2005 American Control Conference
June 8-10, 2005. Portland, OR, USA

A simplified LMI approach to ℓ_1 Controller Design

Anirudh Oberoi
National Semiconductor Corporation
Pittsford, NY 14534 USA
Anirudh.Oberoi@nsc.com

Juan C. Cockburn
Department of Computer Engineering
Rochester Institute of Technology
Rochester, NY 14623 USA
cockburn@ce.rit.edu

***Abstract*— In this paper a new linear matrix inequality (LMI) for peak-to-peak gain minimization is derived. The new LMI defines a convex upper bound on the non-convex $*$-norm rendering the ℓ_1-controller design problem convex and facilitating its integration in multiobjective design problems. It also allows the use of projection formulas which simplifies the numerical implementation of the design algorithm.**

I. INTRODUCTION

Linear Matrix Inequalities (LMIs) provide a unified framework for the analysis and design of control systems [**?**], [**?**], [**?**], [**?**]. The basic idea is to reduce a control system design problem to a convex optimization problem with constraints where the objective function to be minimized is given by a convex combination of the norms of certain closed loop transfer functions and the constraints capture various performance specifications and control objectives. The constraints are Linear Matrix Inequalities, which are actually affine in their variables, that allow the designer to trade-off conflicting objectives.

The ℓ_1-norm is used to capture time-domain constraints like the peak-to-peak gain. It characterizes the effect of signals about which only very limited information, such as a bound on their amplitude, is available. An example of such a situation is the minimization of the absolute value of a disturbance in time.

The LMI design framework for the ℓ_1-control of finite dimensional discrete time systems was introduced in [**?**], [**?**], [**?**] using the method of elimination of variables [**?**]. The method of elimination of variables linearizes the design problem and hence makes it fit for convex optimization. A different approach is taken in this paper in order to linearize the ℓ_1-design problem and make it suitable for convex minimization of multiobjective problems. This approach relies on the use of projection formulas given in [**?**]. The design process optimizes an upper bound on the $*$-norm and hence the ℓ_1-norm, through LMI optimization of a new set of LMIs. The advantage of this approach is the simplicity of the implementation of the algorithm for LMI optimization as compared to the elimination of variables approach. The cost incurred is some additional conservativeness as an upper-bound on the $*$-norm is being minimized.

This paper is divided into four sections. Section 2 describes the general framework for the analysis and design of control systems based on the LMI approach. Section 3 articulates results related to the LMIs for ℓ_1-control and arrives at a new additional LMI which renders the objective function convex. The projection formulas from [**?**] are listed in Section 4. The utility of the approach is illustrated through a simple design example in Section 5 which is followed by a summary of the design process in Section 6.

II. PROBLEM SETUP

A large class of feedback control problems can be represented by the standard feedback structure shown in Figure **??**. where the *generalized plant* G represents the fixed

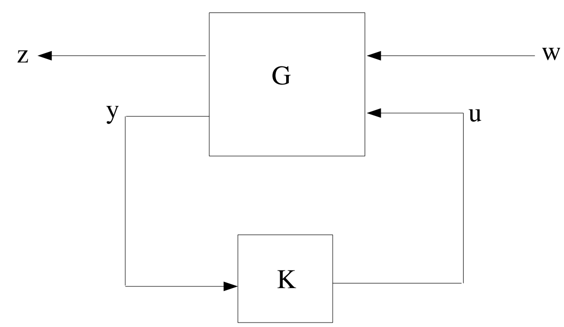

Fig. 1. Standard Feedback Structure

parts of the system, including a model of the plant, performance weights and uncertainty weights and the *controller* K is the free parameter to be designed. The signals u, w, y and z are vector valued signals where $u \in \mathbf{R}^{n_u}$ is the control input,$w \in \mathbf{R}^{n_w}$ is the external input to the system representing reference and noise signals, $y \in \mathbf{R}^{n_y}$ is the measured output and $z \in \mathbf{R}^{n_z}$ the regulated output or error signal that captures design objectives such as tracking error and control energy.

All input-output maps are causal, linear time-invariant and finite dimensional so they can be represented by a rational transfer matrix functions or state-space equations.

