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# Coordination of a Group of Agents with a Leader—Part I: General Case

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*Abstract*— **This paper studies coordinated control of a group of agents with a leader based on a coupled phase oscillator model. The leader is unaffected by the other agent members while each member is influenced by the leader and the other members with the same coupling strengths. By using coupled oscillators theory, it is shown that the group dynamics depends on the motion of the leader and the coupling strengths among all agents. Two types of collective motions occur generally, depending on different ranges of the coupling strengths. One is that all the member agents will move in the same direction while the other is that all the member agents move in a way such that the centroid of the group approaches a fixed position. In each case, all the member agents eventually move in the same manner if the directions of the motion are neglected. We also present the analytical results for two special cases of weak and strong couplings. Numerical simulations are worked out to demonstrate the theoretical analysis. The results suggest potential approaches to control a group motion by steering the motion of the leader and adjusting appropriate coupling patterns. This is of practical interest in applications of multiagent systems.**

## I. INTRODUCTION

In recent years, collective motion and self-organized behavior of swarms, agents, and particles have become a major objective in many fields such as ecology and theoretical biology [1], [2], physics [3]–[7], and control engineering [8]–[23]. The studies focus on understanding the general mechanisms and operational principles of such coordinated cooperative phenomena as well as their potential applications in certain engineering problems such as control of multi-robots, traffic flows. An interesting issue in this direction is the collective behavior of swarms or agent groups with leaders. For example, in [10], moving reference points were viewed as virtual leaders used to manipulate the geometry of autonomous vehicle group and direct the motion of group. The cohesion of the members of swarms following an edge-leader was analyzed in [11], [12]. Ref. [18] also investigated leaderless/leader coordination of mobile autonomous agents using nearest neighbor rules.

In a recent paper [21], the collective motion of a self-propelled particle group has been analyzed by viewing the particle group as a coupled phase oscillator system. It was shown that under sufficiently large coupling strength, the particles either move in parallel or maintain the centroid of the particle group motionless eventually. It also presented formation control design. However, the model of [21] did not consider the role of a leader in the particle group. Recently, networks of coupled identical nonlinear oscillators were analyzed using nonlinear contraction theory in [5]. Collective dynamics of a particle group with an independent leader was investigated using coupled oscillators theory in [6], in which each member of the particle group is coupled with the leader and the other members with different coupling strengths.

In this paper, we investigate the dynamics of an agent group with an independent leader. In the agent group, each member is coupled with the leader and the other members with the same coupled strengths. We show that the leader's dynamics will significantly influence the collective motion of agent group. Particularly, in some coupling strength ranges, the evolution of the agent members will eventually be the same as that of the leader as time elapsing when the motion directions are disregarded. We also discuss the weak/strong coupling cases using a time-scale separation technique. Our results suggest the possibility of controlling the agent group by steering the motion of the leader. This is of practical interest in applications such as control of multi-robots or autonomous vehicles.

The paper is organized as follows. Section II presents a coupled oscillator model of the agent group. Collective dynamics is analyzed in Section III. Section IV gives numerical simulations of the theoretical results. Some conclusions are drawn in Section V.

*Notations:* $\mathbb{R}$ represents the real number set. The formula $\theta_0 \to \theta_0^*(+)$ means $\theta_0 > \theta_0^*$ and $\theta_0 \to \theta_0^*$ and the formula $\theta_0 \to \theta_0^*(-)$ denotes $\theta_0 < \theta_0^*$ and $\theta_0 \to \theta_0^*$.

## II. AGENT MODEL

We consider a group of $N+1$ identical agents (of unit mass) moving in the plane at unit speed, in which an agent indexed by 0 is assigned as the "leader" and the other agents indexed by 1–$N$ are referred to as "members". The leader is unaffected by the members while each member is influenced by the leader and the other members. A continuous-time kinematic model of the $N+1$ agents is described as follows.

$$\begin{aligned} &\dot{r}_0 = e^{i\theta_0}, \\ &\dot{\theta}_0 = u, \\ &\dot{r}_k = e^{i\theta_k}, \\ &\dot{\theta}_k = \tfrac{K}{N+1}\textstyle\sum_{j=0}^{N} \sin(\theta_j - \theta_k), \quad 1 \le k \le N. \end{aligned} \tag{1}$$

