Analytical Design of Takagi-Sugeno Fuzzy Control Systems

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***Abstract*—Based on the properties analysis of Takagi-Sugeno (T-S) fuzzy systems with two-overlapped fuzzy partition (TFP) inputs, an analytical design approach of T-S fuzzy control systems is proposed in this paper. This approach includes two parts: (1) A new method to check the stability of a T-S fuzzy control system which only requires to find a common positive-definite matrix in each maximal overlapped-rules group (MORG). This method can reduce the conservatism and difficulty of the former stability analysis methods. (2) The systematic design of T-S fuzzy controllers by means of the parallel distributed compensation (PDC) and linear matrix inequalities (LMI). The simulation results of a nonlinear mass-spring-damper system show that this approach is effective.**

I. INTRODUCTION

RECENTLY, fuzzy control has been successfully applied to a variety of industrial processes such as bath chemical reactors, cement kilns, etc. However, we still need efficient analytical design methods to improve systems performance. Based on Takagi-Sugeno (T-S) fuzzy models [1], many scholars have been studying the stability analysis and systematic design of T-S fuzzy systems. Tanaka and Sugeno proposed a design and a stability method for fuzzy systems via the Lyapunov direct method [2]. It is required that for all local linear models a common positive-definite matrix P must be found to satisfy the Lyapunov equation. In many cases, it is difficult to find a common positive-definite matrix P when the number of rules for the fuzzy system is large. Cao *et al*. proposed the global model of a T-S fuzzy system with m rules to m local linear uncertain subsystems, and utilized the quadratic stabilization of linear uncertain systems to discuss the stability of fuzzy systems [3]. Johansson *et al*. proposed a stability condition of fuzzy systems that searches for different quadratic Lyapunov functions in different operating regions to constitute a continuous Lyapunov function in global state space [4]. Zhang *et al*. employed the maximum defuzzifier to avoid the difficulty of finding a common positive-definite matrix P [5]. These approaches relax the stability condition of [2] to some extent, but each approach has its limitations. Since Cao *et al*. treat T-S fuzzy systems as a kind of linear uncertain system and don't make the best use of the structural information of the rule's premises, the upper boundaries of local subsystems are not easily found. Johansson *et al*. make use of the structural information of the rule's premises, however, there are many situations where a continuous Lyapunov function is too restrictive. Compared with the center-average defuzzifier commonly used in practice, the maximum defuzzifier adopted by Zhang *et al*. doesn't make use of the information of the rules with lesser firing strength. Therefore, although there has been much progress made in recent years, the stability analysis and systematic design of T-S fuzzy systems have not been solved perfectly because these systems are essentially nonlinear and complex.

In this paper, we use the structural information in the rule base to decrease the conservatism and difficulty of the fuzzy control system design. The two-overlapped fuzzy partition (TFP) is defined by summarizing the common characteristic of the input variable membership functions of most fuzzy systems. Based on the properties of fuzzy systems with TFP inputs, we propose a new method for stability analysis of T-S fuzzy control systems with TFP inputs by constructing a piecewise smooth quadratic (PSQ) Lyapunov function. Using the methods of parallel distributed compensation (PDC) [6] and linear matrix inequalities (LMI) [7, 8], we present an analytical design approach of T-S fuzzy control systems, which is demonstrated by the simulation results of a nonlinear mass-spring-damper system.

This paper is organized as follows. The definitions and properties about T-S fuzzy systems with TFP inputs are presented in Section II. In Section III we introduce a PSQ Lyapunov function as a tool for analyzing the stability of close-loop T-S fuzzy systems with TFP inputs. In Section IV, an analytical design approach of T-S fuzzy control systems is discussed in detail, and applied to a nonlinear mass-spring-damper system. The conclusions are given in the last part of the paper.

