2005 American Control Conference
June 8-10, 2005. Portland, OR, USA

Fault Detection of Uncertain Systems Based on Probabilistic Robustness Theory

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Abstract— **This paper studies the fault detection problem of uncertain linear systems with arbitrary uncertainty structure. With the aid of probabilistic robustness technique, an approach is developed to determine parameters of observer-based residual generators by incorporating probability distribution of model uncertainty in the design. One advantage of the approach is that it needs no assumptions on the structure of model uncertainty. Examples are given to illustrate the proposed design procedure.**

I. INTRODUCTION

With the increasing requirement of modern complex control systems on safety and reliability, model-based fault detection and isolation (FDI) technique has attracted much attention during the last three decades [1]-[4]. The basic idea of model-based FDI is to generate analytical redundancy with the help of a mathematical model of the supervised system. The fault indicating signal, called usually residual, is generated by comparing measured outputs with their estimations. A number of model-based approaches, for instance, observer-based approach, parity space scheme, parameter estimation method, etc., have been proposed to the FDI of linear and nonlinear systems. Applications have been found in automobile industry, process industry, transportation systems, aerospace and aeronautics, etc.

The performance of model-based FDI systems relies on accurate models of the system. However, models are never perfect. Besides the faults, in most cases the residual signal is also influenced by disturbances, noises and even control inputs due to existence of multiplicative model uncertainty. Thus, the main challenge to model-based FDI approaches is to distinguish the change of the residual signal caused by the faults from that caused by non-fault factors. While additive model uncertainty has been extensively treated, only a few approaches to handle multiplicative model uncertainty have been developed and most of them are restricted to particular kinds of model uncertainty, such as, polytopic uncertainty, structured norm-bounded uncertainty or LFT uncertainty [1]-[9].

This paper aims to develop a new approach to the design of robust FD systems for linear systems subject to multiplicative uncertainty. It is assumed that the probabilistic distribution of the bounded uncertainty is known in advance but there is no restriction on the way how the uncertainty influences system matrices.

This work is inspired by the recent development of probabilistic robustness theory, which provides a new philosophy of control system analysis and synthesis. Though the idea of probabilistic robustness originates at the beginning of the eighties [10], characterized by the introduction of concepts like "probability of instability" [11], [12], it is only recently that the probabilistic robustness theory has been intensively investigated and developed [13]-[20]. In this framework, based on random samples of uncertainty generated according to its distribution, the probability of performance can be estimated. The accuracy of the estimate can be guaranteed with a specified confidence level by taking enough amount of samples. Approaches have also been developed to solve controller synthesis problem, i.e. to find a controller which meets specification on robustness performance in a probabilistic sense. The approaches to controller synthesis can be divided into two major classes: learning theory based approach and sequential stochastic approach. The basic idea of learning theory based approach is to take random samples both in the uncertainty space and in the controller parameter space. By estimating the performance achieved by each trial (sample) of controller parameter, the best one will be selected. In sequential stochastic approach, the basic idea is to use the subgradient method to iteratively update the controller parameter based on random samples of uncertainty. While the learning theory based approach is conceptually straightforward, its performance strongly depends on efficient generation of samples in the controller parameter space. The sequential approach can overcome the difficulty of sampling controller parameters and will be employed in this paper.

In the field of fault detection, based on probabilistic robustness theory, [21] has developed a false alarm rate guaranteed scheme of threshold selection for a given residual generator. A quantitative relation has been established between the threshold and the false alarm rate. The proposed approach builds a bridge between the conventional statistical testing based and norm-based evaluation methods.

This paper will apply probabilistic robustness theory to the design of robust FD systems for uncertain linear time-invariant (LTI) systems with *arbitrary uncertainty structure*. After problem formulation in Section II, the suggested solution is presented in Section III. Two examples are given

This work was supported in part by the European Union under Grant IST-2001-32122 IFATIS.

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in Section IV to illustrate the proposed design procedure.