The objective is to design a controller K such that the closed loop transfer function from w to z satisfies certain design specifications. In order to apply the LMI approach to this problem, it is necessary to represent the generalized plant and controller in state variable form. In this work only finite dimensional, discrete-time, linear time-invariant (DLTI) systems will be considered.

The state-space equations for the generalized plant G are given by

$$\begin{aligned} x(k+1) &= \mathcal{A}x(k) + \mathcal{B}_w w(k) + \mathcal{B}u(k) \\ z(k) &= \mathcal{C}_z x(k) + \mathcal{D}_{zw} w(k) + \mathcal{D}_z u(k) \\ y(k) &= \mathcal{C}x(k) + \mathcal{D}_w w(k) \end{aligned} \quad (1)$$

and for the controller K

$$\begin{aligned} x_{\mathcal{K}}(k+1) &= \mathcal{A}_K x(k) + \mathcal{B}_K y(k) \\ u(k) &= \mathcal{C}_K x(k) + \mathcal{D}_K y(k) \end{aligned} \quad (2)$$

For multiobjective optimization it may be necessary to penalize different subsets of input and output signals with different norms. Let $\mathcal{T}$ denote the closed loop map from w to z. Then, the design objectives can be formulated in terms of

$$\mathcal{T}_j = \mathcal{L}_j \mathcal{T} \mathcal{R}_j \quad (3)$$

where $\mathcal{T}_j$ is the transfer function from a particular subset of inputs w and outputs z. The matrices $\mathcal{L}_j$ and $\mathcal{R}_j$ select the appropriate I/O channels (this notation is adopted from [?]).

Given the generalized plant (1) and controller (2) model, the state-space equations of the closed-loop system are

$$\begin{aligned} x_{cl}(k+1) &= \mathcal{A}_{cl} x(k) + \mathcal{B}_{cl} w(k) \\ z(k) &= \mathcal{C}_{cl} x(k) + \mathcal{D}_{cl} w(k) \end{aligned} \quad (4)$$

where

$$\left(\begin{array}{c|c} \mathcal{A}_{cl} & \mathcal{C}_{cl} \\ \hline \mathcal{B}_{cl} & \mathcal{D}_{cl} \end{array}\right) = \left(\begin{array}{cc|c} \mathcal{A}+\mathcal{B}\mathcal{D}_K\mathcal{C} & \mathcal{B}\mathcal{C}_K & \mathcal{B}_w+\mathcal{B}\mathcal{D}_K\mathcal{D}_w \\ \mathcal{B}_K\mathcal{C} & \mathcal{A}_K & \mathcal{B}_K\mathcal{D}_w \\ \hline \mathcal{C}_z+\mathcal{D}_z\mathcal{D}_K\mathcal{C} & \mathcal{D}_z\mathcal{C}_K & \mathcal{D}_{zw}+\mathcal{D}_z\mathcal{D}_K\mathcal{D}_w \end{array}\right) \quad (5)$$

In order to select specific input and output channels for capturing different performance objectives the following variables are defined

$$\begin{aligned} &\mathcal{B}_j \doteqdot \mathcal{B}_w\mathcal{R}_j, \quad \mathcal{C}_j \doteqdot \mathcal{L}_j\mathcal{C}_z, \quad \mathcal{D}_j \doteqdot \mathcal{L}_j\mathcal{D}_{zw}\mathcal{R}_j \\ &\mathcal{E}_j \doteqdot \mathcal{L}_j\mathcal{D}_z, \quad \mathcal{F}_j \doteqdot \mathcal{D}_w\mathcal{R}_j \end{aligned}$$

Hence $\mathcal{T}_j(z) = \mathcal{L}_j\mathcal{T}\mathcal{R}_j$ has the following state-space realization

$$\left(\begin{array}{c|c} \mathcal{A}_{cl} & \mathcal{B}_j \\ \hline \mathcal{C}_j & \mathcal{D}_j \end{array}\right) = \left(\begin{array}{cc|c} \mathcal{A}+\mathcal{B}\mathcal{D}_K\mathcal{C} & \mathcal{B}\mathcal{C}_K & \mathcal{B}_j+\mathcal{B}\mathcal{D}_K\mathcal{F}_j \\ \mathcal{B}_K\mathcal{C} & \mathcal{A}_K & \mathcal{B}_K\mathcal{F}_j \\ \hline \mathcal{C}_j+\mathcal{E}_j\mathcal{D}_K\mathcal{C} & \mathcal{E}_j\mathcal{C}_K & \mathcal{D}_j+\mathcal{E}_j\mathcal{D}_K\mathcal{F}_j \end{array}\right) \quad (6)$$

