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Mixed Sensitivity $\mathcal{H}_\infty$ Control of a Three-Tank-System

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Abstract.-A mixed sensitivity $\mathcal{H}_\infty$ control methodology is tested on a Three-Tank-System laboratory prototype. The methodology is based on a stable pre-compensator and its dual post-compensator which allows to solve a mixed sensitivity problem. Frequency responses show that it is possible to shape the open loop transfer function by the control parameters. Time responses are given under flow leak, close connection valve and sensor failure.

Keywords.- Mixed Sensitivity $\mathcal{H}_\infty$ Control, Uncertainty, Disturbance Attenuation, Optimization, Loop-Shaping, Three-Tank-System

I. Introduction

We consider robust controllers which allow to control the system under external disturbance and/or uncertainties, when these are within an admissible set and give more insight for the engineer due to the frequency design specifications [10]. Among the set of stabilizing compensators [8][7][10], the robust performance and robust stability problems can be solved simultaneously [3][4] and the pre-compensator can be implemented by output feedback. These problems can be conservatively combined on a single $\mathcal{H}_\infty$ norm specification [10], *i.e.*, mixed sensitivity problem, and solved using classical algorithms [1]. The methodology of [3][4] is used because it gives explicit formulas for the double coprime factorization and it solves the mixed sensitivity problem without an augmented system. The first explicit formulas for a double coprime factorization in state space were given by [5].

We consider a linearized model of a Three-Tank-System laboratory prototype. A stable $\mathcal{H}_\infty$ Pre-compensator and stable dual Post-compensator are designed in a control-observer scheme using the methodology of [3][4]. The free parameters of these compensators are fixed to a constant value, which minimize an intersection function of the Sensitivity function in low frequencies and the complementary sensitivity function in high frequencies [3][4]. These allows to get low complexity compensators. Stability is guaranteed for the overall system, including a left inverse of the input matrix and a right inverse of the output matrix [4]. A pre-filter for the input reference is added to avoid saturations of the plant input.

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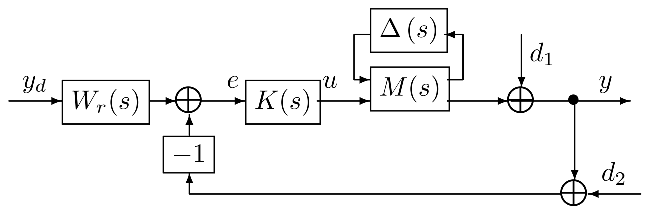

Fig. 1. Control Scheme

Section II gives the problem formulation. A review of the methodology of [3][4] is summarized in section III. In section IV we present a realization of the pre-compensator and give more properties for the open loop transfer function. Section V gives the design of the shape of the open loop transfer function by the tuning of the control parameters and experiments for the time response under flow leak, close connection valve and sensor failure of the Three-Tank-System.

Notation: I_p be the $p \times p$ identity matrix; $A_l := \lim_{s \longrightarrow 0} A(s)$ and $A_h := \lim_{s \longrightarrow \infty} A(s)$ denote the classical asymptotic Bode approximations of a matrix $A(s) \in \Re(s)$, in low and high frequencies, respectively, being $\Re(s)$, the set of rational functions in s with real coefficients; $\cong$ means that only the leading terms are kept; and $\tilde{\rightarrow}$ means $\rightarrow$ for the leading terms.

II. Problem Formulation

Let $K(s)$ be a stabilizing pre-compensator in the control scheme of Figure 1, where $y_\Delta(t)$ and $u_\Delta(t)$, are the input and output of the uncertainty of the plant $\Delta(s)$, $(M(s), \Delta(s))$ is a linear fractional representation of the uncertainty LTI system, $y_d(t)$ is the input reference, $W_r(s)$ is a pre-filter, $d_i(t)$, $i = 1, 2$ are external disturbances, $u(t)$ is the plant input, $y(t)$ is the regulated signal, and $e(t)$ is the error signal.

Note that the aim of the pre-filter $W_r(s)$ of figure 1 is to avoid saturations of the plant input. So, it is not needed to consider a filtered output of the plant input into an augmented system.