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II. T-S Fuzzy Systems with Two-overlapped Fuzzy Partition Inputs

A. The Model of T-S Fuzzy Plants

The continuous T-S fuzzy dynamic model is described by fuzzy IF-THEN rules, which represent local linear input-output relations of nonlinear systems. A continuous-time plant can be represented by fuzzy rules as follows:

R_i: IF $x_1(t)$ is F_1^i and … and $x_n(t)$ is F_n^i

$$\text{THEN } \dot{x}(t) = A_i x(t) + B_i u(t),\ i=1,2,\ldots,l \qquad (1)$$

where R_i denotes the ith fuzzy inference rule, l denotes the number of inference rules, F_j^i (j=1,2,…,n) denote input fuzzy sets, $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^{\mathrm{T}}$, $x \in R^n$, denotes the state vector, $u \in R$ denotes the input variable, (A_i, B_i) represents the ith local model parameters of the fuzzy system.

By using singleton fuzzifier, product inference and center-average defuzzifiier, the following dynamic global model can be obtained:

$$\dot{x}(t) = \sum_{i=1}^{l} w_i (A_i x(t) + B_i u(t)) \Big/ \sum_{i=1}^{l} w_i \qquad (2)$$

where $w_i = \prod_{k=1}^{n} F_k^i(x_k(t))$ is the firing strength of the ith rule.

Let $h_i = w_i \Big/ \sum_{i=1}^{l} w_i$, (2) can be rewritten as:

$$\dot{x}(t) = \sum_{i=1}^{l} h_i \{A_i x(t) + B_i u(t)\} \qquad (3)$$

B. Two-overlapped Fuzzy Partition (TFP)

Assume that the fuzzy membership functions studied in this paper are continuous functions in the universe of discourse.

Definition 1: A cluster of fuzzy sets {F_i, i=1,2,…,k } are said to be a two-overlapped fuzzy partition in the universe X if each of F_i (i=1,2,…,k) overlaps with its neighboring fuzzy sets, but doesn't overlap with its non-neighboring fuzzy sets. k is said to be the number of fuzzy partition of X.

Definition 2: For a state input $x=[x_1, x_2, \ldots, x_n]^{\mathrm{T}}$, all the rules of a fuzzy system which satisfy the firing strength $w_i>0$ constitute an overlapped-rules group. The state input x is said to be an operating point of this overlapped-rules group. The aggregate of operating points of an overlapped-rules group is said to be the operating region of this overlapped-rules group.

Definition 3: An overlapped-rules group with the largest amount of rules is said to be a maximal overlapped-rules group (MORG).

The essential character of TFP is that each fuzzy set only overlaps with its neighboring fuzzy sets. The standard fuzzy partition (SFP) proposed by Xiu and Ren [9] has this character, therefore the SFP is a special case of the TFP.

In Fig.1, we can see that (a) is a SFP and also a TFP, (b) is a TFP but not a SFP, for the 3 fuzzy sets are not satisfied the condition of full-overlapped; (c) is not a TFP, for there are 3 fuzzy sets overlapped on (x_1, x_2).

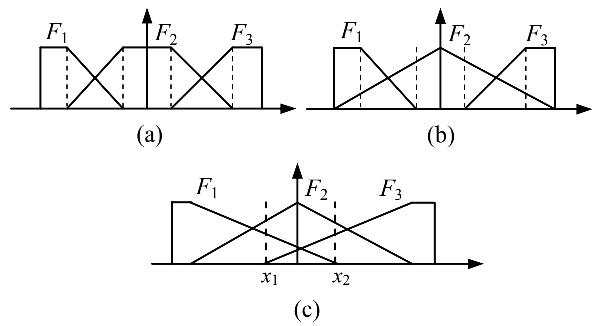

Fig.1 Illustrations of TFPs and non-TFPs

To illustrate the concept of MORGs, we take a T-S fuzzy system with two-inputs and single-output (TISO) as follows:

R_{ij}: IF x_1 is F_1^i and x_2 is F_2^j THEN $\dot{x} = A_{ij} x$,

i=1,2,3; j=1,2,3.

where x_1 and x_2 are input variables of the fuzzy system.

The fuzzy partition of x_1 is $\{F_1^i, i=1,2,3\}$, and the fuzzy partition of x_2 is $\{F_2^j, j=1,2,3\}$. $\{F_1^i\}$ and $\{F_2^j\}$ employ trapezoidal membership functions and conform to the conditions of TFP. The sketch map of the system's input space is shown in Fig. 2.