II. PROBLEM FORMULATION

A. System description

This paper studies the fault detection problem of uncertain discrete LTI systems described by

$$\begin{aligned} x(k+1) &= A(\Delta)x(k) + B(\Delta)u(k) + E_f f(k) \\ y(k) &= C(\Delta)x(k) + D(\Delta)u(k) + F_f f(k) \end{aligned} \quad (1)$$

where $x \in \mathbf{R}^n$ denotes the state vector, $u \in \mathbf{R}^p$ the control input vector, $y \in \mathbf{R}^m$ the measured output vector, and $f \in \mathbf{R}^{k_f}$ the vector of faults to be detected, E_f, F_f are known constant matrices, $A(\Delta), B(\Delta), C(\Delta), D(\Delta)$ are system matrices dependent on *unknown but bounded* parameter vector

$$\Delta = \left[\begin{array}{cccc} \delta_1 & \delta_2 & \cdots & \delta_l \end{array} \right]' \in R^l$$

The way how Δ enters into matrices A, B, C, D may be very complex. It is assumed that the probability distribution of Δ is known and denoted by $f_\Delta(\Delta)$.

B. Analysis

Let

$$\left[\begin{array}{cc} A(\Delta) & B(\Delta) \\ C(\Delta) & D(\Delta) \end{array} \right] = \left[\begin{array}{cc} A_o & B_o \\ C_o & D_o \end{array} \right] + \left[\begin{array}{cc} F_a(\Delta) & F_b(\Delta) \\ F_c(\Delta) & F_d(\Delta) \end{array} \right]$$

where F_a, F_b, F_c, F_d are unknown Δ-dependent matrices, A_o, B_o, C_o, D_o are constant matrices representing some nominal behavior of the system. For instance, A_o, B_o, C_o, D_o can be defined by

$$\left[\begin{array}{cc} A_o & B_o \\ C_o & D_o \end{array} \right] = \left[\begin{array}{cc} A(\bar{\Delta}) & B(\bar{\Delta}) \\ C(\bar{\Delta}) & D(\bar{\Delta}) \end{array} \right] \quad (2)$$

where $\bar{\Delta}$ denotes the mean value of Δ that can be computed according to the probability distribution $f_\Delta(\Delta)$.

Generally speaking, a model-based FDI system is composed of two parts: residual generator and residual evaluator [1]-[4]. An observer-based residual generator can be constructed as [1]-[4]

$$\begin{aligned} \hat{x}(k+1) &= A_o\hat{x}(k) + B_o u(k) + L(y(k) - \hat{y}(k)) \\ \hat{y}(k) &= C_o\hat{x}(k) + D_o u(k) \\ r(k) &= W(y(k) - \hat{y}(k)) \end{aligned} \quad (3)$$

where $r \in \mathbf{R}^{k_r}$ is the residual signal, L and W are, respectively, the observer gain matrix and the weighting matrix to be designed. The residual will be evaluated by the following logic

$$\begin{cases} \|r\|_2 \le J_{th} \;\Rightarrow\; \text{no fault} \\ \|r\|_2 > J_{th} \;\Rightarrow\; \text{alarm} \end{cases} \quad (4)$$

where J_{th} is called threshold, which is often set as

$$J_{th} = \sup_{f=0,\, \Delta} \|r\|_2 \quad (5)$$

The dynamics of residual generator (3) is governed by

$$\begin{aligned} \left[\begin{array}{c} x(k+1) \\ e(k+1) \end{array} \right] &= \left[\begin{array}{cc} A & O \\ F_a - LF_c & A_o - LC_o \end{array} \right] \left[\begin{array}{c} x(k) \\ e(k) \end{array} \right] \\ &+ \left[\begin{array}{c} B \\ F_b - LF_d \end{array} \right] u(k) + \left[\begin{array}{c} E_f \\ E_f - LF_f \end{array} \right] f(k) \\ r(k) &= \left[\begin{array}{cc} F_c & C_o \end{array} \right] \left[\begin{array}{c} x(k) \\ e(k) \end{array} \right] + F_d u(k) + F_f f(k) \end{aligned} \quad (6)$$

where the estimation error $e = x - \hat{x}$. As can be seen from (6), the residual signal is influenced not only by faults but also by control inputs u due to the existence of model uncertainty. Moreover, the residual dynamics is internally stable only if $A(\Delta)$ is stable for any Δ. Therefore, in the following, it is assumed that (A_o, C_o) is observable and that $A(\Delta)$ is stable for any Δ.

