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Modeling and Control Based on Generalized Fuzzy Hyperbolic Model

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***Abstract*— In this paper, a novel generalized fuzzy hyperbolic model (GFHM) is proposed. First, the definition of the GFHM is proposed and the approximation capability of the GFHM is discussed. The GFHM is proved to be an universal approximator by Stone-Weierstrass theorem. Further, this fuzzy model is used as identifier for nonlinear dynamic systems and the back-propagation training algorithm is given. Finally, the adaptive fuzzy control scheme based on the GFHM is presented, which can guarantee that the closed-loop system is globally asymptotically stable. The simulation results show the applicability of the modeling scheme and the effectiveness of the proposed adaptive control scheme.**

I. INTRODUCTION

For many real systems, which are highly complex and inherently nonlinear, conventional modeling approaches often cannot be applied whereas the fuzzy approach might be a viable alternative. In [1], a fuzzy basis function network is used to approximate an unknown nonlinear function. The disadvantage of this fuzzy model is that some important dynamical behavior of the system cannot be represented. On the other hand, a closed-loop system in the T-S fuzzy model was presented in [2], [3]. The control law must be expressed by a T-S fuzzy model, which is hardly used to other kinds of nonlinear systems. Also, there are some closely related neural-fuzzy control approaches [4], [5], [6].

Recently, Zhang and Quan proposed a new fuzzy hyperbolic model (FHM) in [7]. The fuzzy hyperbolic model is a nonlinear model that is very suitable for expressing nonlinear dynamic properties. However, due to the structural characteristic of the FHM, it cannot approximate some nonlinear continuous functions to arbitrary accuracies, that is, it is not an universal approximator. In this paper, we extend the result of [7] and propose a generalized fuzzy hyperbolic model (GFHM) that can be shown to be an universal approximator. Moreover, we use the GFHM equipped with the back-propagation training algorithm as identifier for nonlinear dynamic systems. Finally, an adaptive control law based the GFHM is designed.

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II. THE GENERALIZED FUZZY HYPERBOLIC MODEL

In this section, we first define generalized fuzzy hyperbolic model (GFHM), and then analyze its approximation capabilities.

A. The Definition of the GFHM

In [7], the membership function of the FHM, P_x and N_x, is defined as:

$$\mu_{P_x}(x) = e^{-\frac{1}{2}(x-k)^2}, \quad \mu_{N_x}(x) = e^{-\frac{1}{2}(x+k)^2} \quad (1)$$

where $k > 0$.We can see that only two fuzzy sets are used to represent the input variable, and that the fuzzy sets cannot cover the whole input space. The model cannot be an universal approximator.

Now, we transform the input variable $\bar{x}$ as follows:

$$x_i = \bar{x} - d_i \quad (2)$$

where $i = 1, \ldots, w$(w is a positive integer), and d_i is a constant. We call the input variables after transformation, $x_i = \bar{x} - d_i (i = 1, \ldots, w)$, generalized input variables. We can see that after the linear transformation of $\bar{x}$, the fuzzy sets may cover the whole input space if w is large enough.

Before defining the GFHM, we first give the definition of fuzzy rule base for inferring the GFHM.

Definition 1: Given a plant with n input variables $\bar{x} = (\bar{x}_1(t), ..., \bar{x}_n(t))^T$, and an output variable y, we call the fuzzy rule base a type of a generalized fuzzy hyperbolic rule base if it satisfies the following conditions:

1) Every fuzzy rule takes the following form:

R^l: IF $(\bar{x}_1 - d_{11})$ is $F_{x_{11}}$ and ... and $(\bar{x}_1 - d_{1w_1})$ is $F_{x_{1w_1}}$ and $(\bar{x}_2 - d_{21})$ is $F_{x_{21}}$ and ... and $(\bar{x}_n - d_{nw_n})$ is $F_{x_{nwn}}$

$$\text{THEN} \quad y^l = c_{F_{11}} + \cdots + c_{F_{1w_1}} + c_{F_{21}} + \cdots + c_{F_{nwn}} \quad (i = 1, \cdots, 2^m) \quad (3)$$

where $w_z (z = 1, \ldots, n)$ are the numbers of transforming $\bar{x}_z$, and $d_{zj} (z = 1, \ldots, n,\ j = 1, \ldots, w_z)$ are the constants where $\bar{x}_z$ is transformed, $F_{x_{zj}}$ are fuzzy sets of $\bar{x}_z - d_{zj}$, which include P_x (Positive) and N_x (Negative) subsets. $c_{F_{zj}}$ are constants corresponding to $F_{x_{zj}}$.