Let $P(s)$ be the nominal plant of $(M(s), \Delta(s))$. Then, the problem to solve is to find a proper, real rational and stable controller $K(s)$ that stabilizes internally the uncertainty LTI system and minimizes,

$$J_1 := \left\| \begin{bmatrix} S_{ol} \\ T_{oh} \end{bmatrix} \right\|_\infty \tag{1}$$

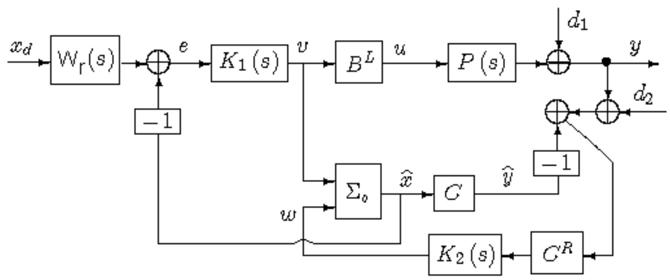

Fig. 2. Pre-Compensator and Dual Post-compensator Scheme

where $S_o(s) = [I + P(s) K(s)]^{-1}$ and $T_o(s) = S_o(s) P(s) K(s)$. The previous criterion is the mixed sensitivity problem [10] of attenuation of $\|d_1(t)\|_2$ in low frequencies and attenuation of $\|d_2(t)\|_2$ and $\|\Delta(s)\|_\infty$ in high frequencies over the output $\|y(t)\|_2$.

We select a pre-compensator $K(s) \in \Re\mathcal{H}_\infty$ stabilizing $P(s)$ using Youla Bongiorno Kučera (YBK) parametrization [7][8],

$$K(s) = \left(Y(s) - R(s)\widetilde{N}_p(s)\right)^{-1}\left(X(s) + R(s)\widetilde{D}_p(s)\right) \tag{2}$$

where $R(s) \in \Re\mathcal{H}_\infty$ and $\det\left(Y(s) - R(s)\widetilde{N}_p(s)\right) \neq 0$, and the $\mathcal{H}_\infty$ optimal value for the parameter $R(s)$, given by the intersection function of [3][4], without an augmented system. Alternatively, the optimal value of $R(s)$ can be accomplished using an inner-outer factorization of the plant and the Nehari Theorem for the general solution of the standard $\mathcal{H}_\infty$ control problem [10]. This solution has similar difficulties to the $\mathcal{H}_2$ control theory due to the high-order involved Riccati equations [10]. Also, there are some state space solutions of the Riccati equations [10] or Loop-Shaping technics [10]. Note that non-linear algebraic equations are involved to solve the Riccati equations. Also, this procedure generates big and complex expressions if we want analytic expressions. Analytic expressions are needed to solve the mixed sensitivity problem using the parameter $R(s)$ because this solution depends of the weights for the pole placement problem. A solution is to iterate between these two problems.

Suppose that $P(s)$ satisfies the parity interlacing property (PIP) [7], then from Youla Bongiorno Kučera (YBK) parametrization [7][8], $K(s) \in \Re\mathcal{H}_\infty$, exists, stabilizing $P(s)$. We select a stable $K(s)$, in the family of stabilizing controllers to get strong stability [7]. This requirement is very important for loop breaking and for increasing the closed-loop bandwidth.

Next, a review of $\mathcal{H}_\infty$ Control Procedure of [4] is summarized.

III. $\mathcal{H}_\infty$ Control Review

Consider the control scheme of figure 2 [4], where, $P(s) = C(sI - A)^{-1} B$ is a strictly proper nominal plant, being $A \in \Re^{n \times n}$, B^L denotes a left inverse of the matrix input $B \in \Re^{n \times m}$, C^R a right inverse of the matrix output $C \in \Re^{p \times n}$, $v(t)$ the output of the stabilizing Pre-compensator $K_1(s)$, $w(t)$ the output of the Dual Post-compensator $K_2(s)$, $\widehat{x}(t)$ the estimated state, $\widehat{y}(t)$ the estimated output, $x_d(t)$ the state input reference, $e_x(t)$ the state error signal, and Σ_o is given by:

$$\Sigma_o : \begin{cases} \dot{\widehat{x}}(t) = & A\widehat{x}(t) + v(t) + w(t) \\ \phi(t) = & \widehat{x}(t) \end{cases} \tag{3}$$

Sufficient stability conditions for $K_i(s)$, $i = 1, 2$ in the scheme of Figure 2, are given by,

Theorem 1 [4]. *Consider a strictly proper plant $P(s)$ in the scheme of Fig. 2 where the dynamic filter Σ_o is given by (3) and (A, B, C) is a stabilizable and detectable realization of $P(s)$ being $A \in \Re^{n \times n}$, and $B \in \Re^{n \times m}$ and $C \in \Re^{p \times n}$. Suppose that $P(s)$ satisfies the Parity Interlacing Property (PIP) [7] and that $K_i(s)$, $i = 1, 2$ are given by :*

$$\begin{aligned} K_i(s) &= A + \frac{(a_i + r_i)s + a_i^2}{s + a_i - r_i} I_n, \quad i = 1, 2 \\ r_i &\cong a_i\left(1 - \frac{\gamma_{\min} a_i}{(w_h + 1)\|A\|_\infty}\right), \quad i = 1, 2 \end{aligned} \tag{4}$$