From different viewpoints, the FD problem can be posed, among others, as:

Problem 1: Given $\alpha > 0$, find parameter L and W of residual generator (3), so that

$$\|G_{ru}\|_\infty < \alpha \quad (7)$$

Problem 2: Given $\alpha > 0$ and $\beta > 0$, find parameter L and W of residual generator (3), so that

$$\begin{aligned} \|G_{ru}\|_\infty &< \alpha \\ \|G_{rf}\|_- &> \beta \end{aligned} \quad (8)$$

The physical meaning behind Problem 1 is to attenuate influence of non-fault factors. Problem 2 is a multi-objective design, which considers not only the robustness of the FD system to non-fault factors but also the sensitivity of the FD system to faults. Note that the transfer function matrices G_{ru}, G_{rf} in Problem 1 and 2 are dependent on the bounded uncertainty Δ. Till now, solutions of these problems are restricted to certain kinds of uncertainty structure. For instance, norm-bounded model uncertainty has been treated by [5], [7], [8], while LFT uncertainty has been handled by [9].

The *basic idea* of this paper is to make use of the probabilistic robustness theory to remove the assumption on uncertainty structure. The algorithm is guaranteed to converge to a feasible solution, if there exists one, in finite steps with probability 1 under some mild assumptions. The price to be paid is that it is difficult to compute the needed iteration steps in advance.

For a better explanation of the basic idea, in this paper, only Problem 1 is considered. We would like to remark that an extension of the results to problems formulated from viewpoint of Problem 2 and some other kinds of norm-based design is possible and worthy of further efforts.

III. SOLUTION

In this section, we shall present an approach to find the solution to Problem 1 for system (1) with arbitrary uncertainty structure by exploring the sequential subgradient approach. To avoid a trivial solution, the weighting matrix W is set as an identity matrix.

A. Formulation of the constrain as LMI

As the first step, constraint (7) is formulated as an LMI. To this aim, the well-known bounded real lemma of discrete linear time-invariant (LTI) systems is introduced [22].

Lemma 1 Given $\alpha > 0$ and a discrete LTI system $G(z) = (A, B, C, D)$. The system is stable and $\|G(z)\|_\infty < \alpha$, if and only if there exists a symmetric matrix X, such that the following LMI holds

$$\begin{bmatrix} -X & XA & XB & O \\ A'X & -X & O & C' \\ B'X & O & -\alpha^2 I & D' \\ O & C & D & -I \end{bmatrix} < 0 \tag{9}$$

Based on Lemma 1, we obtain the following Lemma 2, which builds the basis for the development of the algorithm.

Lemma 2 Given $\alpha > 0$ and system (6). The system is stable and $\|G_{ru}(z)\|_\infty < \alpha$, if there exist matrices X_1, X_2, P and a scalar $\epsilon > 0$, such that

$$\begin{aligned} & V(X_1, X_2, P, \Delta) \\ &= \begin{bmatrix} -X_1 + \epsilon I & O & X_1 A(\Delta) & O & X_1 B(\Delta) & O \\ * & -X_2 + \epsilon I & X_2 F_a(\Delta) - P F_c(\Delta) & X_2 A_o - P C_o & X_2 F_b(\Delta) - P F_d(\Delta) & O \\ * & * & -X_1 & O & O & F_c'(\Delta) \\ * & * & * & -X_2 & O & C_o' \\ * & * & * & * & -\alpha^2 I & F_d'(\Delta) \\ * & * & * & * & * & -I \end{bmatrix} \\ &\leq O \end{aligned} \tag{10}$$

Proof: (10) follows readily from lemma 1 by assuming that

$$X = \begin{bmatrix} X_1 & O \\ O & X_2 \end{bmatrix} \tag{11}$$

and letting $P = X_2 L$.