2) The constants $c_{F_{zj}}$ $(z = 1, \ldots, n,\ j = 1, \ldots, w_z)$ in the "THEN" part correspond to $F_{x_{zj}}$ in the "IF" part, that is, if there is $F_{x_{zj}}$ in the "IF" part, $c_{F_{zj}}$ must appear in the "THEN" part. Otherwise, $c_{F_{zj}}$ does not appear in the "THEN" part.

3) There are $s = 2^m$ fuzzy rules in the rule base, where $m = \sum_{i=1}^{n} w_i$, that is, all the possible P_x and N_x combinations of input variables in the "IF" part and all the linear combinations of constants in the "THEN" part.

Theorem 1: For a multi-input single-output system, $y = f(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$, given a generalized fuzzy hyperbolic rule base as in Definition 1, define the generalized input variables as follows:

$$\begin{aligned}
x_1 &= \bar{x}_1 - d_{11} \\
&\cdots \\
x_{w_1} &= \bar{x}_1 - d_{1w_1} \\
&\cdots \\
x_{w_1+1} &= \bar{x}_2 - d_{21} \\
x_{w_1+w_2} &= \bar{x}_2 - d_{2w_2} \\
&\cdots \\
x_m &= \bar{x}_n - d_{nw_n}
\end{aligned}$$

where $m = \sum_{i=1}^{n} w_i$, $w_z(z = 1, \ldots, n)$ are the numbers to be transformed about $\bar{x}_z$, $d_{zj}(z = 1, \ldots, n,\ j = 1, \ldots, w_z)$ are the constants where $\bar{x}_z$ is transformed. If we define the membership function of the generalized input variables P_x and N_x as (1), then we can derive the following model:

$$\begin{aligned}
y &= \sum_{i=1}^{m} \frac{c_{P_i} e^{k_i x_i} + c_{N_i} e^{-k_i x_i}}{e^{k_i x_i} + e^{-k_i x_i}} \\
&= \sum_{i=1}^{m} p_i + \sum_{i=1}^{m} q_i \frac{e^{k_i x_i} - e^{-k_i x_i}}{e^{k_i x_i} + e^{-k_i x_i}} \\
&= P + Q \tanh(K_x x) \qquad (4)
\end{aligned}$$

where $p_i = (c_{P_i} + c_{N_i})/2$, $q_i = (c_{P_i} - c_{N_i})/2$, $P = \sum_{i=1}^{m} p_i$, $Q = [q_1, \ldots, q_m]$ is constant vector, $\tanh(K_x x)$ is defined by $\tanh(K_x x) = \left[\ \tanh(k_1 x_1),\ \cdots,\ \tanh(k_m x_m)\ \right]^T$, $x = [x_1, \ldots, x_m]^T$, $K_x = diag[k_1, \ldots, k_m$, we call (4) the GFHM.

Proof: By applying the product-inference rule, the singleton fuzzifier, and the center of gravity defuzzifier to the generalized fuzzy hyperbolic rule base, we have:

$$y = U/V$$

where

$$\begin{aligned}
U &= (c_{P_1} + c_{P_2} + \cdots + c_{P_m}) \mu_{P_1} \mu_{P_2} \cdots \mu_{P_m} + \cdots \\
&\quad + (c_{N_1} + c_{N_2} + \cdots + c_{N_m}) \mu_{N_1} \cdots \mu_{N_m} \\
V &= \mu_{P_1} \mu_{P_2} \cdots \mu_{P_m} + \mu_{N_1} \mu_{P_2} \cdots \mu_{P_m} + \ldots \\
&\quad + \mu_{N_1} \cdots \mu_{N_m}
\end{aligned}$$

Then,

$$\begin{aligned}
y &= U/V \\
&= \sum_{i=1}^{m} \frac{c_{P_i} \mu_{P_i} + c_{N_i} \mu_{N_i}}{\mu_{P_i} + \mu_{N_i}} \\
&= \sum_{i=1}^{m} \frac{c_{P_i} e^{-\frac{1}{2}(x_i - k_{x_i})^2} + c_{N_i} e^{-\frac{1}{2}(x_i + k_i)^2}}{e^{-\frac{1}{2}(x_i - k_i)^2} + e^{-\frac{1}{2}(x_i + k_i)^2}} \qquad (5)
\end{aligned}$$