In complex notation, the vector $r_k \in \mathbb{C} \approx \mathbb{R}^2$ (the punctured complex plane) denotes the position of agent $k$ and the angle $\theta_k$ denotes the direction of its (unit) velocity vector

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$e^{i\theta_k} = \cos\theta_k + i\sin\theta_k, 0 \le k \le N$. The term $u$ is control input applied to regulate the direction of the leader. Here we take

$$u = F(\theta_0)$$

with $F(\theta_0)$ a smooth function defined in an interval $I \subseteq \mathbb{R}$. When $F(\theta_0)$ is not identical with 0, we assume that all the equilibrium points of $\dot{\theta}_0 = F(\theta_0)$ are hyperbolic. $K \neq 0$ is a real parameter and $|K|$ quantifies the coupling strength among the agents. Throughout the paper, we assume that the motion of the leader does not have finite escape time.

Since $\theta_0$ is independent of all $\theta_k$ while each $\theta_k$ depends on $\theta_0$ for $k = 1,...,N$, the dynamics of $\theta_k$ are in general more complex than that concerned in [21]. In this paper, we will analyze the dynamics of the model (1) by using coupled oscillators theory [24].

## III. DYNAMICS ANALYSIS

Observe that in model (1), the right side of the equations does not depend on the position variable $r$ (here we omit the index) and $r$ are dependent on $\theta$. Hence, we only need to analyze the dynamics of $\theta$ in the sequel and then examine the dynamics of $r$ when necessary. For the sake of convenience, denote $\bar{\theta}_k = \theta_k - \theta_0, k = 0,1,...,N$. Then for $k = 1,...,N$, the dynamics of $\bar{\theta}_k$ are described as follows.

$$\dot{\bar{\theta}}_k = \frac{K}{N+1}\left[\left(\sum_{j=1}^{N}\sin\bar{\theta}_j\right)\cos\bar{\theta}_k - \left(1+\sum_{j=1}^{N}\cos\bar{\theta}_j\right)\sin\bar{\theta}_k\right] - F(\theta_0). \tag{2}$$

Notice that (2) is similar to the general reducible phase oscillator model in [24], where the authors presented a reduction process and analyzed the dynamics of the model using Lyapunov second method. Their method and results have also been used in [21]. This paper employs similar arguments to that of [24] to analyze the dynamics of $\bar{\theta}_k$. We will make use of the following notations:

$$\begin{aligned} \bar{g} &= \frac{1}{N}\sum_{j=1}^{N}\cos\bar{\theta}_j, \quad \bar{h} = \frac{1}{N}\sum_{j=1}^{N}\sin\bar{\theta}_j, \\ p_{\bar{\theta}} &= \frac{1}{N}\sum_{j=1}^{N}e^{i\bar{\theta}_j} = \bar{g} + i\bar{h}, \\ P_{\bar{\theta}} &= \frac{1}{N+1}\sum_{j=0}^{N}e^{i\bar{\theta}_j} = \frac{1}{N+1}\left(1+N\bar{g}+iN\bar{h}\right), \\ R &= \frac{1}{N+1}\sum_{k=0}^{N}r_k. \end{aligned}$$

Clearly, $R$ is the position vector of the centroid of the group and its derivative $\dot{R} = P_{\bar{\theta}}e^{i\theta_0}$ is the moment of the group.

To analyze the dynamics of $\bar{\theta}_k$, we first need to examine the dynamics of $\theta_0$.

### A. Dynamics of $\theta_0$

For convenience, we rewrite the equation of $\theta_0$ as

$$\dot{\theta}_0 = F(\theta_0). \tag{3}$$

We consider the following two cases.

If $F(\theta_0) \equiv 0$, then $\theta_0$ is a constant. So the leader keeps moving along a straight line with a constant slope.