The control design problem is to find the dynamic control law $(\mathcal{A}_K, \mathcal{B}_K, \mathcal{C}_K, \mathcal{D}_K)$ that satisfies certain set of specifications on the general transfer function described by equation (6). These set of specifications are captured by weighted system norms. The weighted system norms are minimized in order to satisfy the specification set.

In the sequel, it is assumed that the pair $(\mathcal{A}, \mathcal{B})$ is controllable and the pair (A, C) is detectable. The minimal realization of a transfer function $\mathcal{T}(z) = \mathcal{D}+\mathcal{C}(zI-\mathcal{A})^{-1}\mathcal{B}$ will be denoted as $\left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right)$

III. LMI CONSTRAINTS FOR ℓ_1 CONTROL

Sometimes in practical applications exogenous inputs (disturbances, noise, etc.) are encountered which are persistent in nature and the only information available about these signals is their peak amplitude. The ℓ_∞ norm provides a suitable way to characterize the effect of such signals. It can be used to capture the peak amplitude of certain output signals in a control system design problem. For instance, the design specifications might impose a bound on the peak value of the tracking error. Also, in cases where the plant has a maximum input rating, the output of the controller has to be bounded in amplitude in order to avoid saturation of the plant. Such constraints can be captured by the ℓ_∞ norm of the transfer function from the input to the output signal of interest. A good description of the importance of the ℓ_∞ signal norm is given in [?]. The ℓ_∞ norm of a sequence $w(k)$ is defined as

$$\|w\|_\infty = \max_{1 \leqslant \jmath \leqslant n_w} \sup_k |w_\jmath(k)|$$

For a FDLTI discrete time system with impulse response $\mathcal{H}$, the output $z(k)$ is given by convolution as follows

$$z(k) = (\mathcal{H}w)(k) = \sum_{m=0}^{t} \mathcal{H}(k-m)w(m)$$

If the input and output to the system are amplitude bounded signals, *e.g.*, belong to ℓ_∞, then the norm induced on the system is the ℓ_1 norm defined as

$$\|\mathcal{H}\|_1 = \max_{1 \leqslant \imath \leqslant n_z} \sum_{\jmath=1}^{n_w} \sum_{k=0}^{\infty} |\mathcal{H}_{\imath\jmath}| \quad (7)$$

Good performance is reflected by a small ℓ_1-norm. Since this norm is difficult to compute it is often not used in direct minimization. Instead an upper bound on it that is easier to compute, called the star norm, will be used as shown below. The peak value of signals can also be captured by the pointwise Euclidean norm [?], [?]

$$\|w\|_{\infty,e} = \sup_k |w_\jmath^T(k) w_\jmath(k)|$$

The norm induced by the peak pointwise Euclidean norm is denoted by $\|\mathcal{H}\|_{\infty,e}$. For SISO systems this norm is equal to the ℓ_1-norm while for MIMO systems

$$\frac{1}{\sqrt{n_w}} \|\mathcal{H}\|_1 \leq \|\mathcal{H}\|_{\infty,e} \leq \sqrt{n_z} \|\mathcal{H}\|_1$$

The LMI approach to ℓ_1 norm minimization is based on minimizing $\|\mathcal{H}\|_{\infty,e}$ using the following important result.