Remark 1 Assumption (11) will introduce some conservatism in the design. If a post filter $R(z)$ instead of a weighting matrix W is used in the residual generator (3) [23], then a less conservative result can be achieved.

B. Preliminary of sequential subgradient approach

The sequential subgradient approach has been proved to be efficient in finding solutions to LMI, BMI with unknown or varying parameters and employed in robust H_2, H_∞ controller design. In this subsection, we briefly describe the mechanism of this kind of solution algorithm (see [14]-[18], [20] and the references therein).

Suppose that a solution X that satisfies the LMI $V(X, \Delta) \leq O, \forall \Delta$ is to be searched. For certain kinds of structured uncertainty this is a well-studied problem and can be easily solved. However, for general uncertainty structure (for instance, nonlinear dependency of known matrices on Δ), the solution can be obtained by (i) overbounding the uncertainty by a structured one and then solving it, or (ii) employing the randomization algorithm and looking for solutions satisfying the LMIs at the samples of uncertainty. The former may introduce conservatism by overbounding. The latter needs to solve a large amount of LMIs simultaneously.

The sequential subgradient approach handles the problem in an alternative way by [14], [15]:

- setting initial value X^0 of the unknown X,
- generating a random sample Δ^k of Δ according to the known probability distribution of Δ,
- updating X^k based on subgradient of the *convex* objective function with respect to the unknown X.

It is proven that the algorithm converges in finite steps with probability 1, i.e.

$$\Pr\{\exists k_0 < \infty, \text{ s.t. } V(X^k, \Delta) \leq O, \forall \Delta \text{ and } \forall k \geq k_0\} = 1$$

if the following two conditions holds: (i) The solution set is nonempty, and (ii) the probability that the LMI is not satisfied for some Δ is nonzero, as long as X is not a feasible solution. Even if a feasible solution is not found, a good approximately feasible candidate can be obtained through the algorithm [15], [16].

C. Computation of subgradient

In this subsection, we apply the above introduced sequential subgradient approach to find observer gain matrix L that satisfies (10) for arbitrary uncertainty structure.

Let the objective function be defined as

$$v(X_1, X_2, P, \Delta) = \left\| V^+(X_1, X_2, P, \Delta) \right\|_F \tag{12}$$

where V^+ denotes the projection of symmetric matrix V onto the space of positive semi-definite matrix, and $\|V^+\|_F$ denotes the Frobenius norm of matrix V^+. If $V \leq 0$, then $V^+ = O$ and $v = 0$. Otherwise, $V^+ \geq O$ and $v > 0$. The function $v(X_1, X_2, P, \Delta)$ is a convex scalar function of the unknowns X_1, X_2, P. If a set of unknown X_1, X_2, P can be found such that $v = 0$, then a feasible solution of (10) is found. Given a symmetric matrix V, the projection V^+ can be computed via solving an eigenvalue-eigenvector problem. Partition matrix V^+ as $[V_{ij}^+]$, $i, j = 1, \cdots, 6$, corresponding to the dimensions of the blocks in (10).

Theorem The subgradients of $v(X_1, X_2, P, \Delta)$ defined by (12) and (10) with respect to X_1, X_2, P are

$$\begin{aligned} & \partial_{X_1} v(X_1, X_2, P, \Delta) \\ &= -V_{11}^+ - V_{33}^+ + A(\Delta) V_{31}^+ + V_{31}^{+\prime} A'(\Delta) \\ &\quad + B(\Delta) V_{51}^+ + V_{51}^{+\prime} B'(\Delta) \\ & \partial_{X_2} v(X_1, X_2, P, \Delta) \\ &= -V_{22}^+ - V_{44}^+ + F_a(\Delta) V_{32}^+ + A_o V_{42}^+ + F_b(\Delta) V_{52}^+ \\ &\quad + V_{32}^{+\prime} F_a'(\Delta) + V_{42}^{+\prime} A_o' + V_{52}^{+\prime} F_b'(\Delta) \\ & \partial_P v(X_1, X_2, P, \Delta) \\ &= -2 V_{32}^{+\prime} F_c'(\Delta) - 2 V_{42}^{+\prime} C_o' - 2 V_{52}^{+\prime} F_d'(\Delta) \end{aligned}$$