From (5), we have

$$\begin{aligned}
y &= \sum_{i=1}^{m} \frac{c_{P_i} e^{-\frac{1}{2}x_i^2 - \frac{1}{2}k_i^2 + k_i x_i} + c_{N_i} e^{-\frac{1}{2}x_i^2 - \frac{1}{2}k_i^2 - k_i x_i}}{e^{-\frac{1}{2}x_i^2 - \frac{1}{2}k_i^2 + k_i x_i} + e^{-\frac{1}{2}x_i^2 - \frac{1}{2}k_i^2 - k_i x_i}} \\
&= \sum_{i=1}^{m} \frac{c_{P_i} e^{k_i x_i} + c_{N_i} e^{-k_i x_i}}{e^{k_i x_i} + e^{-k_i x_i}}
\end{aligned}$$

let $p_i = (c_{P_i} + c_{N_i})/2$, $q_i = (c_{P_i} - c_{N_i})/2$,

$$\begin{aligned}
y &= \sum_{i=1}^{m} p_i + \sum_{i=1}^{m} q_i \frac{e^{k_i x_i} - e^{-k_i x_i}}{e^{k_i x_i} + e^{-k_i x_i}} \\
&= P + Q \tanh(k_x x)
\end{aligned}$$

which is the same as (4)

∎

Remark 1: The differences between the GFHM and the FHM are:

1) The input variables of the GFHM are generalized input variables, which are transformed from original input variables.

2) After the linear transformation of $\bar{x}$, we may choose the number of fuzzy rules arbitrarily until the model approximates a nonlinear function at an arbitrary accuracy.

3) c_{P_i} and c_{N_i} are unnecessary to be opposite number to each other, and we can choose them arbitrarily.

From the above description, we can see that the GFHM is the generalization of the FHM. When we take c_{P_i} opposite to c_{N_i}, we can obtain the FHM.

B. *Approximation Capability of the GFHM*

In the following, we will show that the GFHM can uniformly approximate any nonlinear function over U to any degree of accuracy if U is compact, that is, GFHM is an universal approximator.

Lemma 1 (Stone-Weierstrass theorem): [8] Let Z be a set of real continuous functions on a compact set U. If 1) Z is an algebra, i.e., the set Z is closed under addition, multiplication, and scalar multiplication; 2) Z separates points on U, i.e., for every $x, y \in U, x \neq y$, there exists $f \in Z$ such that $f(x) \neq f(y)$; and 3) Z vanishes at no point on U, i.e., for each $x \in U$ there exists $f \in Z$ such that $f(x) \neq 0$, then the uniform closure of Z consists of all real continuous functions on U; i.e., (Z, d_∞) is dense in $(C[U], d_\infty)$.

Next, we prove that the generalized fuzzy hyperbolic system is an "universal approximator".

Theorem 2: For any given real continuous $g(x)$ on the compact set $U \subset R^n$ and arbitrary $\varepsilon > 0$, there exists $f(x) \in Y$ such that

$$\sup_{x \in U} |g(x) - f(x)| < \varepsilon \tag{6}$$

Proof: First, we prove that (Y, d_∞) is algebra. Let $f_1, f_2 \in Y$. We can write them as

$$f_1(x) = \sum_{i_1=1}^{m_1} \frac{c^1_{P_{i_1}} e^{k^1_{i_1} x_{i_1}} + c^1_{N_{i_1}} e^{-k^1_{i_1} x_{i_1}}}{e^{k^1_{i_1} x_{i_1}} + e^{-k^1_{i_1} x_{i_1}}} \tag{7}$$

$$f_2(x) = \sum_{i_2=1}^{m_2} \frac{c^2_{P_{i_2}} e^{k^2_{i_2} x_{i_2}} + c^2_{N_{i_2}} e^{-k^2_{i_2} x_{i_2}}}{e^{k^2_{i_2} x_{i_2}} + e^{-k^2_{i_2} x_{i_2}}} \tag{8}$$

We have

$$\begin{aligned} & f_1(x) + f_2(x) \\ = & \sum_{i_1=1}^{m_1} \frac{c^1_{P_{i_1}} e^{k^1_{i_1} x_{i_1}} + c^1_{N_{i_1}} e^{-k^1_{i_1} x_{i_1}}}{e^{k^1_{i_1} x_{i_1}} + e^{-k^1_{i_1} x_{i_1}}} \\ & + \sum_{i_2=1}^{m_2} \frac{c^2_{P_{i_2}} e^{k^2_{i_2} x_{i_2}} + c^2_{N_{i_2}} e^{-k^2_{i_2} x_{i_2}}}{e^{k^2_{i_2} x_{i_2}} + e^{-k^2_{i_2} x_{i_2}}} \\ = & \sum_{z=1}^{m_1+m_2} \frac{c_{P_z} e^{k_z x_z} + c_{N_z} e^{-k_z x_z}}{e^{k_z x_z} + e^{-k_z x_z}} \end{aligned} \tag{9}$$

