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Consensus Algorithms are Input-to-State Stable

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Abstract— **In many cooperative control problems, a shared knowledge of information provides the basis for cooperation. When this information is different for each agent, a state of noncooperation can result. Consensus algorithms ensure that after some time the agents will agree on the information critical for coordination, called the coordination variable. In this paper we show that if the coordination algorithm is input-to-state stable where the input is considered to be the discrepancy between the coordination variable known to each vehicle, then cooperation is guaranteed when a consensus scheme is used to synchronize information. A coordinated timing example is shown in simulation to illustrate the notions of stability when a coordination algorithm is augmented with a consensus strategy.**

I. INTRODUCTION

Replacing large, expensive, monolithic vehicles with teams of networked vehicles, promises less expensive, more capable systems. In addition, there are applications where a team of vehicles can accomplish objectives that would be impossible for a single vehicle [1], [2], [3]. To a large extent, the ability of team members to coordinate hinges on their agreement upon a set of information that we call the *coordination variable* [4]. When this information is the same between team members, centralized coordination algorithms (replicated on each agent) can be used to achieve cooperation in a decentralized manner. Unfortunately, in most real-world applications, perfect synchronization is not possible necessitating algorithms that ensure that team members eventually come to a consensus on the value of the coordination variable. Many good approaches exist to the consensus problem with varying levels of assumed agent connectivity. In the most general case, agents must coordinate under dynamically changing interaction topologies [5], [6], [7], [8]. In [6], necessary and/or sufficient conditions are shown to ensure asymptotic consensus in the case of discrete-time update schemes. In a companion paper [9], a consensus scheme motivated by the Kalman filter is presented and shown to guarantee asymptotic consensus and explicitly account for relative agent reliability. In this paper we show that the Kalman consensus scheme is input-to-state stable (ISS) with respect to communication noise, and use this fact to design cooperative timing strategies for unmanned air vehicles (UAVs).

UAV cooperative timing problems have been investigated recently in the context of battlefield scenarios where the UAVs are required to converge to the boundary of a radar detection area to maximize the element of surprise [10], [11], [12], [13], [14]. Cooperative timing problems also arise in refueling scenarios, fire and hazardous material monitoring, moving area of regard problems, and continuous surveillance problems. In this paper we will investigate a simplified cooperative timing problem that must be accomplished in the presence of an unreliable, dynamically changing communication topology.

In the case of cooperative timing problems, the coordination information is the time-over-target for the whole team. We are particularly interested in the relationship between the consensus algorithm and the cooperative control scheme. Specifically, if the action of each UAV is based on the dynamically changing, local instantiation of the perceived time-over-target, will the team cooperation objective still be achieved?

The main contribution of this paper is to derive sufficient conditions for the coordination scheme when it is used in connection with an asymptotically stable consensus algorithm. Specifically, we wish to investigate overall system behavior when a cooperative control scheme, designed to be stable when the coordination variable is known *a priori*, is instead, given an estimate of the coordination variable by a consensus scheme. The application of these ideas will be investigated in the context of cooperative timing scenarios.

This paper is organized as follows. An overview of Kalman consensus scheme is given in Sections II. The Kalman consensus scheme is shown to be input-to-state stable (ISS) in Section III and this is used to derive a design principle for distributed cooperation algorithms. These principles are applied to a cooperative timing example in Section IV.

II. KALMAN CONSENSUS

In a companion paper, we present a Kalman-filter-inspired technique for consensus seeking [9]. The purpose of the Kalman consensus scheme is to explicitly account for relative agent reliability while at the same time obtaining consensus in the presence of a dynamically changing communication topology. Some of the main results are presented here to facilitate the analysis later of the stability properties of Kalman consensus. As a matter of notation, we are considering *asymptotic consensus* in the sense that consensus is said to be achieved asymptotically if $\|\xi_i(t) - \xi_j(t)\| \to 0$ as $t \to \infty$ for each pair of agents (i, j), where ξ_i is the i^{th} agent's estimate of the coordination variable whose value all agents must agree upon.