Lemma 3.1: [?], [?], [?] For a proper, stable FDLTI system $\mathcal{T}(z) = \mathcal{D} + \mathcal{C}(zI - \mathcal{A})^{-1}\mathcal{B}$

$\|\mathcal{H}\|_{\infty,e}^2 \leq \|\mathcal{H}\|_*^2 \triangleq \mathcal{V}(\alpha) = \inf_{\alpha \in (0,1)} \{\eta :\ni \sigma > 0\}$ $s.t$

$$\begin{pmatrix} \alpha\sigma\mathcal{Q}^{-1} & 0 & \mathcal{C}^T \\ 0 & (\eta-\sigma)\mathcal{I} & \mathcal{D}^T \\ \mathcal{C} & \mathcal{D} & \mathcal{I} \end{pmatrix} > 0 \text{ for } \mathcal{Q} > 0 \text{ satisfying} \quad (8)$$

$$\frac{1}{1-\alpha} \mathcal{A}\mathcal{Q}\mathcal{A}^T - \mathcal{Q} + \mathcal{B}\mathcal{B}^T \leq 0 \quad (9)$$

Moreover, $\mathcal{V}(\alpha)$ is quasi-convex function for $\alpha \in (0, 1 - \rho^2(\mathcal{A}))$, where $\rho(\mathcal{A})$ denotes the spectral radius of $\mathcal{A}$.
Hence the ℓ_1- norm can be minimized by minimizing the $*$- norm (read star- norm) subject to conditions (8) and (9) as the optimal $*$- norm is an upper bound on the optimal ℓ_1- norm.
The next step is to convert these conditions into LMIs. This is done by first working on condition (8) by performing a congruence transformation with $diag(\frac{\mathcal{I}}{\sqrt{\sigma}}, \frac{\mathcal{I}}{\sqrt{\sigma}}, \sqrt{\sigma})$

$$\begin{pmatrix} \frac{\mathcal{I}}{\sqrt{\sigma}} & 0 & 0 \\ 0 & \frac{\mathcal{I}}{\sqrt{\sigma}} & 0 \\ 0 & 0 & \sqrt{\sigma} \end{pmatrix} \begin{pmatrix} \alpha\sigma\mathcal{Q}^{-1} & 0 & \mathcal{C}^T \\ 0 & (\eta-\sigma)\mathcal{I} & \mathcal{D}^T \\ \mathcal{C} & \mathcal{D} & \mathcal{I} \end{pmatrix} \begin{pmatrix} \frac{\mathcal{I}}{\sqrt{\sigma}} & 0 & 0 \\ 0 & \frac{\mathcal{I}}{\sqrt{\sigma}} & 0 \\ 0 & 0 & \sqrt{\sigma} \end{pmatrix} > 0$$

$$\Rightarrow \begin{pmatrix} \alpha\mathcal{Q}^{-1} & 0 & \mathcal{C}^T \\ 0 & \frac{(\eta-\sigma)}{\sigma} & \mathcal{D}^T \\ \mathcal{C} & \mathcal{D} & \sigma \end{pmatrix} > 0 \tag{10}$$

Now performing the change of variables

$$\mu = \frac{\eta}{\sigma}, \quad \nu = \sigma \tag{11}$$

The original problem in Lemma 3.1 is transformed into the following problem

$$\|\mathcal{H}\|_*^2 = \inf_{\alpha\in(0,1),\nu>0,\mu>0,\mathcal{Q}>0} \{\eta = \mu\nu\} \quad s.t. \tag{12}$$

$$\begin{pmatrix} \alpha\mathcal{Q}^{-1} & 0 & \mathcal{C}^T \\ 0 & (\mu-1)\mathcal{I} & \mathcal{D}^T \\ \mathcal{C} & \mathcal{D} & \nu \end{pmatrix} > 0$$

$$\frac{1}{1-\alpha}\mathcal{A}\mathcal{Q}\mathcal{A}^T - \mathcal{Q} + \mathcal{B}\mathcal{B}^T \le 0 \tag{13}$$

With $\mathcal{Q}^{-1} = \mathcal{P}$, transforming (9) with a congruence with $\mathcal{P}$ and using Schur's complement and completion of squares the $*$-norm minimization problem becomes

$$\|\mathcal{H}\|_*^2 = \inf_{\alpha\in(0,1),\nu>0,\mu>0,\mathcal{P}>0} \{\eta = \mu\nu\} \quad s.t. \tag{14}$$

$$\begin{pmatrix} \alpha\mathcal{P} & 0 & \mathcal{C}^T \\ 0 & (\mu-1)\mathcal{I} & \mathcal{D}^T \\ \mathcal{C} & \mathcal{D} & \nu \end{pmatrix} > 0$$

$$\begin{pmatrix} -\mathcal{P} & \mathcal{P}\mathcal{A} & \mathcal{P}\mathcal{B} \\ \mathcal{A}^T\mathcal{P} & (\alpha-1)\mathcal{P} & 0 \\ \mathcal{B}^T\mathcal{P} & 0 & -\mathcal{I} \end{pmatrix} < 0 \tag{15}$$