if $v(X_1, X_2, P, \Delta) > 0$, and

$$\begin{aligned} \partial_{X_1} v(X_1, X_2, P, \Delta) &= 0 \\ \partial_{X_2} v(X_1, X_2, P, \Delta) &= 0, \ \partial_P v(X_1, X_2, P, \Delta) = 0 \end{aligned}$$

if $v(X_1, X_2, P, \Delta) = 0$.

Proof: Let parameters X_1, X_2, P subject to small changes $\delta X_1, \delta X_2, \delta P$, respectively. Then

$$V(X_1+\delta X_1, X_2+\delta X_2, P+\delta P, \Delta) = V(X_1, X_2, P)+\delta V$$

where

$$\begin{aligned} \delta V &= \Lambda_1 H_1 + H_1'\Lambda_1 + \Lambda_2 H_2 + H_2'\Lambda_2 + H_3 + H_3' \\ H_1 &= \begin{bmatrix} -\frac{I}{2} & O & A & O & B & O \\ O & O & O & O & O & O \\ O & O & -\frac{I}{2} & O & O & O \\ O & O & O & O & O & O \\ O & O & O & O & O & O \\ O & O & O & O & O & O \end{bmatrix} \\ H_2 &= \begin{bmatrix} O & O & O & O & O & O \\ O & -\frac{I}{2} & F_a(\Delta) & A_o & F_b(\Delta) & O \\ O & O & O & O & O & O \\ O & O & O & -\frac{I}{2} & O & O \\ O & O & O & O & O & O \\ O & O & O & O & O & O \end{bmatrix} \\ H_3 &= \begin{bmatrix} O & O & O & O & O & O \\ O & O & -\delta P F_c & -\delta P C_o & -\delta P F_d & O \\ O & O & O & O & O & O \\ O & O & O & O & O & O \\ O & O & O & O & O & O \\ O & O & O & O & O & O \end{bmatrix} \\ \Lambda_1 &= diag\{\delta X_1, O, \delta X_1, O, O, O\} \\ \Lambda_2 &= diag\{O, \delta X_2, O, \delta X_2, O, O\} \end{aligned}$$

Due to the differentiability of v, there is [16], [20]

$$\begin{aligned} & v(X_1+\delta X_1, X_2+\delta X_2, P+\delta P, \Delta) \\ = & \left\| [V(X_1, X_2, P) + \delta V]^+ \right\|_F \\ = & v(X_1, X_2, P) + \langle V^+(X_1, X_2, P), \delta V \rangle + o(\|\delta V\|_F) \end{aligned}$$

It can be derived that

$$\begin{aligned} & \langle V^+(X_1, X_2, P), \delta V \rangle \\ & = \langle V^+(X_1, X_2, P), \Lambda_1 H_1 \rangle + \langle V^+(X_1, X_2, P), H_1'\Lambda_1 \rangle \\ & + \langle V^+(X_1, X_2, P), \Lambda_2 H_2 \rangle + \langle V^+(X_1, X_2, P), H_2'\Lambda_2 \rangle \\ & + \langle V^+(X_1, X_2, P), H_3 \rangle + \langle V^+(X_1, X_2, P), H_3' \rangle \\ & = \langle H_1 V^+(X_1, X_2, P), \Lambda_1 \rangle + \langle V^+(X_1, X_2, P) H_1', \Lambda_1 \rangle \\ & + \langle H_2 V^+(X_1, X_2, P), \Lambda_2 \rangle + \langle V^+(X_1, X_2, P) H_2', \Lambda_2 \rangle \\ & + \langle V^+(X_1, X_2, P), H_3 \rangle + \langle V^+(X_1, X_2, P), H_3' \rangle \end{aligned}$$

