta K},$$

provided

$$\Delta K < \frac{\chi^{-1}(P)\,\alpha\theta}{2B}.$$

As a result of (20), for $z \in I\!R^n \setminus \mathcal{S}_\theta$,

$$V(z(t)) < V(z(t_0))e^{-(1-\theta)\alpha(t-t_0)}$$

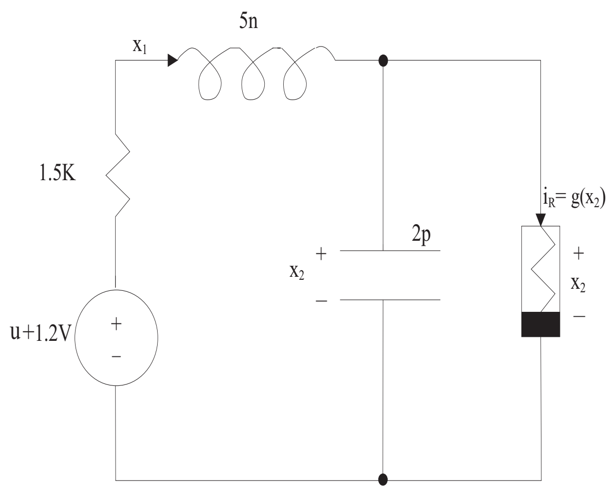

Fig. 2. Circuit with nonlinear resistor

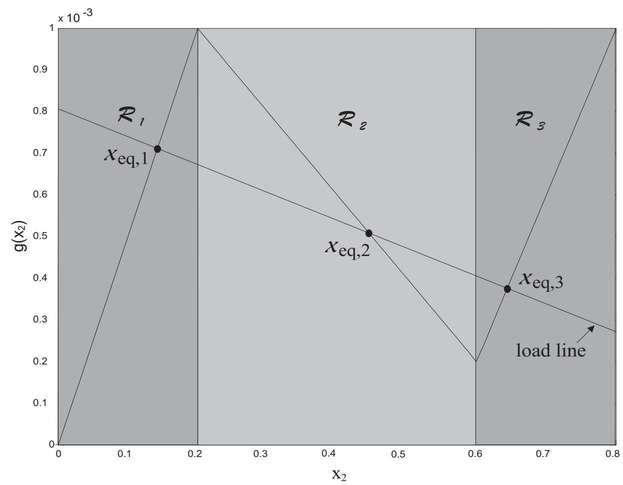

Fig. 3. Nonlinear resistor characteristic.

Using the relation $\sigma_{min}(P)\|z\|^2 \leq V(z) \leq \sigma_{max}(P)\|z\|^2$ we can conclude that for $z \in I\!R^n \setminus \mathcal{S}_\theta$,

$$\|z(t)\| \leq \|z(t_0)\|\chi^{\frac{1}{2}}(P)e^{-0.5(1-\theta)\alpha(t-t_0)}.$$

Thus, there will be a positive and finite time t_1^θ such that $z(t_1^\theta) \in \mathcal{S}_\theta$ for any positive constant $\theta < 1$. Note that $\mathcal{S}_\theta \subseteq \Omega$. This can be proved by contradiction. Assume that it is not true that $\mathcal{S}_\theta \subseteq \Omega$. Then, there exists at least one $z_0 \in \mathcal{S}_\theta$ for which $z_0^T P z_0 > \sigma_{max}(P)\mu_\theta^2$, a contradiction. By the same reasoning that led to (20), $\dot{V} \leq 0$ at the boundary of Ω and, therefore, Ω is an invariant set for system (17). Consequently, since $z(t_1^\theta) \in \mathcal{S}_\theta \subseteq \Omega$, $z(t) \in \Omega$ for all $t \geq t_1^\theta$ and for all $0 < \theta < 1$. □

Remark 2: Note that the result in the theorem roughly states that for feedback gain matrices whose norms do not differ substancially for different regions in the partition of the domain, the closed-loop trajectories converge to a region around the equilibrium point in the presence of bounded noise in the state measurements. Moreover, the size of the region depends on the noise bound, which makes perfect sense from a practical point of view. □

V. EXAMPLE

This example considers a circuit with a nonlinear resistor taken from [2] and shown in figure 2. With time expressed in 10^{-10} seconds, the inductor current in mA and the capacitor voltage in Volts, the dynamics are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -30 & -20 \\ 0.05 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 24 \\ -50g(x_2) \end{bmatrix} + \begin{bmatrix} 20 \\ 0 \end{bmatrix} u.$$

Following [2], the characteristic of the nonlinear resistor $g(x_2)$ is defined to be the piecewise-affine function shown in figure 3 which generates the polytopic regions

$$\begin{aligned} \mathcal{R}_1 &= \{x \in I\!R^2 \mid -L < x_2 < 0.2\}, \\ \mathcal{R}_2 &= \{x \in I\!R^2 \mid 0.2 < x_2 < 0.6\}, \\ \mathcal{R}_3 &= \{x \in I\!R^2 \mid 0.6 < x_2 < L\}, \end{aligned}$$

where $L = 2 \times 10^4$. The (exact) ellipsoidal covering is

$$E_1 = \frac{2}{0.2+L}e_1, \; E_2 = \frac{2}{0.6-0.2}e_2, \; E_3 = \frac{2}{L-0.6}e_3$$

$$\tilde{f}_1 = \frac{L-0.2}{L+0.2} \quad \tilde{f}_2 = -\frac{0.6+0.2}{0.6-0.2} \quad \tilde{f}_3 = -\frac{L+0.6}{L-0.6},$$