where $r_i \in [-a_i, a_i]$, $i = 1, 2$, $\gamma_{\min} = [1 + \lambda_{\max}(YX)]^{0.5}$, being Y and X the solutions of the Riccati equations $A^T X + XA - X^2 + I_n = 0$ and $AY + YA^T - Y^2 + I_n = 0$. Let, $B = \begin{bmatrix} 0 \\ B_1 \end{bmatrix}$, $C = \begin{bmatrix} 0 & C_1 \end{bmatrix}$ or $C = \begin{bmatrix} C_2 & 0 \end{bmatrix}$, $B^L = \begin{bmatrix} G_1 & B_1^{-1} \end{bmatrix}$, $C^R = \begin{bmatrix} G_2 \\ C_1^{-1} \end{bmatrix}$ or $C^R = \begin{bmatrix} C_2^{-1} \\ G_2 \end{bmatrix}$ being $B_1 \in \Re^{m \times m}$ and $C_i \in \Re^{p \times p}$, $i = 1, 2$, non-singular matrices, and G_1 and G_2 be functions only of g_i, $i = 1, 2$, respectively, being g_i, $i = 1, 2$ the free-parameters corresponding to the kernel of B^L and C^R, respectively. Then, the overall system of Fig. 2 is stable if, for $i = 1, 2$,

$$\begin{aligned} r_i \gg a_i(1 - a_i), \quad \|B_1 G_1\|_\infty \leq 1, \quad \|G_2 C_i\|_\infty \leq 1 \\ \|E_1(\rho_1 A + I_n)\|_\infty \cong 1, \quad \|(\rho_2 A + I_n) E_2\|_\infty \cong 1 \end{aligned} \tag{5}$$

■

Proof. For the proof we refer the reader to Lemma 2 and Theorem 1 in [4]. ■

Since $P(s)$ satisfies the Parity Interlacing Property (PIP) [7], then a stable controller exists, stabilizing $P(s)$. So, it is possible to get strong stability for the overall system of figure 2. This requirement is very important for loop breaking and for increasing the close loop bandwidth.

Theorem 1 achieves robust stability and robust performance [10][4], *i.e.*, Loop-Shaping criterion [10]. The optimization allows to fix the free parameter of $K_i(s)$, $i = 1, 2$ to a constant value [4] in order to get a low complexity controller. Also, an advantage of the observer of figure 1 is that it allows to have a Post-compensator as fast as the Dual Pre-compensator.

A four-parameter tuning procedure for the Pre-compensator and Dual Post-compensator is:

Procedure 1 [4]. *Consider a strictly proper nominal plant* $P(s) = C(sI_n - A)^{-1} B$ *in the scheme of Fig. 2 where the dynamic filter* Σ_o *is given by (3). Suppose that* $P(s)$ *satisfies the PIP [7] and that* $K_i(s)$, $i = 1, 2$ *are given by (4). Let* B^L *and* C^R *be functions only of* g_i, $i = 1, 2$, *respectively. For a desired time response of the regulated output, attenuation of* $d_1(t)$ *or admissible uncertainties*, i.e., *for a given* a_1,

1. *Select* $a_2 = a_1 > 0$,
2. *Find the largest* g_i, $i = 1, 2$, *and the lowest* w_h, *satisfying the sufficient stability conditions of Theorem 1, (5),* i.e., *select* g_i, $i = 1, 2$, *and* w_h *such that* $\|E_1(\rho_1 A + I_n)\|_\infty$ *and* $\|(\rho_2 A + I_n) E_2\|_\infty$ *are close to 1 and* r_i *is close to* a_i, $i = 1, 2$, *where* $\rho_i = \dfrac{a_i - r_i}{a_i^2}$, $i = 1, 2$,
3. *If possible select* $x_d \in \operatorname{Im} B$ *to assure that* $\lim_{r_i \to a_i} \widehat{x}_{ss} \overset{\sim}{\to} x_{ss}$, $i = 1, 2$, *if not Theorem 1 only guarantees that* $\lim_{r_i \to a_i} e_{ss} \cong 0$, $i = 1, 2$, *and*
4. *If needed, decrease* g_i *to reduce the magnitudes of* $u(s) = B^L K_1(s) e(s)$ *and* $w(s) = K_2(s) C^R C[x(s) - \widehat{x}(s)]$, *or to reduce* $e_x(t)$. *Alternatively, use a pre-filter* $W_r(s)$ *for the reference or a saturation function for the magnitude of* $u(t)$.

In the following, we give a realization of the compensators which is used to control the three-tank system.

IV. Realization of the Compensators

Corollary 1. *A realization of the pre-compensator and post-compensator of Theorem 1 is given by:*

$$\begin{array}{ll} A_i = -(a_i - r_i) I_n, & B_i = I_n \\ C_i = r_i^2 I_n, & D_i = A + (a_i + r_i) I_n \end{array} \tag{6}$$

for $i = 1, 2$.