The objective function for the $*$-norm minimization is still non-convex. This non-convex objective can be upper-bounded by a convex objective function by noting that

$$\mu\nu \le \mu^2 + \nu^2 \text{ for } \mu > 0,\ \nu > 0 \tag{16}$$

Thus, minimizing $\mu^2 + \nu^2$ minimizes $\mu\nu$. Minimizing $\mu^2 + \nu^2$ is equivalent to minimizing φ where

$$\varphi > \mu^2 + \nu^2 \tag{17}$$

The condition specified by inequality (17) is convex and is captured by the following LMI

$$\begin{pmatrix} \varphi & \mu & \nu \\ \mu & 1 & 0 \\ \nu & 0 & 1 \end{pmatrix} > 0 \tag{18}$$

Hence, the new upper bound on the ℓ_1 norm is minimized by solving the following semi-definite optimization problem

$$\inf_{\alpha\in(0,1),\mathcal{P}>0,\nu>0,\mu>0,\varphi>0} (\varphi) \quad s.t. \tag{19}$$

$$\begin{pmatrix} \alpha\mathcal{P} & 0 & \mathcal{C}^T \\ 0 & (\mu-1)\mathcal{I} & \mathcal{D}^T \\ \mathcal{C} & \mathcal{D} & \nu \end{pmatrix} > 0 \tag{20}$$

$$\begin{pmatrix} -\mathcal{P} & \mathcal{P}\mathcal{A} & \mathcal{P}\mathcal{B} \\ \mathcal{A}^T\mathcal{P} & (\alpha-1)\mathcal{P} & 0 \\ \mathcal{B}^T\mathcal{P} & 0 & -\mathcal{I} \end{pmatrix} < 0 \tag{21}$$

$$\begin{pmatrix} \varphi & \mu & \nu \\ \mu & 1 & 0 \\ \nu & 0 & 1 \end{pmatrix} > 0 \tag{22}$$

Note that the matrix inequalities are still nonlinear. This problem is solved by doing a line search over $\alpha \in (0, 1)$. For a fixed value of α the resulting LMI-optimization is solved for $\mathcal{P}$.

IV. Linearization through Projection into a higher dimensional space

The LMIs derived above are called analysis LMIs as they specify constraints on closed loop matrices. he design problem however poses additional complications since the introduction of the controller variables $\mathcal{A}_K$, $\mathcal{B}_K$, $\mathcal{C}_K$ and $\mathcal{D}_K$ breaks down the linearity of the matrix inequalities as these variables appear together with the Lyapunov matrix $\mathcal{P}$ in a nonlinear fashion. This problem was solved by a change of variables described in [?] that transforms the original problem back into a convex LMI problem. Let n be the number of states of the plant $\mathcal{A}$ and k be the order of the controller and partition $\mathcal{P}$ and $\mathcal{P}^{-1}$ as follows

$$\mathcal{P} = \begin{pmatrix} \mathbf{Y} & \mathcal{N} \\ \mathcal{N}^T & * \end{pmatrix}, \qquad \mathcal{P}^{-1} = \begin{pmatrix} \mathbf{X} & \mathcal{M} \\ \mathcal{M}^T & * \end{pmatrix} \tag{23}$$

where $\mathbf{X}$ and $\mathbf{Y}$, are $n \times n$ and symmetric while $\mathcal{N}$ and $\mathcal{M}$, are $k \times n$ where k is the order of the controller. Define

$$\mathbf{\Pi}_1 = \begin{pmatrix} \mathbf{X} & \mathcal{I} \\ \mathcal{M}^T & 0 \end{pmatrix}, \qquad \mathbf{\Pi}_2 = \begin{pmatrix} \mathcal{I} & \mathbf{Y} \\ 0 & \mathcal{N}^T \end{pmatrix} \tag{24}$$