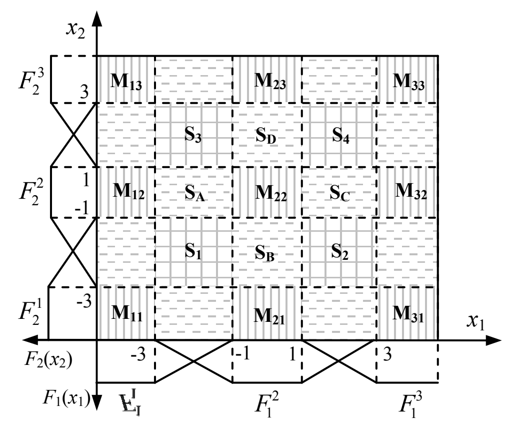

Fig.2 The sketch map of a TISO system's input space

In Fig.2 we can see that there are 25 overlapped-rules groups in this system:

S_1—S_4 are the operating regions of 4 MORGs, where

S_1 includes 4 rules denoted as R_{11}, R_{12}, R_{21}, R_{22};

S_2 includes 4 rules denoted as R_{21}, R_{22}, R_{31}, R_{32};

S_3 includes 4 rules denoted as R_{12}, R_{13}, R_{22}, R_{23};

S_4 includes 4 rules denoted as R_{22}, R_{23}, R_{32}, R_{33}.

M_{11}—M_{33} are the operating regions of sole-rule groups. Each of the other 12 overlapped-rules groups includes 2 rules, for example, S_{A} includes 2 rules denoted as R_{12}, R_{22}.

C. Properties of Fuzzy Control Systems with TFP Inputs

The following propositions indicate two important properties of fuzzy control systems with TFP inputs:

Proposition 1: If the input variables of a fuzzy control system adopt TFPs, then the number of rules in a MORG is 2^n, and the number of MORGs is $\prod_{i=1}^{n}(m_i-1)$. where n denotes the number of input variables, m_i denotes the number of fuzzy partition of the ith input variable.

Proof: Let n denote the number of input variables, the n system denotes a system with n input variables.

If n=1, then the number of the input fuzzy partition m_1 is the number of rules. Since the system input variable employs a TFP, the number of overlapped fuzzy sets is 2, and there are m_1-1 overlapped-rules groups. We can obtain that there are 2 rules in each MORG, and the number of MORGs is m_1-1. The conclusion holds.

Assume that the conclusion holds when $n=k$, that is, the number of rules included in MORGs is 2^k, and the number of MORGs is $\prod_{i=1}^{k}(m_i-1)$.

Consider the case of $n=k+1$. Assume that the number of fuzzy partition of the k+1th input variable is m_{k+1}. The $n=k+1$ system is formed by adding an input variable to the $n=k$ system, that is, adding an "and" connecting item to the premises of the $n=k$ system's fuzzy rules. Hence the number of rules in the $n=k+1$ system is m_{k+1} times the number in the $n=k$ system. Since the k+1th input variable employs a TFP, the number of overlapped fuzzy sets is 2, and there are $m_{k+1}-1$ overlapped fuzzy set groups. Each of the 2 overlapped fuzzy sets connects with the premises of every rules of a MORG of the $n=k$ system by an "and" to form 2^k rules of the $n=k+1$ system. Therefore, the number of rules in a MORG of the $n=k+1$ system is $2\times2^k=2^{k+1}$.

Since each overlapped fuzzy sets group of the k+1th input variable combines with each MORG of the $n=k$ system to form a MORG of the $n=k+1$ system, the number of MORGs of the $n=k+1$ system with TFP inputs is $(m_{k+1}-1)\prod_{i=1}^{k}(m_i-1)=\prod_{i=1}^{k+1}(m_i-1)$. The conclusion holds.

By deduction, the proof is now completed.

Proposition 2: If the input variables of a fuzzy control system adopt TFPs, then all the rules in an overlapped-rules group must be included in a MORG.

Proof: Let n denote the number of input variables, the n system denotes a system with n input variables.

If n=1, a non-MORG includes one rule, and a MORG includes 2 rules. It is clear that each non-MORG is included in a MORG, thus the conclusion holds.

Assume that the conclusion holds when $n=k$, that is, each non-MORG is included in a MORG when $n=k$.