Proof. From figure 2, $v(s) = K_1(s) e(s)$. Let define the auxiliary state $\xi(s) := \dfrac{1}{s + a_i - r_i} e(s)$, $i = 1, 2$ for which A_i and B_i, $i = 1, 2$, follow. Then, from (4) we have $v(s) = Ae(s) + \left[(a_i + r_i) s + a_i^2\right] \xi(s)$, $i = 1, 2$. So, $v(t) = Ae(t) + \left[(a_i + r_i) \dot{\xi}(t) + a_i^2 \xi(t)\right]$, $i = 1, 2$. Thus, C_i and D_i, $i = 1, 2$, follow substituting $\dot{\xi}(t) = -(a_i - r_i) \xi(t) + e(t)$, $i = 1, 2$. ■

Next section gives the shape of the open loop transfer function and the experimental results.

V. Three-Tank-System

The experimental results shows that as higher is a_i, $i = 1, 2$ or as close is r_i to a_i, $i = 1, 2$, then, $\|L_o(s)\|_\infty$ is increased in low frequencies, achieving robust performance [4][10], where $L_o(s) = P(s) K(s)$ is the open loop transfer function. Also, as smaller is a_i, $i = 1, 2$ or bigger is w_h, $\|L_o(s)\|_\infty$ is decreased in high frequencies, achieving robust stability [4][10]. This Loop-Shaping criterion is the criterion J_1, (1). As higher are a_i, $i = 1, 2$, the time response is decreased because $s = -a_i$, $i = 1, 2$ are the closed loop poles, also, $\|S_o(s)\|_\infty$ is decreased where $S_o(s)$ is the sensitivity function. So, as higher are a_i, $i = 1, 2$, the set of admissible uncertainties and the attenuation of $\|d_1(t)\|_2$ are increased. We take into account that $\|T_o(s)\|_\infty$ increases as $\|S_o(s)\|_\infty$ decreases due to $T_o(s) = I_n - S_o(s)$, which implies a compromise between the attenuation of $\|d_2(t)\|_2$ and $\|d_1(t)\|_2$. Also, as higher is w_h, the attenuation of $\|d_2(t)\|_2$ is increased in a high frequency bandwidth. We use the pre-filter $W_r(s)$ to avoid saturations of the plant input.

Fig. 3. Three-Tank-System

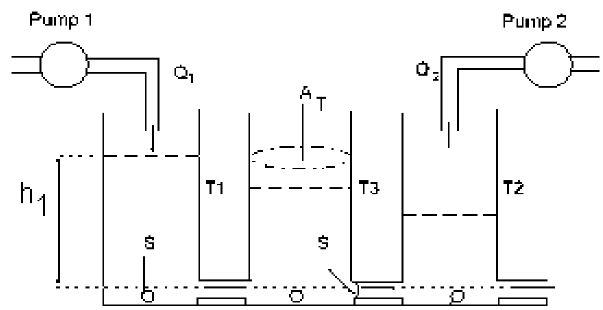

Fig. 4. Three-Tank-System Scheme

The Three-Tank-System laboratory prototype is shown in figure 3 and its scheme in figure 4. The following is a non-linear model of a Three-Tank-System,

$$\begin{aligned} \frac{dh(t)}{dt} &= \frac{1}{A_T} \begin{bmatrix} Q_1(t) - Q_{13}(t) - Q_{10}(t) \\ Q_2(t) + Q_{32}(t) - Q_{20}(t) \\ Q_{13}(t) - Q_{32}(t) - Q_{30}(t) \end{bmatrix} \\ y(t) &= \begin{bmatrix} h_1(t) \\ h_2(t) \end{bmatrix} \\ Q_{13}(t) &= az_{13} S \operatorname{sign}(h_{13}(t)) (2g|h_{13}(t)|)^{1/2} \\ Q_{32}(t) &= az_{32} S \operatorname{sign}(h_{32}(t)) (2g|h_{32}(t)|)^{1/2} \\ Q_{i0}(t) &= az_{i0} S (2gh_i(t))^{1/2}, \quad i = 1, \ldots, 3 \end{aligned} \tag{7}$$