where $\mathcal{P}\,\mathbf{\Pi}_1 = \mathbf{\Pi}_2$. The linearizing change of variables is as follows:

$$\begin{aligned} \hat{A} &\triangleq \mathcal{N}\mathcal{A}_K\mathcal{M}^T + \mathcal{N}\mathcal{B}_K\mathcal{C}\mathbf{X} + \mathbf{Y}\mathcal{B}\mathcal{C}_K\mathcal{M}^T \\ &\quad + \mathbf{Y}(\mathcal{A} + \mathcal{B}\mathcal{D}_K\mathcal{C})\mathbf{X} \\ \hat{B} &\triangleq \mathcal{N}\mathcal{B}_K + \mathbf{Y}\mathcal{B}\mathcal{D}_K \\ \hat{C} &\triangleq \mathcal{C}_K\mathcal{M}^T + \mathcal{D}_K\mathcal{C}\mathbf{X} \\ \hat{D} &\triangleq \mathcal{D}_K \end{aligned} \tag{25}$$

For full row rank $\mathcal{M}$ and $\mathcal{N}$ and a given set of matrices ($\hat{A}$,$\hat{B}$,$\hat{C}$,$\hat{D}$,$\mathbf{X}$ and $\mathbf{Y}$) the controller state matrices ($\mathcal{A}_K$,$\mathcal{B}_K$,$\mathcal{C}_K$,$\mathcal{D}_K$) can be recovered from equation (25). If $\mathcal{M}$ and $\mathcal{N}$ are square ($\mathcal{A}_K$,$\mathcal{B}_K$,$\mathcal{C}_K$,$\mathcal{D}_K$) are unique. Hence, dynamic feedback design in the present context shall be restricted to full-order controllers (n=k) to keep the design constraints convex.

It is also possible to synthesize reduced order controllers. However, the price to be paid is an additional non-convex rank minimization constraint that is coupled with the design constraints [**?**], [**?**], [**?**]. Some methods to deal with this non-convex problem can be found in [**?**].
The following identities derived from equation (25) and (24) validate the change of variables described in equation (25)

$$\mathcal{A}(\phi) \triangleq \mathbf{\Pi}_1^T(\mathcal{P}\mathcal{A})\mathbf{\Pi}_1 = \mathbf{\Pi}_2^T\mathcal{A}\mathbf{\Pi}_1 = \begin{pmatrix} \mathcal{A}\mathbf{X}+\mathcal{B}\hat{C} & \mathcal{A}+\mathcal{B}\hat{D}\mathcal{C} \\ \hat{A} & \mathbf{Y}\mathcal{A}+\hat{B}\mathcal{C} \end{pmatrix} \quad (26)$$

$$\mathcal{B}(\phi) \triangleq \mathbf{\Pi}_1^T(\mathcal{P}\mathcal{B}_j) = \mathbf{\Pi}_2^T\mathcal{B}_j = \begin{pmatrix} \mathcal{B}_j+\mathcal{B}\hat{D}\mathcal{F}_j \\ \mathbf{Y}\mathcal{B}_j+\hat{B}\mathcal{F}_j \end{pmatrix} \quad (27)$$

$$\mathcal{C}(\phi) \triangleq \mathcal{C}_j\mathbf{\Pi}_1 = \begin{pmatrix} \mathcal{C}_j\mathbf{X}+\mathcal{E}_j\hat{C} & \mathcal{C}_j+\mathcal{E}_j\hat{D}\mathcal{C} \end{pmatrix} \quad (28)$$

$$\mathcal{P}(\phi) \triangleq \mathbf{\Pi}_1^T\mathcal{P}\mathbf{\Pi}_1 = \mathbf{\Pi}_1^T\mathbf{\Pi}_2 = \begin{pmatrix} \mathbf{X} & \mathcal{I} \\ \mathcal{I} & \mathbf{Y} \end{pmatrix} \quad (29)$$