It is easy to see that (9) has the same form as (4), that is, $f_1 + f_2 \in Y$. In the same way, we can get

$$\begin{aligned} & f_1(x) \cdot f_2(x) \\ = & \sum_{i_1=1}^{m_1} \frac{c^1_{P_{i_1}} e^{k^1_{i_1} x_{i_1}} + c^1_{N_{i_1}} e^{-k^1_{i_1} x_{i_1}}}{e^{k^1_{i_1} x_{i_1}} + e^{-k^1_{i_1} x_{i_1}}} \\ & \times \sum_{i_2=1}^{m_2} \frac{c^2_{P_{i_2}} e^{k^2_{i_2} x_{i_2}} + c^2_{N_{i_2}} e^{-k^2_{i_2} x_{i_2}}}{e^{k^2_{i_2} x_{i_2}} + e^{-k^2_{i_2} x_{i_2}}} \\ = & \sum_{i_1, i_2=1}^{m_1, m_2} \Bigg(\frac{c^1_{P_{i_1}} c^2_{P_{i_2}} e^{k^1_{i_1} x_{i_1}} e^{k^2_{i_2} x_{i_2}} + c^1_{P_{i_1}} c^2_{N_{i_2}} e^{k^1_{i_1} x_{i_1}} e^{-k^2_{i_2} x_{i_2}}}{(e^{k^1_{i_1} x_{i_1}} + e^{-k^1_{i_1} x_{i_1}})(e^{k^2_{i_2} x_{i_2}} + e^{-k^2_{i_2} x_{i_2}})} \\ & + \frac{c^1_{N_{i_1}} c^2_{P_{i_2}} e^{-k^1_{i_1} x_{i_1}} e^{k^2_{i_2} x_{i_2}} + c^1_{N_{i_1}} c^2_{N_{i_2}} e^{-k^1_{i_1} x_{i_1}} e^{-k^2_{i_2} x_{i_2}}}{(e^{k^1_{i_1} x_{i_1}} + e^{-k^1_{i_1} x_{i_1}})(e^{k^2_{i_2} x_{i_2}} + e^{-k^2_{i_2} x_{i_2}})} \Bigg) \\ = & \sum_{i_1, i_2=1}^{m_1, m_2} \Bigg(\frac{(c^{1^*}_{P_{i_1}} + c^{2^*}_{P_{i_2}}) e^{k^1_{i_1} x_{i_1}} e^{k^2_{i_2} x_{i_2}} + (c^{1^*}_{P_{i_1}} + c^{2^*}_{N_{i_2}}) e^{k^1_{i_1} x_{i_1}} e^{-k^2_{i_2} x_{i_2}}}{(e^{k^1_{i_1} x_{i_1}} + e^{-k^1_{i_1} x_{i_1}})(e^{k^2_{i_2} x_{i_2}} + e^{-k^2_{i_2} x_{i_2}})} \Bigg) \\ & + \sum_{i_1, i_2=1}^{m_1, m_2} \Bigg(\frac{(c^{1^*}_{N_{i_1}} + c^{2^*}_{P_{i_2}}) e^{-k^1_{i_1} x_{i_1}} e^{k^2_{i_2} x_{i_2}} + (c^{1^*}_{N_{i_1}} + c^{2^*}_{N_{i_2}}) e^{-k^1_{i_1} x_{i_1}} e^{-k^2_{i_2} x_{i_2}}}{(e^{k^1_{i_1} x_{i_1}} + e^{-k^1_{i_1} x_{i_1}})(e^{k^2_{i_2} x_{i_2}} + e^{-k^2_{i_2} x_{i_2}})} \Bigg) \\ = & \sum_{i_1, i_2=1}^{m_1, m_2} \Bigg(\frac{c^{1^*}_{P_{i_1}} e^{k^1_{i_1} x_{i_1}} + c^{1^*}_{N_{i_1}} e^{-k^1_{i_1} x_{i_1}}}{e^{k^1_{i_1} x_{i_1}} + e^{-k^1_{i_1} x_{i_1}}} \\ & + \frac{c^{2^*}_{P_{i_2}} e^{k^2_{i_2} x_{i_2}} + c^{2^*}_{N_{i_2}} e^{-k^2_{i_2} x_{i_2}}}{e^{k^2_{i_2} x_{i_2}} + e^{-k^2_{i_2} x_{i_2}}} \Bigg) \\ = & \sum_{z=1}^{m_1+m_2} \frac{c_{P_z} e^{k_z x_z} + c_{N_z} e^{-k_z x_z}}{e^{k_z x_z} + e^{-k_z x_z}} \end{aligned} \tag{10}$$

where $c^{1^*}_{P_{i_1}}, c^{1^*}_{N_{i_1}}, c^{2^*}_{P_{i_2}}, c^{2^*}_{N_{i_2}}$ satisfy the following equations:

$$\begin{aligned} c^{1^*}_{P_{i_1}} + c^{2^*}_{P_{i_2}} &= c^1_{P_{i_1}} c^2_{P_{i_2}}, \\ c^{1^*}_{P_{i_1}} + c^{2^*}_{N_{i_2}} &= c^1_{P_{i_1}} c^2_{N_{i_2}}, \\ c^{1^*}_{N_{i_1}} + c^{2^*}_{P_{i_2}} &= c^1_{N_{i_1}} c^2_{P_{i_2}}, \\ c^{1^*}_{N_{i_1}} + c^{2^*}_{N_{i_2}} &= c^1_{N_{i_1}} c^2_{N_{i_2}}. \end{aligned} \tag{11}$$