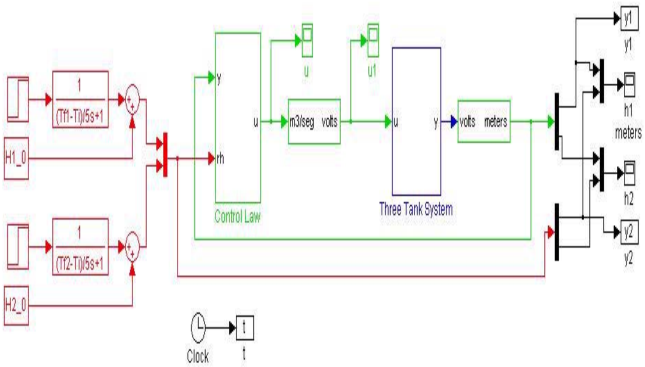

Fig. 5. Real Time Control Scheme

where $h_{13}(t) := h_1(t) - h_3(t)$, $h_{32}(t) := h_3(t) - h_2(t)$, the states $h_i(t)$, $i = 1, \ldots, 3$ are the liquid levels in meters, $Q_i(t)$, $i = 1, 2$ are the supplying flow rates in $m^3/\sec.$, A_T is the section of tank in m^2, az_{13}, az_{32} and az_{i0}, $i = 1, \ldots, 3$ are the outflow coefficients, S is the section of flow leak and connection pipes in m^2 and g is the earth acceleration. By an identification procedure we find the following relations for the pumps and sensors:

$$\begin{array}{l} Q_1(t) = c_{13}A_T\left[c_{11}u_1(t) + c_{12}\right] \\ Q_2(t) = c_{13}A_T\left[c_{21}u_2(t) + c_{22}\right] \\ h_i(t) = -c_{i3}x_i(t) + c_{i4}, \quad i = 1, \ldots, 3 \end{array} \tag{8}$$

where $x_i(t)$, $i = 1, \ldots, 3$ are the liquid levels in volts, $u_i(t)$, $i = 1, 2$ are the supplying flow rates in volts and c_{ij}, $i = 1, \ldots, 3$, $j = 1, \ldots, 4$ are constant parameters.

Let $Q_1(t) = Q_2(t) = 0$ and $az_{10} = az_{20} = az_{30} = 0$. Without flow leaks, the equilibrium point is $h_1(t) = h_2(t) = h_3(t)$. Normally, at least a flow leak in tank 2 exists and $h_1(t) > h_3(t) > h_2(t)$ is satisfied. However, we can have a flow leak in any tank. So, the system tries to reach the equilibrium point without flow leaks and normally we have $h_1(t) \neq h_2(t) \neq h_3(t)$.

Consider the three tank system and the relations (8) in the scheme of figure 5. So, the plant $P(s)$ which is seen by the controller has input in $m^3/\sec.$ and output in meters, that is, the model (7). First, this model is linearized and after, the controller is designed using Theorem 1.

From (7), $\dfrac{\partial Q_{13}(t)}{\partial h_{13}(t)} = az_{13}S\left(\dfrac{g}{2\left|h_{13}(t)\right|}\right)^{1/2}$. This is similar for $Q_{32}(t)$. Since, $\dfrac{\partial Q_{13}(t)}{\partial x_1(t)} = -c_{13}\dfrac{\partial Q_{13}(t)}{\partial h_{13}(t)}$, $\dfrac{\partial Q_{13}(t)}{\partial x_3(t)} = c_{33}\dfrac{\partial Q_{13}(t)}{\partial h_{13}(t)}$, $\dfrac{\partial Q_{23}(t)}{\partial x_2(t)} = -c_{23}\dfrac{\partial Q_{23}(t)}{\partial h_{23}(t)}$ and $\dfrac{\partial Q_{23}(t)}{\partial x_3(t)} = c_{33}\dfrac{\partial Q_{23}(t)}{\partial h_{23}(t)}$, from (7) a linearization around (x_{10}, x_{20}, x_{30}) is given by:

$$\begin{array}{l} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \\ A = \dfrac{-1}{A_T}\left[\begin{array}{ccc} \theta_{13}+\theta_{10} & 0 & -\theta_{13} \\ 0 & \theta_{32}+\theta_{20} & -\theta_{32} \\ -\theta_{13} & -\theta_{32} & \theta_{13}+\theta_{32}+\theta_{30} \end{array}\right] \\ B = \dfrac{1}{A_T}\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{array}\right], \quad C = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \end{array} \tag{9}$$

where:

$$\begin{array}{ll} \theta_{13} & = az_{13}S\left(\dfrac{g}{2\left|h_{10}-h_{30}\right|}\right)^{1/2} \\ \theta_{32} & = az_{32}S\left(\dfrac{g}{2\left|h_{30}-h_{20}\right|}\right)^{1/2} \\ \theta_{i0} & = az_{i0}S\left(\dfrac{g}{2h_{i0}}\right)^{1/2}, \quad i = 1, \ldots, 3 \end{array} \tag{10}$$

We assume that h_{10}, h_{20} and h_{30} are close to the equilibrium point. Also, note that the linear model is symmetric and for the real system, there is an asymmetry due only to (8).