ϕ above refers to the set ($\hat{A}$,$\hat{B}$,$\hat{C}$,$\hat{D}$,$\mathbf{X}\,\mathbf{Y}$). The identities listed above are used to derive synthesis LMIs from the analysis LMIs described in the last section. The analysis LMIs are projected onto the ϕ space using suitable congruence transformations involving the projection matrix $\mathbf{\Pi}_1$ and $\mathbf{\Pi}_1^T$. The original LMIs with blocks $\mathcal{P}$, $\mathcal{P}\mathcal{A}$, $\mathcal{P}\mathcal{B}_j$, $\mathcal{C}_j$ and $\mathcal{D}_j$ are transformed into LMIs with blocks $\mathbf{\Pi}_1^T\mathcal{P}\mathbf{\Pi}_1$, $\mathbf{\Pi}_1^T\mathcal{P}\mathcal{A}\mathbf{\Pi}_1$, $\mathbf{\Pi}_1^T\mathcal{P}\mathcal{B}_j$, $\mathcal{C}_j\mathbf{\Pi}_1$ and $\mathcal{D}_j$. Using the above identities new synthesis LMIs in the variables ($\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$, $\mathbf{X}$, $\mathbf{Y}$) and some auxiliary variables are obtained. The congruence transformations for the ℓ_1-control LMIs are given below:

$$\begin{pmatrix} \mathcal{Q} & 0 & \mathcal{C}_j^T \\ 0 & (\mu-1)\mathcal{I} & \mathcal{D}_j^T \\ \mathcal{C}_j & \mathcal{D}_j & \nu \end{pmatrix} > 0 \Leftarrow \begin{pmatrix} \mathbf{\Pi}_1 & 0 & 0 \\ 0 & \mathcal{I} & 0 \\ 0 & 0 & \mathcal{I} \end{pmatrix} \quad (30)$$

$$\begin{pmatrix} -\mathcal{Q} & \mathcal{Q}\mathcal{A} & \mathcal{Q}\mathcal{B}_j \\ \mathcal{A}^T\mathcal{Q} & (\alpha-1)\mathcal{Q} & 0 \\ \mathcal{B}_j^T\mathcal{Q} & 0 & -\mathcal{I} \end{pmatrix} < 0 \Leftarrow \begin{pmatrix} \mathbf{\Pi}_1 & 0 & 0 \\ 0 & \mathbf{\Pi}_1 & 0 \\ 0 & 0 & \mathcal{I} \end{pmatrix} \quad (31)$$

V. EXAMPLE

One of the motivations to introduce this LMI approach to ℓ_1 design comes from the need to develop a principled approach to noise-shaping feedback coder design, where time-frequency constraints appear naturally. A noise-shaping feedback coder uses a quantizer in a feedback loop to code information continuous in magnitude in the form of a digital word. The controller (or noise-shaping filter) imposes a high pass spectrum on the quantization noise thereby moving artifacts produced by the digitization process into high frequencies. Since this coder is followed by a low-pass filter it provides an effectively was to remove quantization noise. For simplicity the quantizer will be modelled as additive white noise as shown in Figure **??**. The objective is to design a noise-shaping filter (*i.e.*, a controller) to push the quantization noise energy into the high frequency spectrum. At the same time saturation of the quantizer must be prevented. These mixed time-frequency specifications can be effectively captured within the LMI framework.

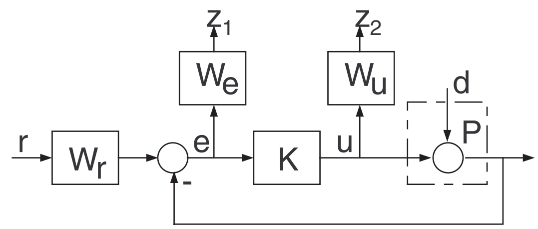

Fig. 2. Feedback Structure

In this example only a pure ℓ_1 design will be shown. The error signals to be minimized are the weighted tracking error, $z_1 = e$, and the controller output, $z_2 = u$, which is the input to the quantizer, *i.e.*, the plant. The exogenous inputs to the generalized plant are the reference signal into the coder, $w_1 = r$, and the white noise disturbance (modelling the quantizer) whose effect on the closed loop response we wish to minimize $w_2 = d$. The sampling frequency is 1 MHz and the reference input bandwidth 100 KHz. To model the input signal the following shaping filter was used

$$W_r(z) = \frac{0.46651z}{z-0.5335}$$

The tracking error was penalize at low frequencies through the following dynamic weight