It is easy to see that (10) is also in the same form as (4), hence, $f_1 \cdot f_2 \in Y$.

Finally, for any constant $c \in R$, we have

$$\begin{aligned} cf(x) &= c \sum_{i=1}^{m} \frac{c_{P_i} e^{k_i x_i} + c_{N_i} e^{-k_i x_i}}{e^{k_i x_i} + e^{-k_i x_i}} \\ &= \sum_{i=1}^{m} \frac{c^*_{P_i} e^{k_i x_i} + c^*_{N_i} e^{-k_i x_i}}{e^{k_i x_i} + e^{-k_i x_i}} \end{aligned} \tag{12}$$

which is again in the same form as (4), hence, $cf_1 \in Y$. Therefore, (Y, d_∞) is an algebra.

Next, we prove that (Y, d_∞) separates points on U. We prove this by constructing a required $f \in Y$ such that $f(x^\circ) \neq f(y^\circ)$ for arbitrarily given $x^\circ, y^\circ \in U$ with $x^\circ \neq y^\circ$. Let $x^\circ = (x^\circ_1, x^\circ_2, \cdots, x^\circ_n)^T$, $y^\circ = (y^\circ_1, y^\circ_2, \cdots, y^\circ_n)^T$. If $x^\circ_i \neq y^\circ_i$, choose input variable as

$$x^*_i = x_i - \frac{x^\circ_i + y^\circ_i}{2} \tag{13}$$

$$k^*_i = \frac{x^\circ_i - y^\circ_i}{2} \tag{14}$$

That is, $x^*_i - k^*_i = x_i - x^\circ$, $x^*_i + k^*_i = x_i - y^\circ$. Then, from (5) we can get

$$\begin{aligned} f(x^\circ) &= \sum_{i=1}^{n} \frac{c_{P_i} e^{-\frac{1}{2}(x^\circ_i - x^\circ_i)^2} + c_{N_i} e^{-\frac{1}{2}(x^\circ_i - y^\circ_i)^2}}{e^{-\frac{1}{2}(x^\circ_i - x^\circ_i)^2} + e^{-\frac{1}{2}(x^\circ_i - y^\circ_i)^2}} \\ &= \sum_{i=1}^{n} \frac{c_{P_i} + c_{N_i} e^{-\frac{1}{2}(x^\circ_i - y^\circ_i)^2}}{1 + e^{-\frac{1}{2}(x^\circ_i - y^\circ_i)^2}} \end{aligned} \tag{15}$$

$$\begin{aligned} f(y^\circ) &= \sum_{i=1}^{n} \frac{c_{P_i} e^{-\frac{1}{2}(y^\circ_i - x^\circ_i)^2} + c_{N_i} e^{-\frac{1}{2}(y^\circ_i - y^\circ_i)^2}}{e^{-\frac{1}{2}(y^\circ_i - x^\circ_i)^2} + e^{-\frac{1}{2}(y^\circ_i - y^\circ_i)^2}} \\ &= \sum_{i=1}^{n} \frac{c_{P_i} e^{-\frac{1}{2}(y^\circ_i - x^\circ_i)^2} + c_{N_i}}{e^{-\frac{1}{2}(y^\circ_i - x^\circ_i)^2} + 1} \end{aligned} \tag{16}$$

Let $c_{P_i=1}, c_{N_i} = 0$. We can derive

$$\begin{aligned} & f(x^\circ) - f(y^\circ) \\ &= \sum_{i=1}^{n} \frac{1}{1 + e^{-\frac{1}{2}(x^\circ_i - y^\circ_i)^2}} - \sum_{i=1}^{n} \frac{e^{-\frac{1}{2}(x^\circ_i - y^\circ_i)^2}}{1 + e^{-\frac{1}{2}(x^\circ_i - y^\circ_i)^2}} \\ &= \frac{1 - \prod_{i=1}^{n} e^{-\frac{1}{2}(x^\circ_i - y^\circ_i)^2}}{1 + \prod_{i=1}^{n} e^{-\frac{1}{2}(x^\circ_i - y^\circ_i)^2}} \end{aligned} \tag{17}$$

Since $x^\circ \neq y^\circ$, there must exist some i such that $x_i^\circ \neq y_i^\circ$. Hence, we have $\prod_{i=1}^{n} e^{-\frac{1}{2}(x_i^\circ - y_i^\circ)^2} \neq 1$, thus, $f(x^\circ) \neq f(y^\circ)$. Therefore, (Y, d_∞) separates points on U.