The Pre-Compensator and Dual Post-Compensator are given from Theorem 1, with

$$B^L = \left[\begin{array}{ccc} A_T & 0 & g_1 \\ 0 & A_T & g_1 \end{array}\right], \qquad C^R = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ g_2 & g_2 \end{array}\right] \tag{11}$$

These compensators were implemented in the scheme of figure 2. The relations of the pumps and sensors are located separately as shown in figure 5.

We select $x_d = \left[\begin{array}{ccc} x_{d1} & x_{d2} & 0 \end{array}\right]^T \in \operatorname{Im} B$. The Three-Tank-System parameters are: $az_{13} = az_{32} = az_{20} = 1$, $az_{10} = az_{30} = 0$ and $h_i(0) = 0$, $i = 1, \ldots, 3$, $S = 0.05 \times 10^{-3}$, $A_T = 0.0154$ m^2, and the pump and sensor parameters are,

$$\begin{array}{ll} c_{11} = 0.0103, & c_{12} = 0.1022 \\ c_{21} = 0.0098, & c_{22} = 0.0774 \\ c_{13} = 0.0338143, & c_{14} = 0.3115872 \\ c_{23} = 0.0336217, & c_{24} = 0.3046245 \\ c_{33} = 0.0335975, & c_{34} = 0.3059215 \end{array} \tag{12}$$

The poles of $P(s)$ are -0.0991, -0.0393 and -0.0035 and the zero is -0.0643. So, $P(s)$ satisfies the PIP. The control parameters are: $y_{d1} = 0.4$ meters, $y_{d2} = 0.3$ meters, $\tau_{r1} = \tau_{r2} = 60$, $h_{ri}(t) = \mathcal{L}^{-1}\left\{\dfrac{y_{di}}{\tau_{ri}s+1}\right\}$ for $t \geq 50$ sec. and $h_{ri}(t) = 0$ for $t < 50$ sec., $i = 1, 2$, where $\mathcal{L}$ denotes the Laplace transform, $W_r(s) = \mathcal{L}\left[\begin{array}{ccc} h_{r1}(t)+h_1(0) & h_{r2}(t)+h_2(0) & 0 \end{array}\right]^T$, $\widehat{x}_1(0) = \widehat{x}_2(0) = \widehat{x}_3(0) = 0$, $a_1 = a_2 = 0.1$, $w_h = 220$, $g_1 = 0.016$, $g_2 = 0.015$, $h_{10} = y_{d1}$, $h_{20} = y_{d2}$, $h_{30} = \dfrac{y_{d1}+y_{d2}}{2}$. For security we

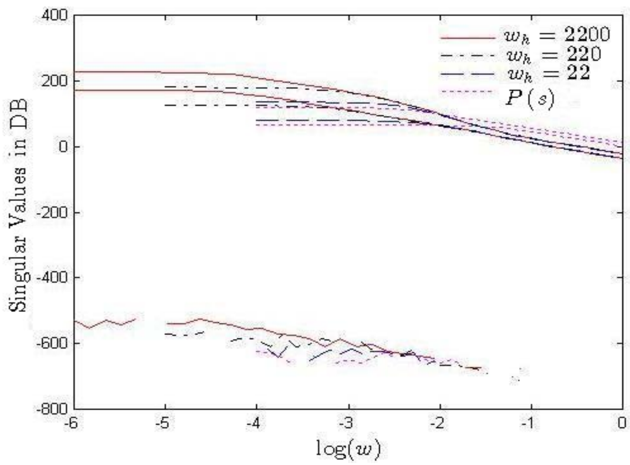

Fig. 6. Singular values of $\sigma(P(s))$ and $\sigma(L_o(s))$ with $w_h = 22$, $w_h = 220$ and $w_h = 2200$ and $a_i = 0.1$, $i = 1, 2$

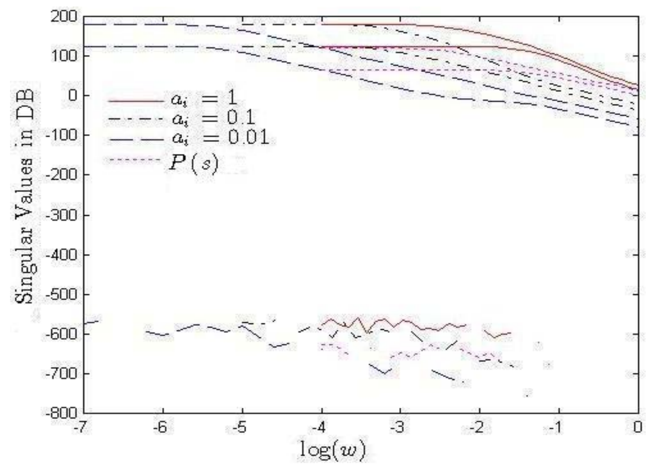

Fig. 7. Singular values of $\sigma(P(s))$ and $\sigma(L_o(s))$ with $a_i = 0.01$, 0.1, 1, $i = 1$, 2, and $w_h = 220$