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The following update equations describe the Kalman consensus scheme for the i^{th} agent:

$$\dot{P}_i = -P_i \left[\sum_j g_{ij}(t)(P_j + \Omega_{ij})^{-1} \right] P_i + Q_i \quad (1)$$

$$K_{ij} = P_i(P_j + \Omega_{ij})^{-1} \quad (2)$$

$$\dot{\xi}_i = \sum_{j=1}^{n} g_{ij}(t) K_{ij} \left((\xi_j + \nu_{ij}) - \xi_i \right) \quad (3)$$

where ξ_i is i^{th} agent's coordination variable and P_i the associated relative uncertainty for ξ_i. $g_{ij}(t)$ captures the connectivity between agent i and j, specifically, when $g_{ij}(t) = 1$, agent i receives communication from agent j, otherwise $g_{ij}(t) = 0$. ν_{ij} is the noise on the communication channel from agent j to i (assumed to be zero-mean Gaussian with covariance Ω_{ij}). Finally, Q_i is the covariance associated with the zero-mean Gaussian random variable which corrupts the state-space model in a typical Kalman filter setting.

Theorem 1: Under switching interaction topologies, the Kalman consensus scheme given in Equations (1)–(3) achieves asymptotic consensus if there exist infinitely many consecutive uniformly bounded time intervals such that the union of the interaction graph across each interval has a spanning tree.

Theorem 1 is proven in [9], but deserves mention here to highlight the conditions under which the Kalman consensus scheme achieves agreement between agents. The central condition (from [15]) is that under dynamically switching communication topologies, a spanning tree of the communication topology graph must be reached infinitely many times. A spanning tree is the least restrictive graph arrangement that includes all agents in a way that allows for consensus. Each time a spanning tree is achieved, the consensus error is driven closer to zero, so if a spanning tree is reached infinitely many times, then each agent's estimate of the coordination variable approaches the others' asymptotically. The proof of Theorem 1 also shows that the transition matrix in each interval in which a spanning tree is reached is indecomposable and aperiodic (SIA), meaning $\lim_{n\to\infty} P^n = \mathbf{1}y^T$, where y is a column vector [16]. SIA matrices are composed of all non-negative entries, have a row sum of 1 and are essentially *averaging* matrices in the sense that a vector operated on by an SIA matrix returns a new vector whose elements are composed of a weighted average of all the entries of the original vector. It is this fact that allows us to conclude uniformity in Section III.

III. Consensus Algorithms are Input-to-state Stable

We are primarily interested in the application of consensus algorithms to cooperative control problem. In this paper we will explore a control architecture where a consensus algorithm is in cascade with a coordination algorithm, as shown in Figure 1. Our purpose in this section is to derive

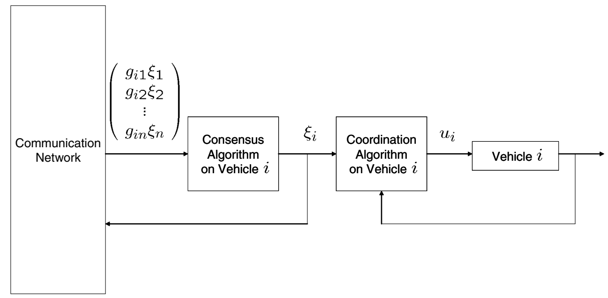

Fig. 1. The control architecture consists of a consensus algorithm in cascade with a coordination algorithm. The consensus algorithm receives information from the communication network to produce a value of the coordination variable ξ_i. The coordination algorithm uses the coordination variable ξ_i to produce a command the the vehicle u_i. We assume that the same consensus and coordination algorithms are implemented on each vehicle.

conditions on the consensus and coordination algorithms that guarantee that the cooperation objective is achieved. Toward that end, rewrite Equation (3) as

$$\dot{\xi}_i = \sum_{j=1}^{n} g_{ij}(t) K_{ij} \left(\xi_j - \xi_i \right) + \sum_{j=1}^{n} g_{ij}(t) K_{ij} \nu_{ij}. \quad (4)$$

Defining the total consensus error vector x as $x_{ij} = \xi_i - \xi_j$ and $x = (x_{11}, x_{12}, \ldots, x_{1n}, x_{21}, \ldots, x_{nn})^T$, we get the state-space model

$$\dot{x} = A(t)x + B(t)\nu \quad (5)$$

where ν is a column vector created by stacking the communication noise terms ν_{ij}, and the elements of $A(t)$ and $B(t)$ are linear combinations of $g_{ij}K_{ij}(t)$ and can be easily constructed from Equation (4).

We now state the main technical result of the paper.