Finally, we prove that (Y, d_∞) vanishes at no points of U. By observing (1) and (4), we simply choose all $c_{P_i} > 0$, $c_{N_i} > 0 (i = 1, \ldots, m)$; that is, any $f \in Y$ with $c_{P_i} > 0$, $c_{N_i} > 0$; serves as the required f.

From(4), it is obvious that Y is a set of real continuous functions on U. The universal approximation theorem is therefore a direct consequence of the Stone-Weierstrass Theorem.

■

Remark 2: There are some distinguishing characteristics in the GFHM:

1) The GFHM is nonlinear model. Unlike the T-S fuzzy model, which is the combination of local linear models, the GFHM is a global nonlinear model.

2) The GFHM is a fuzzy model that can easily be derived from known linguistic information.

3) Just like the T-S fuzzy model, we can design a neural network model to identify the model parameters.

III. FUZZY NONLINEAR IDENTIFIER

Since the GFHM is both universal approximator and the neural network model, we can use the GFHM equipped with the back-propagation training algorithm as identifier for unknown nonlinear function.

Our task is to determine a GFHM $\hat{y}(k+1)$ in the form of (4) such that

$$J = \frac{1}{2}[y(k+1) - \hat{y}(k+1)]^2 \quad (18)$$

is minimized. BP training algorithm of the GFHM is as follows:

$$\begin{aligned} P(k+1) &= P(k) - \eta \frac{\partial J}{\partial \hat{y}} \cdot \frac{\partial \hat{y}(k+1)}{\partial P(k)} \\ &= P(k) + \eta [y(k+1) - \hat{y}(k+1)] \quad (19) \end{aligned}$$

$$\begin{aligned} q_j(k+1) &= q_j(k) - \eta \frac{\partial J}{\partial \hat{y}} \cdot \frac{\partial \hat{y}(k+1)}{\partial q_j(k)} \\ &= q_j(k) + \eta [y(k+1) - \hat{y}(k+1)] \cdot \\ &\quad \tanh[k_j(k)(x_z - d_j(k))] \\ &\quad (j = 1, \cdots, m) \quad (20) \end{aligned}$$

$$\begin{aligned} k_j(k+1) &= k_j(k) - \eta \frac{\partial J}{\partial \hat{y}} \cdot \frac{\partial \hat{y}(k+1)}{\partial k_j(k)} \\ &= k_j(k) + \eta [y(k+1) - \hat{y}(k+1)] q_j(k) \cdot \\ &\quad [x_z - d_j(k)] \sec\mathrm{h}^2 [k_j(k)(x_z - d_j(k))] \quad (21) \end{aligned}$$

$$\begin{aligned} d_j(k+1) &= d_j(k) - \eta \frac{\partial J}{\partial \hat{y}} \cdot \frac{\partial \hat{y}(k+1)}{\partial d_j(k)} \\ &= d_j(k) - \eta [y(k+1) - \hat{y}(k+1)] q_j(k) \cdot \\ &\quad k_j(k) \sec\mathrm{h}^2 [k_j(k)(x_z - d_j(k))] \quad (22) \end{aligned}$$

where η is the rate of training.

Remark 3: In some sense, the identification complexity of the GFHM is much less than of the T-S fuzzy model. When the number of input variables and fuzzy subsets are n and m_i, respectively, the number of the unknown parameters of the T-S fuzzy model is $\sum_{i=1}^{n} 2m_i + \prod_{i=1}^{n} m_i(n+1)$, while the number of the unknown parameters of the GFHM is $1 + (3/2)\sum_{i=1}^{n} m_i$. It is clear that as the number of input variable or (and) fuzzy subsets increases, the increase of the number of the unknown parameters of the T-S fuzzy model is geometric, while that of the GFHM is linear. Therefore, when the number of input variables and fuzzy subsets in the GFHM are equal to those of in the T-S model, the degree of approximate accuracy of the GFHM is higher than that of the T-S fuzzy model.

IV. CONTROL SCHEME

Having derived the GFHM for the plant, the next step is to design a controller that achieves some control objective.