Consider the case of $n=k+1$. The $n=k+1$ system is formed by adding an input variable to the $n=k$ system, that is, adding an "and" connecting item to the premises of the $n=k$ system's fuzzy rules. Since the k+1th input variable employs a TFP, each fuzzy set of the k+1th input variable is overlapped with its neighbors, and the number of overlapped fuzzy sets is 2. Let G^1 denote an overlapped-rules group of the $n=k+1$ system formed by the sole fuzzy set of the k+1th input variable connecting with the premises of rules in an overlapped-rules group of the $n=k$ system by an "and", and G^2 denote an overlapped-rules group of the $n=k+1$ system formed by each of the 2 overlapped fuzzy sets of the k+1th input variable connecting with the premises of rules in an overlapped-rules group of the $n=k$ system by an "and". It is clear that $G^1\subset G^2$, so we only need to study the case of. In the case G^2, it forms a non-MORG of the $n=k+1$ system to connect each of the 2 overlapped fuzzy sets of the k+1th input variable with all premises of rules in a non-MORG of the $n=k$ system by an "and" (for the number of rules in this group is less than 2^{k+1}), and it forms a MORG of the $n=k+1$ system to connect each of the 2 overlapped fuzzy sets of the k+1th input variable with all premises of rules in a MORG of the $n=k$ system by an "and" (for the number of rules in this group is $2\times2^k=2^{k+1}$). Therefore, in the $n=k+1$ system, all rules in a non-MORG must be included in a MORG. The conclusion holds.

By deduction, the proof is now completed.

III. Stability Analysis of Closed-loop T-S Fuzzy Control Systems with TFP Inputs

A. Parallel Distributed Compensation (PDC)

PDC is a simple and natural design technique for T-S fuzzy models (1). In PDC concept, each control rule is distributively designed for the corresponding rule of a T-S fuzzy model. Linear control theory can be used to design the consequent parts of a fuzzy controller, because the consequent parts of T-S fuzzy models are described by linear state equations. The fuzzy controller shares the same fuzzy sets with the fuzzy model in the premise parts.

For (1), let K_i denote the state feedback gain of the ith local model. Thus, the local control laws are

R_i: IF x_1 is F_1^i and … and x_n is F_n^i

$$\text{THEN } u=-K_i x,\ i=1,2,\ldots,l \tag{4}$$

The global model of fuzzy controllers is inferred as follows:

$$u=-\sum_{i=1}^{l}h_i K_i x \tag{5}$$

B. The Extended Lyapunov Stability Theory

A sufficient condition to check the stability of a control system via the extended Lyapunov theory [3, 10] is shown as follows:

Theorem 1: Assume that a system can be described by

$\dot{x} = f(x)$. where $x \in R^n$, $f(x)$ is a $n \times 1$ function vector, and satisfies $f(0)=0$. Suppose that there exists a piecewise smooth quadratic (PSQ) function $V(x)$ such that

(a) $V(0)=0$,

(b) $V(x)>0$, for $x \neq 0$,

(c) $V(x) \to \infty$ as $\|x\| \to \infty$,

(d) $\dot{V}(x) < 0$, for $x \neq 0$. where $\dot{V}(x)$ at the right and left discontinuous points of $V(x)$ are defined as the right-hand derivative and the left-hand derivative respectively.

Then the equilibrium state $x=0$ is asymptotically stable in the large, and $V(x)$ is a PSQ Lyapunov function.

C. Stability Analysis of T-S Fuzzy Control Systems

Based on the properties of fuzzy systems with the TFP inputs, we present a new method for stability analysis of closed-loop T-S fuzzy control systems with the TFP inputs as follows.

Theorem 2: For a fuzzy control system which is described by (1) and (4), if the input variables adopt TFPs, then the equilibrium of a closed-loop fuzzy control system is asymptotically stable in the large if there exists a common positive-definite matrix P_j in each MORG satisfying the following condition (C1) or (C2):

(C1): $G_{ik}^{\mathrm{T}} P_j + P_j G_{ik} < 0$, $i,k \in L_j$; $j=1,2,\ldots,\prod_{p=1}^{n}(m_p - 1)$.