If $F(\theta_0) \not\equiv 0$, then by assumption, all the equilibrium points of $\dot{\theta}_0 = F(\theta_0)$ are hyperbolic. Hence, for any initial value $\theta_0(t_0)$ in $I$, the solution $\theta_0$ has one of the following properties, depending on properties of $F(\theta_0)$ and $\theta_0(t_0)$:

*P1. $\theta_0$ monotonically tends to an equilibrium point of (3);*

*P2. $\theta_0$ monotonically increases and approaches the boundary of I;*

*P3. $\theta_0$ monotonically decreases and approaches the boundary of I.*

When the sign of $F(\theta_0)$ does not change in $I$, either P2 or P3 occurs. Otherwise, there exists at least one equilibrium point of (3) in $I$. In this case, according to the assumption on $F(\theta_0)$, the equilibrium point is either stable or unstable. This implies that as $t \to +\infty$, $\theta_0$ either monotonically tends to the equilibrium point or monotonically approaches the boundary of $I$. We summarize these properties in the following four cases, where $\theta_0^*$ is an equilibrium point of (3):

- $\theta_0 \in P1(+)$ denotes that P1 occurs and $\theta_0 \to \theta_0^*(+)$ as $t \to +\infty$.
- $\theta_0 \in P1(-)$ denotes that P1 occurs and $\theta_0 \to \theta_0^*(-)$ as $t \to +\infty$.
- $\theta_0 \in P2$ denotes that P2 occurs.
- $\theta_0 \in P3$ denotes that P3 occurs.

From this, we have the following Lemma.

**Lemma 1.** For any initial value $\theta_0(t_0) \in I$, the sign of $F(\theta_0)$ is eventually constant as $t \to +\infty$. Specifically, one has $F(\theta_0) > 0$ for $\theta_0 \in P1(-)$ or $\theta_0 \in P2$ and $F(\theta_0) < 0$ for $\theta_0 \in P1(+)$ or $\theta_0 \in P3$.

Next, we will analyze the dynamics of $\bar{\theta}_k$ based on Lemma 1. Observe that (2) includes a term $-K\sin\bar{\theta}_k/(N+1) - F(\theta_0)$ in its right-hand side, which comes from the influence of the leader on every agent member, so the governing equations of $\bar{\theta}_k$ are different from what considered in [21] and [24].

### B. Change of Variables

From the definition of $p_{\bar{\theta}}$, we have

$$\sum_{j=1}^{N}\cos\bar{\theta}_j = N\bar{g}, \quad \sum_{j=1}^{N}\sin\bar{\theta}_j = N\bar{h}.$$

Inserting them into (2), we get

$$\dot{\bar{\theta}}_k = \frac{NK\bar{h}}{N+1}\cos\bar{\theta}_k - \frac{(1+N\bar{g})K}{N+1}\sin\bar{\theta}_k - F(\theta_0) \tag{4}$$

with $k = 1,...,N$. Take the change of variables [24]

$$\tan\left[\frac{1}{2}(\bar{\theta}_k - \Theta)\right] = \sqrt{\frac{1+\gamma}{1-\gamma}}\tan\left[\frac{1}{2}(\psi_k - \Psi)\right] \tag{5}$$

for $k = 1,...,N$, which map $N$-dimensional state vector $(\bar{\theta}_1, \bar{\theta}_2, ..., \bar{\theta}_N)$ to the $(N+3)$-dimensional state vector

$(\gamma, \Theta, \Psi, \psi_1, ..., \psi_N)$. In (5), $\Theta$ and $\Psi$ are two rigid rotation variables, $\gamma$ is a dilation with $0 \le \gamma < 1$, and $\psi_k, k = 1, ..., N$, are new phase variables. All these variables depend on time. Equation (5) is regarded as a way to redistribute the phase variables $\bar{\theta}_k$ on unit circle, for more detailed implication about the transformation we refer to [24].

From (5) and trigonometric identities, one can get two useful formulas:

$$\sin(\bar{\theta}_k - \Theta) = \frac{\sqrt{1-\gamma^2}\sin(\psi_k - \Psi)}{1 - \gamma\cos(\psi_k - \Psi)}, \tag{6}$$

$$\cos(\bar{\theta}_k - \Theta) = \frac{\cos(\psi_k - \Psi) - \gamma}{1 - \gamma\cos(\psi_k - \Psi)}. \tag{7}$$