consider a saturation function for the plant input, *i.e.*, $\text{sat}(u(t)) = \begin{cases} \eta & \text{if} \quad u(t) > \eta \\ u(t) & \text{if} \quad |u(t)| \le \eta \\ -\eta & \text{if} \quad u(t) < -\eta \end{cases}$ where $\eta = 0.11667 \times 10^{-3}$ m^3/sec.. For these parameters $\gamma_{\min} = 1.4117$, $r_i = 0.0995$, $i = 1$, 2, $\|B_1 G_1\|_\infty = 1.0394$, $\|G_2 C_1\|_\infty = 0.0150$, $\|E_1(\rho_1 A + I_n)\|_\infty = 1.0394$ and $\|(\rho_2 A + I_n) E_2\|_\infty = 1.0267$. Then, the sufficient stability conditions of Theorem 1 (5), are satisfied.

Let $\widehat{x}(s) \cong x(s)$, so, $P(s) = (sI_n - A)^{-1} BB^L$. Figure 6 show that as higher is w_h, $\sigma(L_o(s))$ is increased in low frequencies, where $L_o(s) = P(s) K_1(s)$. So, as expected $\sigma(S_{ol})$ decrease and we achieve robust performance. Also, the slope at the crossover frequency is increased and the phase angle decrease. Thus, increasing w_h, we have better attenuation of $\|d_1(t)\|_2$, small stationary state error and the set of admissible uncertainties is also bigger, the price to pay is a small phase angle.

Figure 7 show that as smaller is a_1, $\sigma(L_o(s))$ is decreased in high frequencies. So, $\sigma(T_h)$ decrease and we achieve robust stability. Also, the phase angle is increased. Thus, decreasing a_1, we have better attenuation of d_2, less control magnitude and bigger phase angle but the price to pay is a slower time response and small attenuation of d_1.

The following results were obtained using MATLAB-SIMULINK environment and PC card DAC98 for the Three-Tank-System DTS200 in the control scheme of figure 5. Figures 8 and 9 show the experiments for a 100% flow leak in tank 3 at $t = 300$ sec., *i. e.*, $az_{30} = 1$ at $t = 300$ sec. while figures 10 and 11 shows the experiments for a 100% close connection valve between tank 2 and tank 3 at $t = 300$ sec. , *i.e.*, $az_{32} = 0$ at $t = 300$ sec. and figures 12 and 13 shows the experiments for a 50% sensor failure in tank 1 at $t = 300$ sec.

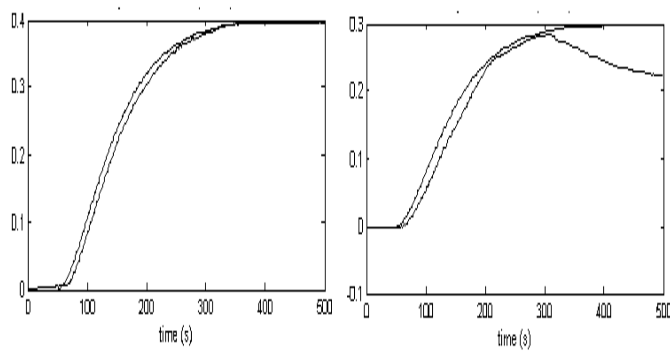

Fig. 8. Outputs $y_1(t)$ and $y_2(t)$, and reference in meters

Figures 8 and 9 show a fast and complete compensation of the flow leak at the regulated output $y_1(t)$, while it is not true for $y_2(t)$ due to saturation of $u_2(t)$. To avoid saturation of $u(t)$ we can reduce w_h or a_i, $i = 1$, 2, however, the stationary state error is increased. Alternatively, we can increase τ_{ri}, $i = 1$, 2, avoiding saturation of $u(t)$ with a more slow time response.

Figures 10 and 11 show that the close of the connection valve is well compensated at $y(t)$ without input saturation. Note that $u_1(t)$ goes to zero for $t > 300$ in figure 11, as expected. Also, always exist some energy in tank 2, in order to compensate the flow leak in this tank.

Figure 12 shows that $y_1(t)$ arrives to a "wrong" stationary state value due to the sensor failure of tank 1. The small oscillations of $y_1(t)$ in figure 12 are due to saturation of $u_1(t)$ as shown by figure 13. Figures 12 and 13 show that the sensor failure of tank 1 is well compensated at $y_2(t)$.