(C2): $G_{ii}^{\mathrm{T}} P_j + P_j G_{ii} < 0$, $i \in L_j$; and

$$\left(\frac{G_{ik}+G_{ki}}{2}\right)^{\mathrm{T}} P_j + P_j\left(\frac{G_{ik}+G_{ki}}{2}\right) < 0,\ i, k \in L_j \text{ and } i<k;$$

$j=1,2,\ldots,\prod_{p=1}^{n}(m_p - 1)$.

where $G_{ik}=A_i-B_iK_k$, L_j={the sequence numbers of rules included in the jth MORG }.

Proof: Assume that the system state input is $x=[x_1, x_2,\ldots, x_n]^{\mathrm{T}}$. There are r overlapped-rules groups in the system, l_j={the sequence numbers of rules included in the jth overlapped-rules group}. S_j ($j=1,2,\ldots,r$) are operating regions of overlapped-rules groups, and satisfy $\bigcup_{j=1}^{r} S_j = S$.

Since in the operating region S_j, there exist $h_i h_k \geq 0$, $\sum_{i\in l_j}\sum_{k\in l_j} h_i h_k = 1$, and $\sum_{i\in l_j} h_i^2 + 2\sum_{i,k\in l_j}^{i<k} h_i h_k = 1$, then the local model of the closed-loop system is

$$\dot{x} = \sum_{i\in l_j} h_i \left\{ A_i x - B_i\left(\sum_{k\in l_j} h_k K_k x\right)\right\}$$

$$= \left\{\sum_{i\in l_j}\sum_{k\in l_j} h_i h_k (A_i - B_i K_k)\right\} x = \left\{\sum_{i\in l_j}\sum_{k\in l_j} h_i h_k G_{ik}\right\}$$

$$= \left\{\sum_{i\in l_j} h_i^2 G_{ii} + 2\sum_{i,k\in l_j}^{i<k} h_i h_k \left(\frac{G_{ik}+G_{ki}}{2}\right)\right\} x \tag{6}$$

If in the operating region S_j, there exists a common positive-definite matrix P_j satisfying (7) or (8) and (9).

$$G_{ik}^{\mathrm{T}} P_j + P_j G_{ik} < 0,\ i, k \in l_j \tag{7}$$

$$G_{ii}^{\mathrm{T}} P_j + P_j G_{ii} < 0,\ i \in l_j \tag{8}$$

$$\left(\frac{G_{ik}+G_{ki}}{2}\right)^{\mathrm{T}} P_j + P_j\left(\frac{G_{ik}+G_{ki}}{2}\right) < 0,\ i,k \in l_j \text{ and } i<k \tag{9}$$

then we select $V_j(x)=x^{\mathrm{T}}P_j x$ as a Lyapunov function in S_j. It can be easily verified that $V_j(x)$ satisfies (a),(b) and (c) of Theorem 1 in S_j.

From (7) or (8) and (9), and with $h_i>0$, for $x\neq 0$, we have

$$\dot{V}_j(x) = \lim_{\Delta\to 0}\frac{1}{\Delta}\left(V_j(x(t+\Delta)) - V_j(x(t))\right) = \dot{x}^{\mathrm{T}} P_j x + x^{\mathrm{T}} P_j \dot{x}$$

$$= \left(\sum_{i\in l_j}\sum_{k\in l_j} h_i h_k G_{ik} x\right)^{\mathrm{T}} P_j x + x^{\mathrm{T}} P_j \left(\sum_{i\in l_j}\sum_{k\in l_j} h_i h_k G_{ik} x\right)$$

$$= \sum_{i\in l_j}\sum_{k\in l_j} h_i h_k \left\{x^{\mathrm{T}}(G_{ik}^{\mathrm{T}} P_j + P_j G_{ik})x\right\}$$

$$= \sum_{i\in l_j} h_i^2 \left\{x^{\mathrm{T}}(G_{ii}^{\mathrm{T}} P_j + P_j G_{ii})x\right\}$$

$$+ 2\sum_{i,k\in l_j}^{i<k} h_i h_k \left\{x^{\mathrm{T}}\left(\left(\frac{G_{ik}+G_{ki}}{2}\right)^{\mathrm{T}} P_j + P_j\left(\frac{G_{ik}+G_{ki}}{2}\right)\right)x\right\} < 0 \tag{10}$$

Thus, $V_j(x)$ also satisfies (d) of Theorem 1 in S_j.