Differentiating (5) with respect to time and considering (4), (6), and (7), we can obtain for $1 \le k \le N$,

$$\begin{aligned}
&-\sqrt{1-\gamma^2}\dot{\psi}_k \\
&+\left[\frac{K}{N+1}(P - \sin\Theta) + \gamma(F(\theta_0) + \dot{\Theta})\right]\cos(\psi_k - \Psi) \\
&-\left[\frac{K}{N+1}(Q + \cos\Theta)\sqrt{1-\gamma^2} + \frac{\dot{\gamma}}{\sqrt{1-\gamma^2}}\right]\sin(\psi_k - \Psi) \\
&+\left[-\frac{K}{N+1}\gamma(P - \sin\Theta) - F(\theta_0) - \dot{\Theta} + \sqrt{1-\gamma^2}\dot{\Psi}\right] \\
&= 0
\end{aligned} \tag{8}$$

with

$$\begin{aligned}
P &= N(\bar{h}\cos\Theta - \bar{g}\sin\Theta) \\
&= \sum_{j=1}^{N} \frac{\sqrt{1-\gamma^2}\sin(\psi_j - \Psi)}{1 - \gamma\cos(\psi_j - \Psi)}, \\
Q &= N(\bar{h}\sin\Theta + \bar{g}\cos\Theta) \\
&= \sum_{j=1}^{N} \frac{\cos(\psi_j - \Psi) - \gamma}{1 - \gamma\cos(\psi_j - \Psi)}.
\end{aligned}$$

If the three variables $(\gamma, \Theta, \Psi)$ satisfy following differential equations

$$\begin{aligned}
\dot{\gamma} &= -(1-\gamma^2)\frac{K}{N+1}(Q + \cos\Theta), \\
\gamma\dot{\Psi} &= -\sqrt{1-\gamma^2}\frac{K}{N+1}(P - \sin\Theta), \\
\gamma\dot{\Theta} &= -\frac{K}{N+1}(P - \sin\Theta) - \gamma F(\theta_0),
\end{aligned} \tag{9}$$

for $\gamma \ne 0$ and

$$\begin{aligned}
0 &= \sum_{j=1}^{N}\sin(\psi_j - \Psi) - \sin\Theta, \\
\dot{\gamma} &= -\frac{K}{N+1}\left[\sum_{j=1}^{N}\cos(\psi_j - \Psi) + \cos\Theta\right], \\
0 &= \dot{\Psi} - \dot{\theta}_0 - \dot{\Theta},
\end{aligned} \tag{10}$$

for $\gamma = 0$, where $\theta_0$ satisfies (3), then the identities (8) reduce to

$$\dot{\psi}_k = 0, \quad k = 1, ..., N. \tag{11}$$

Thus the dynamics analysis of $\bar{\theta}_k$ are converted into analyzing the dynamics of $\gamma, \Theta, \Psi$, and $\psi_k$ determined by (3) and (9)–(11).

### *C. Constraints Imposed on $\psi_k$*

Equation (11) means that the $N$ new phase variables $\psi_k$ are frozen. Hence, $\psi_k$ can be considered as parameters. Under the coordinate transformation (5), the original $N$ variables are converted into 3 new variables and $N$ parameters. Three constraints can be imposed on the initial values of $\gamma, \Theta, \Psi$, and $\psi_k$, $k = 1, ..., N$. So we impose the constraints

$$\sum_{k=1}^{N}\cos\psi_k = 1, \quad \sum_{k=1}^{N}\sin\psi_k = 0 \tag{12}$$

on the initial values of $\psi_k$ and hence on $\psi_k$ for $k = 1, ..., N$. For $\gamma = 0$, under constraints (12), one has

$$\sin\Psi + \sin\Theta = 0. \tag{13}$$

From this we further have

$$\cos\Psi + \cos\Theta = 0 \quad \text{or} \quad \cos\Psi - \cos\Theta = 0.$$

And we impose a constraint on the initial values of $\Theta$ and $\Psi$ such that

$$\cos\Psi + \cos\Theta = 0 \quad \text{for} \quad \gamma = 0. \tag{14}$$

Otherwise the last two equations in (9) are singular. Thus, we have $\dot{\gamma} = 0$ for $\gamma = 0$, which means that $\gamma = 0$ is an equilibrium point of $\gamma$.