Theorem 2: Under the hypothesis of Theorem 1, the Kalman consensus scheme given by Equations (1), (2), and (5) is input-to-state stable.

The proof of this theorem requires the following two lemmas.

Lemma 3: Under the hypothesis of Theorem 1, if the communication error ν is zero, then the consensus error x is uniformly stable.

Proof: As shown in [9], the transition matrix associated with the coordination variable dynamics is SIA. When $g_{ij}(t) = 1$, the i^{th} coordination variable is updated to a weighted average of all agents' coordination variables communicating with i. Since a weighted average can never be greater (or smaller) than any one of the components in the average, the updated ξ_i must be within $[\min(\xi_j), \max(\xi_j)]$. Since all agents that receive communication with other agents use the same averaging scheme, $\xi_i(t) \in [\min(\xi_j(t_0)), \max(\xi_j(t_0))]$ for all t and i. Then

$$\|x\|_\infty \leq \|x(t_0)\|_\infty, \quad \text{for} \quad t \geq t_0.$$

■

Lemma 4: The norm of $B(t)$ in Equation (5) is bounded.

Proof: Since $B(t)$ is composed of linear combinations of $K_{ij}(t)$, if $\|K_{ij}(t)\|$ is bounded for each (i,j), then $\|B(t)\|$ will also be bounded. Referring to Equation (2) and recalling that $\Omega_{ij} > 0$ and $P_j(t) > 0$, then $\|K_{ij}\|$ will be bounded if $\|P_i(t)\|$ is bounded. Using Equation (1) and noting that $P_i > 0$, Q_i is bounded and $-P_i \left[\sum_j g_{ij}(t)(P_j + \Omega_{ij})^{-1}\right] P_i \leq -P_i(P_i + \Omega_{ii})^{-1}P_i$, we see that P_i is uniformly bounded. ■

Proof of Theorem 2: By Lemma 3, the Kalman consensus error is uniformly stable. By Theorem 1, $\|\xi_i - \xi_j\| \to 0$ as $t \to \infty$ for all (i,j). Since each element of $x \to 0$ then $\|x\| \to 0$ as $t \to \infty$ and we conclude uniform asymptotic stability. Any linear system that is uniformly asymptotically stable is also uniformly exponentially stable [17]. Additionally, linear uniformly exponentially stable systems with $\|B(t)\| < \beta$ for finite β are bounded-input bounded-output stable [18]. Since the Kalman consensus error governed by Equation (5) is a linear uniformly asymptotically stable system with $\|B(t)\|$ bounded, it is ISS. ■

Corollary 5: If the continuous-time consensus schemes presented in [19], [15], [5], and [20] are augmented with communication noise, then the representation of these schemes that is equivalent to Equation (5) is ISS.

Proof: The difference between each of these schemes and Equation 3 is that the consensus gain $K_{ij}(t)$ is time invariant. Therefore from the proof of Theorem 2 it is clear that they are ISS. ■

Referring to Figure 1 we see that the combination of the communication network and the consensus scheme is an ISS system. The cascade combination of two ISS systems is also ISS [21]. Therefore if the feedback loop containing the coordination algorithm and the i^{th} vehicle is ISS from the consensus error to the cooperation objective, then the total system will be ISS from the communication noise to the cooperation objection. This concept is shown schematically in Figure 2 and can be summarized by the following Theorem.

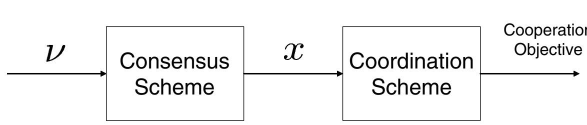

Fig. 2. The distributed cooperative control problem can be thought of as a cascade connection between the consensus algorithm and the coordination algorithm. If both are ISS, then the cascade system will be ISS.

Theorem 6: Given a cascade interconnection between a coordination algorithm and a consensus scheme that is ISS from the communication noise to the consensus error. If a coordination scheme is ISS from the consensus error to the cooperation objective then the interconnection is ISS from the communication noise to the cooperation objective.

The major implication of Theorem 6 is that communication noise cannot disrupt overall team cooperation. If a coordination algorithm is ISS and is driven by a consensus algorithm that is implemented over noisy communication channels (communication via sensing, for example), then Theorem 6 states that the error in the cooperation objective will be bounded and related to the power of the noise in the communication. When there is significant communication noise, then the cooperation objective will still be achieved, albeit loosely.