The characteristic function of S_j is defined by

$$\lambda_j(x) = \begin{cases} 1, & x \in S_j \\ 0, & \text{other} \end{cases},\quad \sum_{j=1}^{r}\lambda_j(x) = 1 \tag{11}$$

Then the global model of the closed-loop T-S fuzzy control system can be constructed as follows:

$$\dot{x} = \sum_{j=1}^{r}\lambda_j(x)\left(\sum_{i\in l_j}\sum_{k\in l_j} h_i h_k G_{ik} x\right) \tag{12}$$

Let $P = \sum_{j=1}^{r}\lambda_j P_j$, (13)

A global Lyapunov function can be constructed as: $V(x)=x^{\mathrm{T}}Px$. From (13) we have

$$V(x) = x^{\mathrm{T}}\left(\sum_{j=1}^{r}\lambda_j P_j\right)x = \sum_{j=1}^{r}\lambda_j x^{\mathrm{T}} P_j x = \sum_{j=1}^{r}\lambda_j V_j(x) \tag{14}$$

It can be easily verified that $V(x)$ satisfies (a),(b) and (c) of Theorem 1. $V(x)$ is a PSQ function, the derivatives at the right and left discontinuous points of $V(x)$ are defined as the right-hand derivative and left-hand derivative respectively. From the definition of a derivative and (10), for $x\neq 0$, we have

$$\dot{V}(x) = \frac{dV}{dt} = \lim_{\Delta\to 0}\frac{1}{\Delta}\left(V(x(t+\Delta)) - V(x(t))\right)$$

$$= \lim_{\Delta \to 0} \frac{1}{\Delta} \left(\sum_{j=1}^{r} \lambda_j V_j (x(t+\Delta)) - \sum_{j=1}^{r} \lambda_j V_j (x(t)) \right)$$

$$= \sum_{j=1}^{r} \lambda_j \left(\lim_{\Delta \to 0} \frac{1}{\Delta} \left(V_j (x(t+\Delta)) - V_j (x(t)) \right) \right) = \sum_{j=1}^{r} \lambda_j \dot{V}_j < 0$$

So $V(x)$ also satisfies (d) of Theorem 1. $V(x)$ is a PSQ Lyapunov function of the closed-loop fuzzy control system described by (1) and (4). Therefore, we can reach a conclusion as follows: for a fuzzy control system which discribed by (1) and (4), if the input variables adopt TFPs, then the equilibrium of the closed-loop fuzzy control system is asymptotically stable in the large if there exists a common positive-definite matrix P_j in each overlapped-rules group satisfying (7) or (8) and (9).

From Proposition 2, if the input variables of a fuzzy control system adopt TFPs, all the rules of an overlapped-rules group must be included in a MORG. So we only require to find a common positive-definite matrix P_j in each MORG satisfying (C1) or (C2).

The proof is now completed.

IV. Analytical Design and Its Application of T-S Fuzzy Control Systems with TFP Inputs

A. *Analytical Design of T-S Fuzzy Controllers with TFP Input*

If the T-S fuzzy plant is locally controllable, i.e. (A_i, B_i), i=1,…,l , are controllable pairs, the feedback control gains K_i, (i=1,2,…,l) can be derived from Ackermann's formula. By using linear system theory, we can assign the eigenvalues of (A_i–B_iK_i) to get a desired performance. Therefore, the analytical design procedure of a fuzzy controller is as follows:

Step 1: For a T-S fuzzy plant described by (1), verify that all the local linear subsystems are controllable. That is,

rank (B_i, A_iB_i,…, $A_i^{n-1}B_i$)=n, i=1,…,l.

Step 2: Using linear system theory, assign the eigenvalues of each (A_i–B_iK_i) according to the desired performance.

Step 3: Calculate the feedback gain K_i of each local linear subsystem via Ackermann's formula.

Step 4: Find all the MORGs of the T-S fuzzy system according to Proposition 1, and apply Theorem 2 to check the stability of the closed-loop T-S fuzzy control system. If the system is not stable, go back to step 2 to reassign the eigenvalues of each (A_i–B_iK_i) till satisfying Theorem 2. In this step, the search for a common P_j in each MORG approaches as a convex optimization problem in terms of LMI, and LMI-lab tools can be utilized [8].