### *D. V-Function and Dynamics Analysis*

Observe that $\Theta$ is coupled with $\Psi, \gamma$, and $\theta_0$, we choose a $V$-function as follows.

$$\begin{aligned}
V = &(N+1)\left[\sum_{k=1}^{N}\ln\frac{1-\gamma\cos(\psi_k - \Psi)}{\sqrt{1-\gamma^2}} - \gamma(1+\cos\Theta)\right] \\
&-(N+1)(\ln(1-\gamma) - v),
\end{aligned} \tag{15}$$

where if $F(\theta_0) \equiv 0$, then $v = 0$; otherwise, for $K > 0$,

$$v = \begin{cases} 2\theta_0 & \text{if } \theta_0 \in P2, \\ -2\theta_0 & \text{if } \theta_0 \in P3, \\ (\theta_0 - \theta_0^* - 2)^2/2 & \text{if } \theta_0 \in P1(+), \\ (\theta_0 - \theta_0^* + 2)^2/2 & \text{if } \theta_0 \in P1(-), \end{cases} \tag{16}$$

and for $K < 0$,

$$v = \begin{cases} -2\theta_0 & \text{if } \theta_0 \in P2, \\ 2\theta_0 & \text{if } \theta_0 \in P3, \\ (\theta_0 - \theta_0^* + 2)^2/2 & \text{if } \theta_0 \in P1(+), \\ (\theta_0 - \theta_0^* - 2)^2/2 & \text{if } \theta_0 \in P1(-), \end{cases} \tag{17}$$

with $\theta_0$ being the solution of (3) and $\theta_0^*$ an equilibrium point of (3).

Below we analyze the dynamics of (3) and (9) using the derivatives of the $V$-function with respect to time and $\gamma$. The derivative of the $V$-function with respect to time along the solutions of (3) and (9) can be reduced to

$$\dot{V} = KG_1G_2 + KG_3 + (N+1)G_4 \tag{18}$$

with

$$
\begin{aligned}
G_1 &= \sum_{k=1}^{N} \frac{\gamma - \cos(\psi_k - \Psi)}{1-\gamma\cos(\psi_k-\Psi)} \\
&\quad -(1-\gamma^2)(1+\cos\Theta)+1+\gamma, \\
G_2 &= \sum_{k=1}^{N} \frac{\gamma - \cos(\psi_k - \Psi)}{1-\gamma\cos(\psi_k-\Psi)} - \cos\Theta, \\
G_3 &= \left[\sum_{k=1}^{N} \frac{\sqrt{1-\gamma^2}\sin(\psi_k-\Psi)}{1-\gamma\cos(\psi_k-\Psi)} - \sin\Theta\right]^2, \\
G_4 &= \dot{v} - \gamma\sin\Theta F(\theta_0).
\end{aligned}
$$

Obviously, $G_3 \geq 0$ for all $(\gamma, \Theta, \Psi, \theta_0)$ and $G_4 = 0$ if $F(\theta_0) \equiv 0$. We assert that, for all $(\gamma, \Theta, \Psi, \theta_0)$,

*(i)* $G_1 \geq 0, G_2 \geq 0,$ *and* $G_1 = 0$ *if and only if* $\gamma = 0$*;*

*(ii) if* $F(\theta_0) \not\equiv 0$, *then* $G_4 > 0$ (*resp.,* $G_4 < 0$) *for* $K > 0$ (*resp.,* $K < 0$) *as* $t$ *is large enough.*

The proof of the assertion is presented in Appendix. From this, $K\dot{V} > 0$ for all $(\gamma, \Theta, \Psi, \theta_0)$ as $t$ is large enough. This implies that $V$ eventually increases along the solutions of (3) and (9) for $K > 0$ and decreases for $K < 0$. On the other hand, we note that

$$
\frac{\partial V}{\partial \gamma} = \frac{(N+1)G_1}{1-\gamma^2} > 0
$$

for $0 < \gamma < 1$. This implies that $V$ strictly increases with $\gamma$ and vice versa. Thus, along the solutions of (3) and (9),

- if $K > 0$, then $\gamma \to 1$;
- if $K < 0$, then $\gamma \to 0$.

Hence, the dynamics of (9) depend on the sign of $K$. We consider two cases in the following.

For $K > 0$, i.e., $\gamma \to 1$, from (7), we get

$$
\bar{\theta}_k \to \Theta + \pi, \quad \text{i.e.,} \quad \theta_k \to \theta_0 + \Theta + \pi, \quad (\mathrm{mod}(2\pi)) \tag{19}
$$

with $k = 1, \ldots, N$, which means that all the members move in the same direction eventually. Also note that the equations for the position of every member are the same. Hence all members move in the same manner as $t \to +\infty$.