VI. Conclusions

A non saturated $\mathcal{H}_\infty$ control law allows to control the regulated output under external disturbance, guarantee-

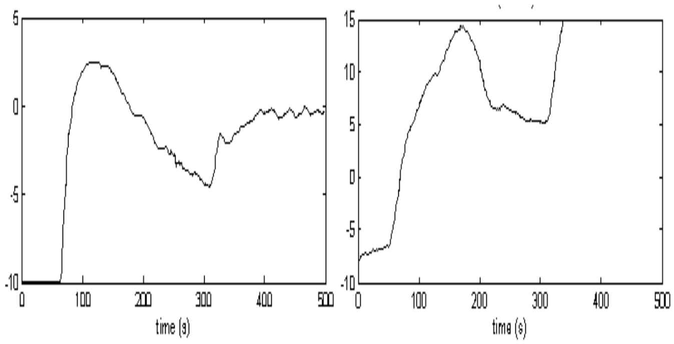

Fig. 9. Inputs $u_1(t)$ and $u_2(t)$ in volts

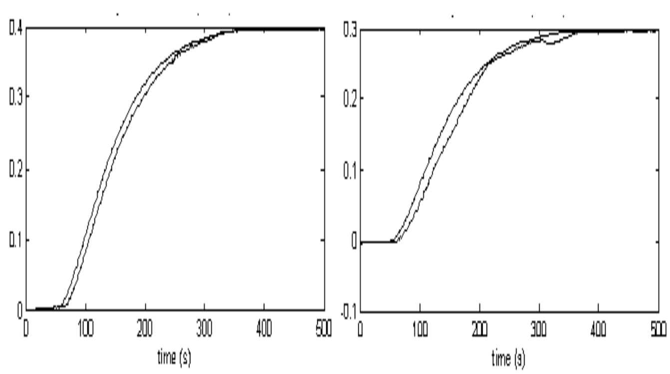

Fig. 10. Outputs $y_1(t)$ and $y_2(t)$, and reference in meters

ing stability. The flow leak, close connection valve and sensor failures are well attenuated at the liquid levels. These robust stability and robust performance are due to the shape of the open loop transfer function getting by means of the tuning of the control parameters. A pre-filter of the state input reference and the tuning of the control parameters allows a non saturated control law.

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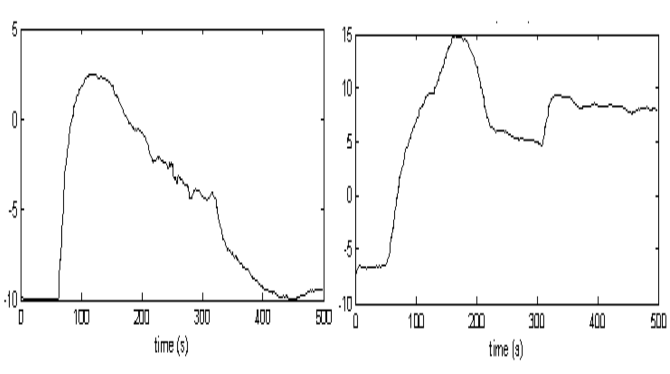

Fig. 11. Inputs $u_1(t)$ and $u_2(t)$ in volts

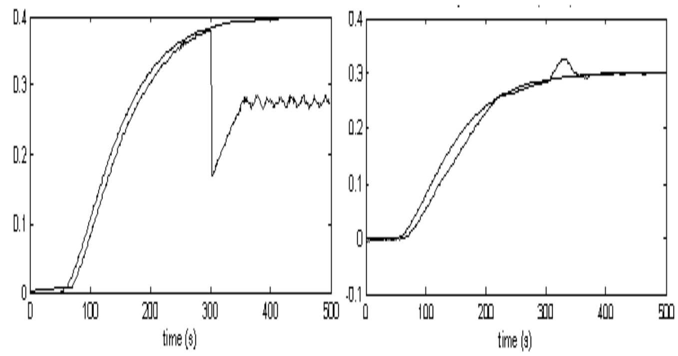

Fig. 12. Outputs $y_1(t)$ and $y_2(t)$, and reference in meters

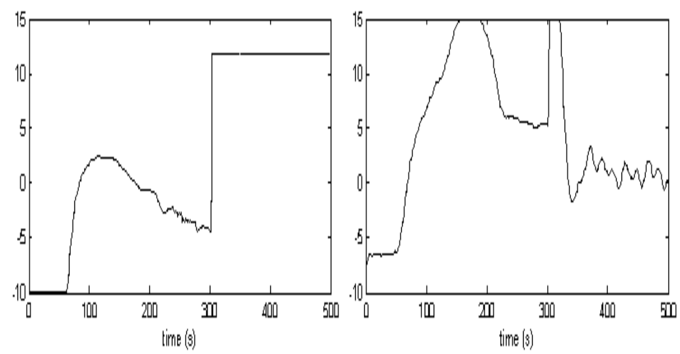

Fig. 13. Inputs $u_1(t)$ and $u_2(t)$ in volts

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