Step 5: Combine the state-feedback controllers of local linear subsystems to a global model of the controller according to (5). Through computer simulations or practical experimentations, if the controller achieves a desired performance, then the design of fuzzy controller is over. Otherwise, go back to step 2 to reassign the eigenvalues of each (A_i–B_iK_i) till the desired performance is achieved.

B. *An Example to Illuminate the Analytical Design Approach of T-S Fuzzy Controllers*

To illustrate the analytical design of T-S fuzzy control systems proposed above, we shall design a fuzzy controller for a nonlinear mass-spring-damper mechanical system used by Tanaka *et al.* [6] as follows:

$$M\ddot{x} + g(x,\dot{x}) + f(x) = \phi(\dot{x})u \tag{15}$$

where M is the mass and u is the force. $g(x,\dot{x}), f(x), \phi(\dot{x})$ are the nonlinear terms with respect to the damper, spring and input.

Assume that $x \in [-1.5,1.5]$, $\dot{x} \in [-1.5,1.5]$, M=1.0, $g(x,\dot{x}) = \dot{x}^3$, $f(x) = 0.01x + 0.1x^3$, $\phi(\dot{x}) = 1 + 0.13\dot{x}^3$. Then (15) can be rewritten as follows:

$$\ddot{x} = -\dot{x}^3 - 0.01x - 0.1x^3 + (1 + 0.13\dot{x}^3)u \tag{16}$$

(16) can be approximated by a T-S fuzzy model as

R_i: IF x_1 is F_1^i and x_2 is F_2^i

$$\text{THEN } \dot{x} = A_i x + B_i u \ , \ i=1,2,\ldots,9 \tag{17}$$

where x_1=x, the fuzzy sets of x_1 are $\{F_{1j}, j = 1,2,3\}$; x_2= $\dot{x}$, the fuzzy sets of x_2 are $\{F_{2k}, k = 1,2,3\}$;

$F_1^1 = F_1^4 = F_1^7 = F_{11}$, $\quad F_1^2 = F_1^5 = F_1^8 = F_{12}$,

$F_1^3 = F_1^6 = F_1^9 = F_{13}$, $\quad F_2^1 = F_2^2 = F_2^3 = F_{21}$,

$F_2^4 = F_2^5 = F_2^6 = F_{22}$, $\quad F_2^7 = F_2^8 = F_2^9 = F_{23}$.

The membership functions of the input variables' fuzzy sets are shown in Fig.3

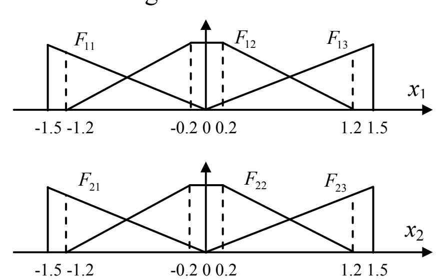

Fig.3 The membership functions of input variables

In Fig.3 we can see that the input variables of the fuzzy system employ TFPs. There are 4 MORGs in this system whose operating regions denoted as S_1—S_4: S_1 includes 4 rules denoted as R_1, R_2, R_4, R_5; S_2 includes 4 rules denoted as R_2, R_3, R_5, R_6; S_3 includes 4 rules denoted as R_4, R_5, R_7, R_8; S_4 includes 4 rules denoted as R_5, R_6, R_8, R_9. The parameter matrixes of the local subsystems are as follows:

$$A_1 = A_3 = A_7 = A_9 = \begin{bmatrix} 0 & 1 \\ -0.235 & -2.25 \end{bmatrix},$$

$$A_2 = A_8 = \begin{bmatrix} 0 & 1 \\ -0.235 & 0 \end{bmatrix}, \ A_5 = \begin{bmatrix} 0 & 1 \\ -0.01 & 0 \end{bmatrix},$$

$$A_4 = A_6 = \begin{bmatrix} 0 & 1 \\ -0.01 & -2.25 \end{bmatrix}, B_1=B_4=B_7=[0, 0.7046]^T,$$

$B_2=B_5=B_8=[0, 1]^T$, $B_3=B_6=B_9=[0, 1.2954]^T$.