For $K < 0$, i.e., $\gamma \to 0$, (7) reduces to

$$
\sin(\bar{\theta}_k - \Theta) \to \sin(\psi_k - \Psi), \quad \cos(\bar{\theta}_k - \Theta) \to \cos(\psi_k - \Psi)
$$

with $k = 1, \ldots, N$. Particularly, for $\gamma = 0$, we have

$$
\sin(\bar{\theta}_k - \Theta) = \sin(\psi_k - \Psi), \quad \cos(\bar{\theta}_k - \Theta) = \cos(\psi_k - \Psi). \tag{20}
$$

Considering (13), (14), and after some algebraic manipulations using trigonometric identities, we get

$$
\sum_{k=1}^{N} \cos\bar{\theta}_k = -1, \quad \sum_{k=1}^{N} \sin\bar{\theta}_k = 0. \tag{21}
$$

It follows that $N\bar{g} = -1, N\bar{h} = 0$, i.e., $P_{\bar{\theta}} = 0$ which is a balance manifold. This balance manifold is equivalent to constraint (12) as $\gamma = 0$. Therefore, it follows that as $\gamma \to 0$, $P_{\bar{\theta}} \to 0$ which implies that the centroid of the group tends to a fixed position as $t \to +\infty$. From (20), we further have

$$
\bar{\theta}_k = \psi_k + \Theta - \Psi, \quad (\mathrm{mod}(2\pi))
$$

with $k = 1, \ldots, N$, i.e.,

$$
\theta_k = \psi_k + \theta_0 + \Theta - \Psi, \quad (\mathrm{mod}(2\pi)), \quad k = 1, \ldots, N. \tag{22}
$$

Also from (9) and (10), it yields that $\theta_0 + \Theta - \Psi$ tends to a constant $C$ as $t \to +\infty$. Thus as as $t \to +\infty$, we have

$$
\theta_k = \psi_k + C, \quad (\mathrm{mod}(2\pi)), \quad k = 1, \ldots, N,
$$

which implies that as $t \to +\infty$, each member eventually moves away from the centroid of the group in a fixed direction, i.e., all members eventually move along straight lines. So all members will move in the same manner as $t \to +\infty$ increases if the directions of the motion are neglected.

From above analysis, we arrive at the following proposition.

**Proposition 1.** *Consider the model (1). If* $K > 0$, *then the members all move in the same direction eventually. If* $K < 0$, *the members eventually move along straight lines with the centroid of the group approaching a fixed point. In each case, all members move in the same manner as* $t \to +\infty$ *if the directions of the motion are neglected.*

Proposition 1 shows that the members eventually tend to form certain formation.

**Remark 1.** In the above analysis, if we impose the following constrains on $\psi_k$ instead of (12),

$$
\sum_{k=1}^{N} \cos\psi_k = -1, \quad \sum_{k=1}^{N} \sin\psi_k = 0,
$$

and

$$
\cos\Psi - \cos\Theta = 0 \quad \text{for} \quad \gamma = 0.
$$

on the initial values of $\Theta$ and $\Psi$ instead of (14), the same results can be obtained.

### *E. Weak/Strong Coupling Cases*

Now, we consider two special cases of weak and strong couplings. Due to the space limitation, we only present the results here. Details can be found in [17].

First, we consider the weak coupling case, i.e., $|K|$ is sufficiently small. In this case, we perform a time-scale separation between the slow dynamics in time-scale $\tau = |K|(t - t_0)$ and the fast dynamics in time-scale $t$. Under the time-scale separation, $\gamma$ and $\Psi$ are slow variables, and $\theta_0$ and $\Theta$ are fast variables. Through analyzing slow and fast dynamics, we have following conclusion.

**Proposition 2.** *In the weak coupling case, each member eventually moves along a fixed straight line as* $t \to +\infty$. *For* $K > 0$, *all these straight lines are parallel and all members move in the same direction eventually. For* $K < 0$, *all members will move away from the centroid along these lines, while leaving the centroid approaching a fixed point as* $t \to +\infty$.