where $e_1 = e_2 = e_3 = [0 \; 1]$. Assume that the affine terms of the control law have magnitude bounded by 0.2 so that $l_0 = [0.2 \;\; 0.2 \;\; 0.2]^T$. The objective is to design a piecewise-affine state feedback controller to stabilize the system to the open loop equilibrium point of $\mathcal{R}_3$

$$x_{cl} = x_{ol}^3 = \begin{bmatrix} 0.3714 \\ 0.6429 \end{bmatrix}.$$

For region $\mathcal{R}_3$ we then must have $m_3 = 0$. A grid of 0.1 increments in the interval $[-0.2, 0.2]$ was selected for each of the parameters $m_1, \; m_2$. Imposing the constraint $Y_1 = Y_2 = Y_3$ and using $\alpha = 1, \; l_1 = 10^{-13}[8 \quad 8]^T$, Algorithm # 1 yields

$$\begin{aligned} K_1 &= \begin{bmatrix} -0.0027 & -1.7647 \end{bmatrix}, \quad m_1 = +0.1, \\ K_2 &= \begin{bmatrix} -0.0027 & -1.7647 \end{bmatrix}, \quad m_2 = -0.1 \\ K_3 &= \begin{bmatrix} -0.0027 & -1.7647 \end{bmatrix}, \quad m_3 = +0.0, \end{aligned}$$

The simulation results for the initial condition $x_1^0 = 0.5, \; x_2^0 = 0.1$ (inside region $\mathcal{R}_1$, which is the region furthest away of the region holding the equilibrium point)

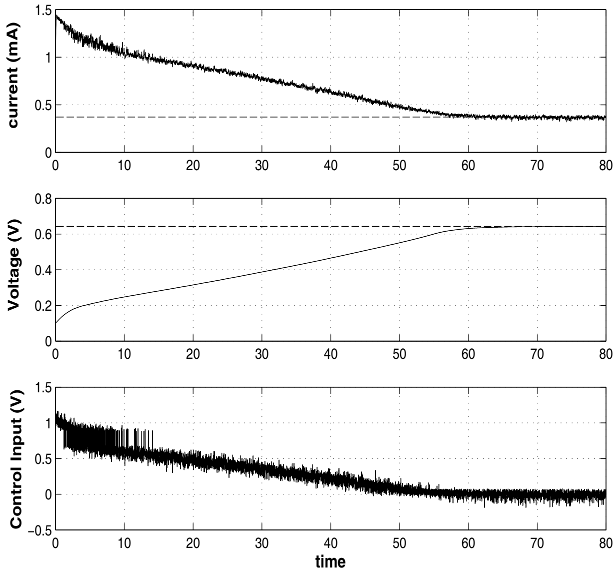

Fig. 4. Simulation Results for $x_1^0 = 0.5,\ x_2^0 = 0.1$

and for measurements of x_2 subject to bounded noise are presented in figure 4. The desired set-points for the state components are represented by the dashed lines and it is clear from the figure that in the presence of noise the state components converge to a region around the set-points. Note that typically in applications the control signal would be filtered before feeding it to the plant input. If either a linear or a piecewise-affine filter is implemented, the approach of this paper can still be used to analyze closed-loop stability in the presence of bounded noise and control signal filtering because the closed-loop system is still piecewise-affine.

VI. Conclusions

Being able to design controllers that are robust to noisy measurements is of fundamental importance in practical applications. This paper proposed a design technique for state feedback control of piecewise-affine systems that is robust to bounded noise in the measurements. More specifically, the paper presented conditions under which a piecewise-affine state feedback controller designed for a noise-free piecewise-affine system still stabilizes the system to a region around the equilibrium point when subject to norm bounded noise in the state measurements . The result presented in the paper explicitly relates the size of the region of convergence to the bound on the noise. Simulation examples illustrate the result showing that Lyapunov-based piecewise-affine state feedback controller synthesis is robust to noise in the measurement of the state.

References

[1] V. D. Blondel and J. N. Tsitsiklis, "Complexity of stability and controllability of elementary hybrid systems," *Automatica*, vol. 35, pp. 479–489, March 1999.

[2] A. Hassibi and S. P. Boyd, "Quadratic stabilization and control of piecewise-linear systems," in *Proceedings of the American Control Conference*, pp. 3659–3664, June 1998.

[3] M. Johansson and A. Rantzer, "Piecewise linear quadratic optimal control," *IEEE Transactions on Automatic Control*, vol. 45, no. 4, pp. 629–637, 2000.

[4] L. Rodrigues and J. How, "Observer-based control of piecewise-affine systems," *International Journal of Control*, vol. 76, pp. 459–477, March 2003.

[5] L. Rodrigues and S. P. Boyd, "Piecewise-affine state feedback for piecewise-affine slab systems using convex optimization", to appear in Systems and Control Letters.

[6] L. Rodrigues and J. How, "Automated control design for a piecewise-affine approximation of a class of nonlinear systems," in *Proceedings of the American Control Conference*, pp. 3189–3194, June 2001.

[7] M. Johansson and A. Rantzer, "Computation of piecewise quadratic lyapunov functions for hybrid systems," *IEEE Transactions on Automatic Control*, vol. 43, pp. 555–559, April 1998.

[8] S. Pettersson, *Analysis and Design of Hybrid Systems*. PhD thesis, Chalmers University of Technology, 1999.

[9] M. Johansson, *Piecewise Linear Control Systems*. PhD thesis, Lund Institute of Technology, 1999.

[10] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, vol. 15 of *Studies in Applied Mathematics*. SIAM, June 1994.

[11] K. C. Goh, J. H. Ly, L. Turand, and M. G. Safonov, "Biaffine matrix inequality properties and computational methods," *Proceedings of the American Control Conference*, pp. 850–855, 1994.