If we select the closed-loop eigenvalues of the 9 local linear subsystems via state-feedback to be: $p_1=p_2=p_3=p_4=p_5=p_6=p_7=p_8=p_9=[-2,-2]$; then the state feedback gains K_i of 9 local linear subsystems can be derived from Ackermann's formula as follows:

K_1 =[5.3435, 2.4837], K_2 =[3.765, 4.0],
K_3 =[2.9064, 1.3509], K_4 =[5.6628, 2.4837],
K_5 =[3.99, 4.0], K_6 =[3.0801, 1.3509],
K_7 =[5.3435, 2.4837], K_8 =[3.765, 4.0],
K_9 =[2.9064, 1.3509],

and we can calculate all $G_{ij}=A_i-B_iK_j$; i=1,2,…,9; j=1,2,…,9.

We have found 4 common positive-definite matrixes in 4 MORGs satisfying the condition (C1) of Theorem 2 via the LMI approach as follows:

$$P_1 = P_3 = \begin{bmatrix} 24.7317 & 2.6665 \\ 2.6665 & 3.4956 \end{bmatrix}, P_2 = P_4 = \begin{bmatrix} 0.801 & 0.0971 \\ 0.0971 & 0.1393 \end{bmatrix}.$$

We have also found 4 common positive-definite matrices in 4 MORGs satisfying the condition (C2) of Theorem 2 via the LMI approach as follows:

$$P_1' = P_3' = \begin{bmatrix} 1.4247 & 0.154 \\ 0.154 & 0.2026 \end{bmatrix}, \ P_2' = P_4' = \begin{bmatrix} 1.3503 & 0.1486 \\ 0.1486 & 0.1824 \end{bmatrix}.$$

So we can conclude that the fuzzy system described by (17) with state feedback controllers $u=-K_ix$, i=1, 2, …, 9, is stable by Theorem 2.

We simulate the system (16) with the fuzzy controller using various initial conditions. The results of simulation show that this system is stable under all initial conditions. The system state responses under the initial condition of $x_0=[-1, -1]^T$ are shown in Fig.4.

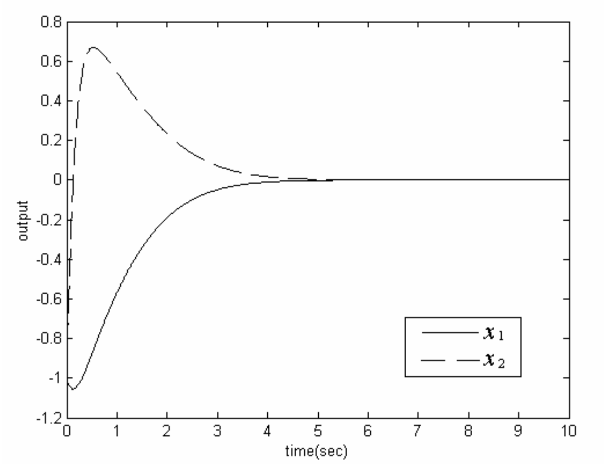

Fig.4 The system state responses under the initial condition of $x_0=[-1, -1]^T$

V. Conclusion

Stability analysis and systematic design are two important issues in analytical design of T-S fuzzy control systems. We summarize the common points of membership functions of most fuzzy control system's input variables, and define the concepts related to a TFP. Based on the properties of a fuzzy control system with TFP inputs, a new sufficient condition to check the stability of a closed-loop T-S fuzzy control system is proposed via the extended Lyapunov theory. This method only requires to find a common positive-definite matrix in each maximal overlapped-rules group, and can greatly reduce the conservatism and difficulty of the former stability analysis approaches. An analytical design approach of a T-S fuzzy controller is proposed and applied to a nonlinear mass-spring-damper mechanical system by using the method of PDC. The simulation result of a mass-spring-damper control system shows that this approach is effective. In practical applications, the fuzzy membership functions of the input variables of most fuzzy control systems satisfy the conditions of TFP. Therefore, the results investigated in this paper can be widely used in other control processes.

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