Next, we consider the strong coupling case, i.e., $|K|$ is large enough. In this case, a time-scale separation between the fast dynamics in time-scale $\tau = |K|(t - t_0)$ and the slow dynamics in time-scale $t$ is enforced. Under this time-scale separation, $\theta_0$ is slow variable and $\gamma, \Theta$, and $\Psi$ are fast variables. We have following conclusion.

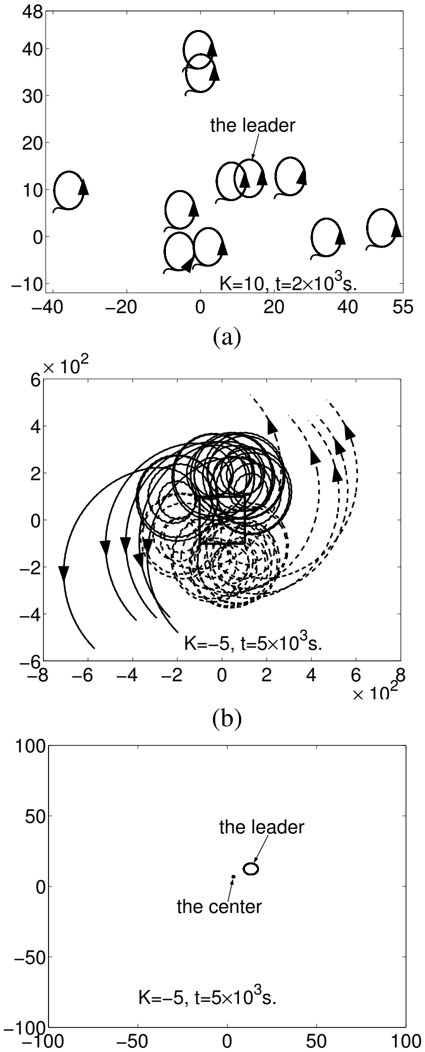
**Proposition 3.** *In the strong coupling case, all member agents move in the same manner as $t \to +\infty$ if the directions are neglected. Particularly, in the case of $K > 0$, if $(N+1)F(\theta_0)/K$ tends to a constant in $[-1,1]$ along solutions of Eq. (3) as $t \to +\infty$, all members and the leader eventually move in the same manner. For $K < 0$, all members eventually move along straight lines and the direction differences $\theta_k - \theta_j$ with $j,k = 1,...,N$, and $j \neq k$ are fixed as $t \to +\infty$.*

## IV. NUMERICAL SIMULATIONS

In this section we will demonstrate our theoretical results by numerical simulations. We have carried out a lot of numerical simulations. Here we display some of them. In the numerical simulations, we choose a leader and ten members. In the figures presented below, the arrows on the curves indicate the directions of the motion. The values of $t$ in the figures indicate the time slots for simulations.

Fig. 1 presents two simulation results for position evolutions of eleven agents described by (1) with $F(\theta_0) = 0.25$ and different values of $K$ as indicated in the figures. In Figs. 1(a)–(c), each curve indicates a trajectory of the motion of an agent and the trajectories of the leader are the same as shown in Fig. 1(a). Figs. 1(a) and (b) show the motion of the agents with $K > 0$ and $K < 0$, respectively. The trajectories in the similar shapes in Figs. 1(a) and (b) indicate that all the members almost move in the same manner if the direction of motion is neglected. Fig. 1(a) also shows that the members and the leader move in the same manner, as predicted in Proposition 3 because of $(N+1)F(\theta_0)/K \equiv 0.275$ for all $t$ in this case. From Fig. 1(b), it can be seen that the agent members are split into two subgroups whose motions are indicated by the dashed and the solid lines, respectively, and that the curvatures of the lines become smaller and smaller with $t$ increasing. This shows that the directions of motion of the member agents change more and more slowly. This agrees with the analytical result of Proposition 1. Fig. 1(c) is an enlargement of the box at the center of Fig. 1(b), in which we only remained the trajectories of the leader and the weighted centroid for the sake of clarity. From Fig. 1(c), we can see that the trajectory of the weighted centroid of the group appears to be a point. This agrees with the analytical result of Proposition 1 that for $K_1 < 0$, the position of the weighted centroid of the group approaches a fixed point eventually.