$$W_e(z) = \frac{0.2696z}{z-0.7304},$$

Similarly, the control effort was penalized at high frequencies through the weight

$$W_u(z) = \frac{0.5217(z-0.9397)}{z-0.0.001867}.$$

Combining these weights with the plant dynamics and the projection formulas listed earlier the following state-space realization for the generalized was obtained:

$$\mathcal{A} = \begin{pmatrix} 0.5335 & 0 & 0 \\ 0.2214 & 0.7304 & 0 \\ 0 & 0 & 0.0019 \end{pmatrix}$$

$$\mathcal{B}_w = \begin{pmatrix} 0.4989 & 0 \\ 0.2070 & -0.4438 \\ 0 & 0 \end{pmatrix}$$

$$\mathcal{C}_z = \begin{pmatrix} 0.1345 & 0.4438 & 0 \\ 0 & 0 & -0.6995 \end{pmatrix}$$

$$\mathcal{D}_{zw} = \begin{pmatrix} 0.1258 & -0.2696 \\ 0 & 0 \end{pmatrix}$$

$$\mathcal{B} = \begin{pmatrix} 0 \\ -0.4438 \\ 0.6995 \end{pmatrix} \qquad \mathcal{C} = \begin{pmatrix} 0.4989 & 0 & 0 \end{pmatrix}$$

$$\mathcal{D}_z = \begin{pmatrix} -0.2696 \\ 0.5217 \end{pmatrix} \qquad \mathcal{D}_w = \begin{pmatrix} 0.4665 & -1.0 \end{pmatrix}$$

The MATLAB LMI Toolbox [**?**] was used to implement the synthesis LMIs.

A pure ℓ_1 design involves a sequence of LMI optimizations for a fixed α and a scalar minimization over α. It was found that the controller that minimizes the ℓ_1-norm from w to z corresponds to $\alpha = 0.46$ where the upper bound on the $*$-norm is minimum. The transfer function of the controller corresponding to this value of α was

$$K(z) = \frac{0.25669(z+0.9816)(z-0.472)(z-0.001867)}{(z-0.08013)(z-0.7304)(z-0.4962)}$$

and the actual ℓ_1-norm achieved was 0.64.

The minimization of the $*$-norm objective over α is captured by Figure 3. This figure compares the non-convex objective representing the $*$-norm with its convex upper bound as a function of α. Figure 3 shows the gap between the two objective functions and thus illustrates the conservative nature of this new ℓ_1-design procedure. However, the simplicity of the implementation makes it a practical procedure for ℓ_1-control synthesis. Furthermore, the real advantage of this approach becomes evident when the proposed simplified ℓ_1-norm LMIs are used in conjunction with $\mathcal{H}_\infty$ and $\mathcal{H}_2$ design constraints thus making the mixed-norm design process simpler to implement and understand. [**?**].

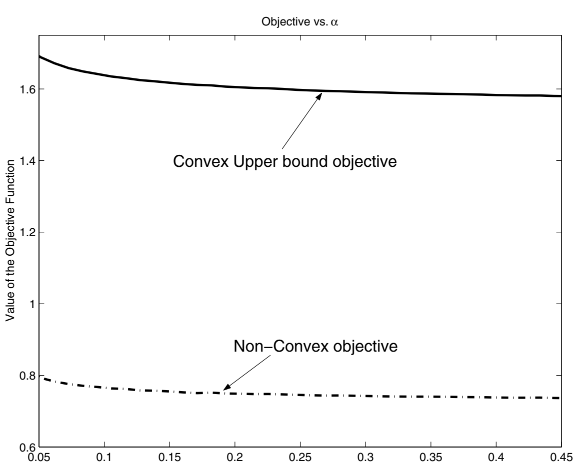

Fig. 3. Objective functions vs. α

VI. Conclusion

In this paper a new LMI for the design of controllers subject to ℓ_1-norm constraints has been presented. The advantage of this LMI is that it simplifies the implementation of the design algorithm and facilitates its integration in multiobjective controller design problems where a convex combination of different norms is optimized. The simplicity comes at the expense of some additional conservativeness since a non-convex objective function is approximated by a convex upper bound.