Consider the discrete nonlinear system:

$$\begin{aligned} y(k+1) &= f(y(k), \cdots, y(k-n+1)) + CU(k) \\ &= f(\bar{X}(k)) + CU(k) \quad (23) \end{aligned}$$

where $f(\bullet)$ is an unknown nonlinear function, $\bar{X}(k) = [y(k), \cdots, y(k-n+1)]^T$, and $u(k) \in R, y(k) \in R$ are the input and output variable of the system, respectively, $U(k) = [u(k), \cdots, u(k-n+1)]^T$, $C = [c_0, c_1, \cdots, c_{m-1}]$, $c_0 \neq 0$, $c_i \in R (i = 0, 1, \cdots, m-1)$. The aim of control is to determine a controller $u(k)$ such that the output $y(k+1)$ of the closed-loop system follows the output $y_m(k+1)$ of the following reference model:

$$y_m(k+1) = A\bar{X}_m(k) + br(k) \quad (24)$$

where $A = [a_0, \ldots, a_{n-1}], a_i \in R \quad (i = 0, 1, \ldots, n-1), b \in R, \bar{X}_m(k) = [y_m(k), \ldots, y_m(k-n+1)]^T, r(k) \in R$ is the input of reference model. That is, we want $e(k+1) = y_m(k+1) - y(k+1)$ to converge to zero as k goes to infinity.

Since $f(\bullet)$ is unknown, the controller cannot easily be implemented. To solve this problem, we identify it by the GFHM. That is, we replace the $f(\bullet)$ by $\hat{f}(\bullet)$ that is in the form of (4). $\Delta f(k) = \hat{f}(\bar{X}(k)) - f(\bar{X}(k))$ is defined as the identification error.

Assumption 1: 1) There exist positive constants M_f, M_e such that $|\Delta f(k)| \leq M_f$, $|e(k)| \leq M_e$; 2) There exists a positive constant $M_{\max}$ such that $M_{\max} = M_f/M_e < 1$.

Theorem 3: Consider the plant described by the difference equation (23), assumption 1 is satisfied, if the adaptive controller is chosen as:

$$\begin{aligned} u(k) &= \frac{1}{c_0} \Big[-\hat{f}(\bar{X}(k)) + A\bar{X}_m(k) + br(k) \\ &\quad -\tilde{C}\tilde{U}(k) + \rho e(k) \Big] \quad (25) \end{aligned}$$

where $\tilde{C} = [c_1, \cdots, c_{m-1}]$, $\tilde{U}(k) = [u(k-1), \cdots, u(k-m+1)]^T$, ρ is feedback gain which is chosen to satisfy

$M_{\max}-1<\rho<-M_{\max}+1$, then the closed-loop system is globally asymptotically stable.

Proof: From (23) and (25), we can get

$$\begin{aligned} y(k+1) &= f(\bar{X}(k)) + c_0 u(k) + \tilde{C}\tilde{U}(k) \\ &= f(\bar{X}(k)) - \hat{f}(\bar{X}(k)) + A\bar{X}_m(k) + br(k) + \rho e(k) \end{aligned} \tag{26}$$

Combining (24) and (26), we have

$$\begin{aligned} e(k+1) &= \hat{f}(\bar{X}(k)) - f(\bar{X}(k)) - \rho e(k) \\ &= \Delta f(k) - \rho e(k) \end{aligned} \tag{27}$$

We choose the Lyapunov function candidate

$$V(k) = \frac{1}{2}e^2(k) \tag{28}$$

then

$$\begin{aligned} \Delta V(k) &= \frac{1}{2}e^2(k+1) - \frac{1}{2}e^2(k) \\ &= \frac{1}{2}[\Delta f^2(k) - 2\rho\Delta f(k)e(k) + \rho^2 e^2(k) - e^2(k)] \\ &= \frac{1}{2}e^2(k)[\rho^2 - 1 - 2\rho\frac{\Delta f(k)}{e(k)} + \frac{\Delta f^2(k)}{e^2(k)}] \end{aligned}$$

Define $M = \Delta f(k)/e(k)$, then

$$\begin{aligned} \Delta V(k) &= \frac{1}{2}e^2(k)(\rho^2 - 2\rho M + M^2 - 1) \\ &= \frac{1}{2}e^2(k)[\rho - (M-1)][\rho - (M+1)] \end{aligned} \tag{29}$$

From Assumption 1, we have

$$\max(M) = M_{\max} < 1,$$
$$\min(M) = M_{\min} = -M_{\max}$$

Let $M_{\max} - 1 < \rho < -M_{\max} + 1$, we get $(\rho - M + 1)(\rho - M - 1) < 0$, then $\Delta V(k) < 0$ holds. By Lyapunov's stability theorem of discrete-time systems, the closed-loop system is globally asymptotically stable. ■