IV. Illustrative Example - Cooperative Timing

Suppose that a team of UAVs, flying at distinct altitudes, is tasked to simultaneously visit a pre-specified location. For simplicity, also assume that paths have been precomputed for each UAV as shown in Figure 3.

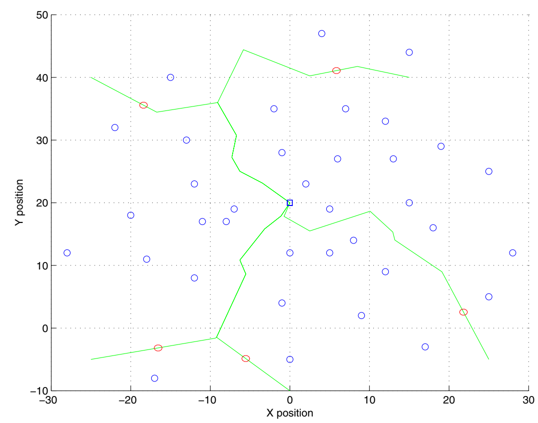

Fig. 3. Cooperative timing scenario with five agents involved.

We will also assume that each UAV has autopilot functionality that maintains the UAV on its pre-defined path, but that the velocity along the path can be adjusted to meet the simultaneous arrival objective. We will assume that the velocity hold autopilot has been designed such that

$$\dot{v}_i = v_i^c - v_i \tag{6}$$

where v_i is the velocity and v_i^c the commanded velocity for the i^{th} UAV. Let L_i denote the length of the path remaining to the target, then

$$\dot{L}_i = -v_i.$$

Given L_i and v_i, the i^{th} UAV can estimate its expected time-of-arrival (ETA) as

$$\tau_i = \frac{L_i}{v_i}.$$

Differentiating, we obtain

$$\begin{aligned} \dot{\tau}_i &= \frac{v_i \dot{L}_i - L_i \dot{v}_i}{v_i^2} \\ &= -1 - \tau_i \left(\frac{v_i^c - v_i}{v_i} \right). \end{aligned} \tag{7}$$

The cooperation objective for this problem is that each UAV arrives at its destination simultaneously, i.e. $\tau_i - \tau_j = 0$ for each (i,j). The coordination variable for this problem is chosen as the arrival time. Therefore ξ_i represents the

i^{th} UAVs understanding of the team arrival time. Clearly, to satisfy the simultaneous arrival objective, the team must come into consensus before the actual arrival time. As in many practical applications, we desire consensus in finite time, but consensus is only guaranteed as $t \to \infty$. However, due to its exponential nature, a consensus algorithm will still be useful in the presence of finite horizon requirements.

Let the commanded velocity to each UAV be

$$v_i^c = v_i + \frac{v_i}{\tau_i}\left(\gamma\tau_i - \gamma\xi_i - 1\right), \tag{8}$$

then Equation (7) reduces to

$$\dot{\tau}_i = -\gamma\tau_i + \gamma\xi_i. \tag{9}$$

Note that

$$\begin{aligned}(\dot{\tau}_i - \dot{\tau}_j) &= -\gamma\tau_i + \gamma\xi_i + \gamma\tau_j - \gamma\xi_j \\ &= -\gamma\left(\tau_i - \tau_j\right) + \gamma\left(\xi_i - \xi_j\right),\end{aligned}$$

and that the system $\dot{\phi} = -\gamma\phi + \gamma u$ is input-to-state stable. In fact we have that

$$|\phi(t)| \leq e^{-\gamma(t-t_0)}\phi(t_0) + \sup_{t_0 \leq \sigma \leq t} |u(\sigma)|.$$

Therefore, the combination of the consensus strategy given by Equations (1)–(3) and the velocity controller given by Equation (8) is input-to-state stable with the input being communication noise and the state consisting of both the consensus discrepancy $\xi_i - \xi_j$ and the UAV arrival discrepancy $\tau_i - \tau_j$.

The cooperative timing scenario was simulated with an unreliable switching communication topology. The team is connected in the graph shown in Fig. 4 where each link is only available 70 percent of the time. When an agent receives communication it updates its estimate of ξ_i, the team estimated time-of-arrival (ETA_{team}), using the Kalman consensus scheme of Section II. In between consensus updates, agents control their velocity using Equation (8) so that the actual time-of-arrival matches the estimate from the consensus algorithm. Five agents were given a single target at which to arrive simultaneously, as in Fig. 3.