Figs. 2(a) and (b) simulate the position evolutions of eleven agents described by (1) with $F(\theta_0) = 0.25$ and $|K| = 0.01$. The trajectories of the leader are the same as shown in Fig. 1(a). It can be seen that the trajectories are almost straight lines as $t$ is large enough. This agree with the analytical results of Proposition 2 that in the weak coupling case, each member eventually moves along a fixed straight line as $t \to +\infty$.




(a)

(b)

(c)

Fig. 1. Phase diagrams of the position dynamics of the agents with $F(\theta_0) = 0.25$ and different values of $K$. The abscissa axis and the ordinate axis represent the first and the second components of the position vector $r$, respectively, in (a)–(c).

## V. CONCLUSIONS

We have considered cooperative control problem of a group of agents with a leader. Analytic results show that the motion of all member agents depends on the motion of the leader and the coupling strengths. In general, two types of collective motion occur, depending on different ranges of the coupling strengths. One is that all members move in the same direction and the other is that all members move leaving the centroid of the group approaching a fixed position. In each case, all member agents move in the same manner eventually. In the special case of weak couplings, all members will move along lines. In another special case

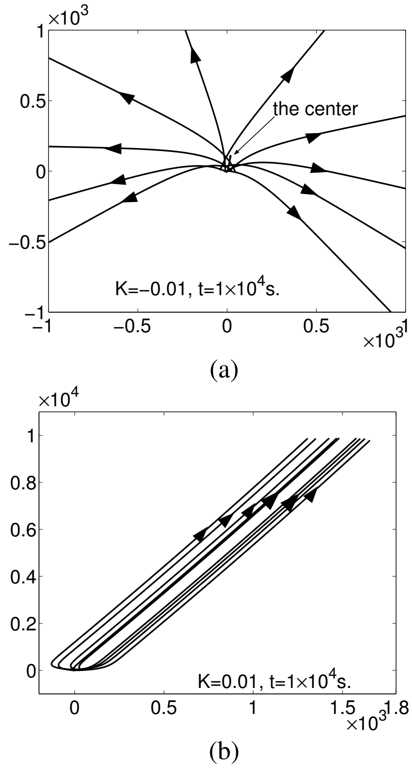



Fig. 2. Phase diagrams of the position dynamics of the particles with $F(\theta_0) = 0.25$ and $|K| = 0.01$. The abscissa axis and the ordinate axis represent the first and the second components of the position vector $r$, respectively, in (a) and (b).

of strong couplings, the members and the leader move in the same manner as time increasing in certain range of the coupling strengths. Numerical simulations agree with the results of this paper very well. These results show that it is possible to control the agent group by steering the motion of the leader. This is of practical interest in applications such as control of multi-robots or autonomous vehicles.

## APPENDIX

**Proof of the assertion**:

First we prove (*i*). We only show $G_1 \geq 0$. $G_2 \geq 0$ can be established similarly. For $\gamma = 0$, from (12) and (13), we get $G_1 = 0$. Since

$$\frac{\partial G_1}{\partial \gamma} = \sum_{k=1}^{N} \frac{1-\cos^2(\psi_k - \Psi)}{(1-\gamma\cos(\psi_k - \Psi))^2} + 2\gamma(1+\cos\Theta) + 1 > 0$$

for all $(\Theta, \Psi, \theta_0)$, $G_1$ strictly increases in $\gamma$. Therefore, $G_1 \geq 0$ for all $(\gamma, \Theta, \Psi, \theta_0)$ and the equality holds if and only if $\gamma = 0$.

Now we prove (*ii*). We only show the case of $K > 0$. The case of $K < 0$ can be done similarly. For $K > 0$, we have

$$\begin{aligned} G_4 &= \dot{v} - \gamma\sin\Theta F(\theta_0) \\ &= \begin{cases} (2-\gamma\sin\Theta)F(\theta_0) & \text{if } \theta_0 \in P2, \\ (-2-\gamma\sin\Theta)F(\theta_0) & \text{if } \theta_0 \in P3, \\ (\theta_0 - \theta_0^* - 2 - \gamma\sin\Theta)F(\theta_0) & \text{if } \theta_0 \in P1(+), \\ (\theta_0 - \theta_0^* + 2 - \gamma\sin\Theta)F(\theta_0) & \text{if } \theta_0 \in P1(-). \end{cases} \end{aligned}$$

According to Lemma 1, $G_4 > 0$ always holds as $t$ is large enough. □

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