V. SIMULATION RESULTS

Consider the discrete nonlinear system:

$$y(k+1) = f[y(k), y(k-1)] + CU(k) \tag{30}$$

where $C = [\ 1 \quad 0.8\ ]$, $U(k) = [\ u(k) \quad u(k-1)\ ]^T$,

$$f[y(k), y(k-1)] = \frac{y(k)y(k-1)[y(k)+2.5]}{1+y^2(k)+y^2(k-1)}$$

is an unknown nonlinear function. The aim of control is to determine a controller $u(k)$ such that the output $y(k+1)$ of the closed-loop system follows the reference model output $y_m(k+1)$ given by:

$$\begin{aligned} y_m(k+1) &= 0.6y_m(k) + 0.2y_m(k-1) + r(k) \\ &= A\bar{X}_m(k) + br(k) \end{aligned}$$

where $A = [\ 0.6 \quad 0.2\ ]$, $b = 1$, $\bar{X}_m(k) = [\ y_m(k) \quad y_m(k-1)\ ]^T$, $r(k) = \sin(2\pi k/25)$.

To apply Theorem 3 to this system, we design adaptive controller

$$u(k) = -\hat{f}(\bar{X}(k)) + A\bar{X}_m(k) + br(k) - \tilde{C}\tilde{U}(k) + \rho e(k) \tag{31}$$

where $\tilde{C} = [0.8]$, $\tilde{U}(k) = [u(k-1)]$, $\rho = 0.25$, the output of identifier is as follows

$$\begin{aligned} \hat{y}(k+1) &= P + Q\tanh(k_x x) \\ &= P + Q\tanh[k_x(\bar{x}_z - d)] \end{aligned} \tag{32}$$

After off-line identification, we can derive the following initial parameters of the GFHM:

$$P = 1.7966,$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y(k) - d_1 \\ y(k) - d_2 \\ y(k-1) - d_3 \end{bmatrix},$$

$$d = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} -0.0135 \\ 1.3630 \\ -0.6249 \end{bmatrix},$$

$$Q = [\ q_1 \quad q_2 \quad q_3\ ] = [\ -0.8234 \quad 2.8879 \quad 0.9175\ ],$$

$$k_x = diag\,[k_1, k_2, k_3] = diag\,[-0.5893, 0.2256, -1.9538],$$

$\eta = 0.1$.We chose the above initial parameters of the GFHM and the adaptive controller in the form of (31), and the result of simulation is shown in Fig.1 and Fig.2. We see that the control was almost perfect.

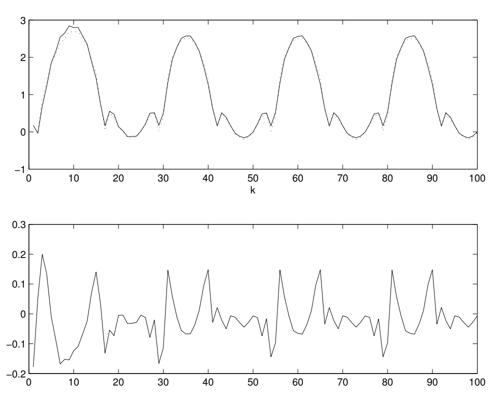

Fig. 1. The nonlinear function $f(\bar{x}(k))$ (solid line) and the result of identifier $\hat{f}(\bar{x}(k))$ (dotted line) and the identification error $\Delta f(k)$

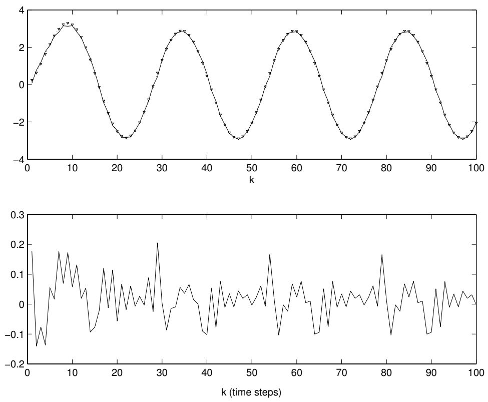

Fig. 2. The output $y(k)$ (solid line) of the close-loop system and the reference trajectory $y_m(k)$ (triangle-downward line) and trajectory error e(k)

VI. CONCLUSIONS

In this paper, a new generalized fuzzy hyperbolic model (GFHM) is proposed. First, the GFHM is proved to be an universal approximator, and can be as an identifier for nonlinear dynamic systems. We compare between the numbers of the unknown parameters of the T-S fuzzy model and that of the GFHM. As the number of input variables or (and) fuzzy subsets increases, the increase of the number of the unknown parameters of the T-S fuzzy model is geometric, while that of the GFHM is linear. The back-propagation training algorithm of the GFHM is given. Finally, the adaptive fuzzy controller is presented, which can guarantee that the closed-loop system is globally asymptotically stable.

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