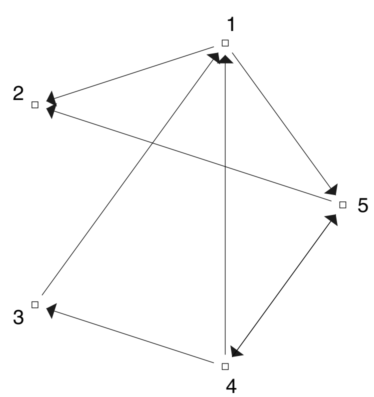

Fig. 4. Union of possible communication topologies.

In the first case, communication noise was set to zero and each agent started with approximately the same confidence in its estimate of the team ETA. The reference team ETA for each vehicle is shown in Fig. 5 and the actual ETA of each vehicle is shown in Fig. 6. As can be seen, each agent in the team achieves agreement using consensus, adjusts its ETA to match the team ETA, and arrives at the target in approximately 20 seconds.

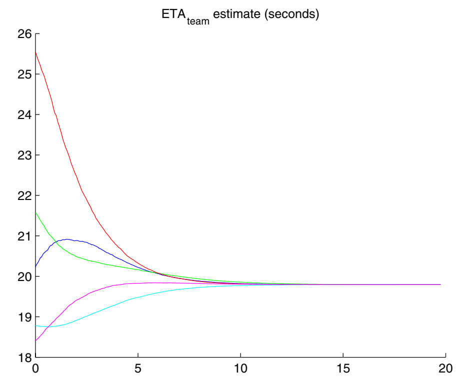

Fig. 5. Reference team ETA for each agent with no communication noise.

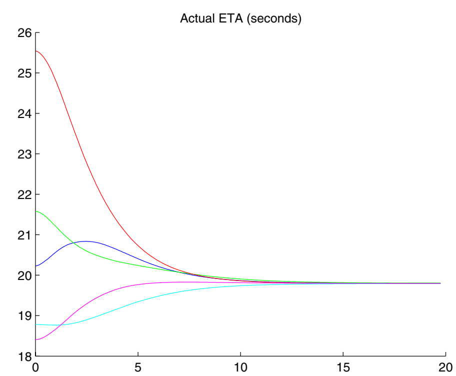

Fig. 6. Actual ETA for each agent with no communication noise.

In the second case, significant communication noise is added. The reference team ETA for each vehicle is shown in Fig. 7 and the actual ETA of each vehicle is shown in Fig. 8. As can be seen, each agent in the team achieves approximate agreement using consensus where the error in agreement is due to the communication noise.

V. Conclusions

This paper has shown that the Kalman consensus scheme presented in [9] is input-to-state stable. As a corollary we get that most of the consensus schemes presented in the literature are also ISS. The input-to-state property of the consensus scheme was used to show that if the consensus scheme is used in cascade with a multiple vehicle coordination algorithm that is also ISS, then the fidelity of the cooperation objective is directly related to the power level of the communication noise.

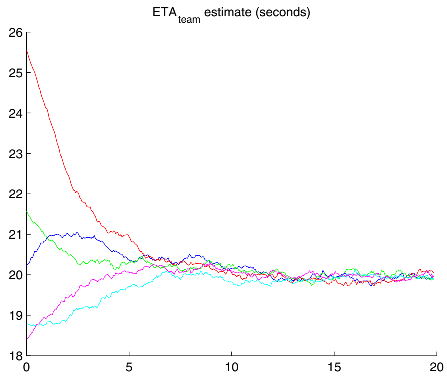

Fig. 7. Reference team ETA for each agent with communication noise.

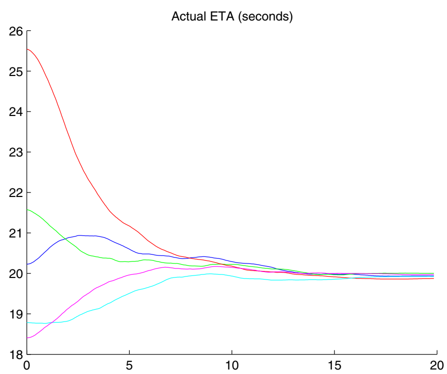

Fig. 8. Actual ETA for each agent with communication noise.

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