Therefore

$$\begin{aligned} & \partial_{X_1} \left\| V^+(X_1, X_2, P) \right\|_F = U_{11} + U_{11}' + U_{33} + U_{33}' \\ & \partial_{X_2} \left\| V^+(X_1, X_2, P) \right\|_F = Q_{22} + Q_{22}' + Q_{44} + Q_{44}' \\ & \partial_P \left\| V^+(X_1, X_2, P) \right\|_F = W_{22} \\ & U_{11} = -\frac{1}{2} V_{11}^+ + A(\Delta) V_{31}^+ + B(\Delta) V_{51}^+ \\ & Q_{22} = -\frac{1}{2} V_{22}^+ + F_a(\Delta) V_{32}^+ + A_o V_{42}^+ + F_b(\Delta) V_{52}^+ \\ & U_{33} = -\frac{1}{2} V_{33}^+, \quad Q_{44} = -\frac{1}{2} V_{44}^+ \\ & W_{22} = -2 V_{32}^{+'} F_c'(\Delta) - 2 V_{42}^{+'} C_o' - 2 V_{52}^{+'} F_d'(\Delta) \end{aligned}$$

The theorem is thus proven.

After obtaining the subgradients, the sequential subgradient approach introduced in the last subsection can be used to find out the solution of Problem 1.

D. Design procedure

In summary, given $\alpha > 0$, an observer-based residual generator in the form of (3) satisfying (7) for system (1) with arbitrary uncertainty structure can be designed as follows:

Step 1 Set the value of ϵ and select an initial value of X_1^0, X_2^0, P^0.

Step 2 Generate a sample of model uncertainty Δ^k according to the known probability distribution $f_\Delta(\Delta)$.

Step 3 Compute the projection $V^+(X_1^k, X_2^k, P^k, \Delta^k)$ and the value of

$$v(X_1^k, X_2^k, P^k, \Delta^k) = \left\| V^+(X_1^k, X_2^k, P^k, \Delta^k) \right\|_F$$

Step 4 Compute subgradients $\partial_{X_1} v(X_1^k, X_2^k, P^k, \Delta^k)$, $\partial_{X_2} v(X_1^k, X_2^k, P^k, \Delta^k)$, $\partial_P v(X_1^k, X_2^k, P^k, \Delta^k)$ according to the theorem.

Step 5 Calculate the value of $\lambda^k = \frac{\alpha^k}{\beta^k}$, where

$$\begin{aligned} \beta^k = & \Big(\left\| \partial_{X_1} v(X_1^k, X_2^k, P^k, \Delta^k) \right\|_F^2 \\ & + \left\| \partial_{X_2} v(X_1^k, X_2^k, P^k, \Delta^k) \right\|_F^2 \\ & + \left\| \partial_P v(X_1^k, X_2^k, P^k, \Delta^k) \right\|_F^2 \Big)^{1/2} \\ \alpha^k = & \frac{v(X_1^k, X_2^k, P^k, \Delta^k)}{\beta^k} + r \end{aligned}$$

with $r > 0$ being the radius of a ball inside the feasible solution set.

Step 6 If $v(X_1^k, X_2^k, P^k, \Delta^k) = 0$, let $X_1^{k+1} = X_1^k$, $X_2^{k+1} = X_2^k$, $P^{k+1} = P^k$. Otherwise, update the variables X_1^k, X_2^k, P^k by

$$\begin{aligned} X_1^{k+1} &= X_1^k - \frac{\alpha^k}{\beta^k} \partial_{X_1} v(X_1^k, X_2^k, P^k, \Delta^k) \\ X_2^{k+1} &= X_2^k - \frac{\alpha^k}{\beta^k} \partial_{X_2} v(X_1^k, X_2^k, P^k, \Delta^k) \\ P^{k+1} &= P^k - \frac{\alpha^k}{\beta^k} \partial_P v(X_1^k, X_2^k, P^k, \Delta^k). \end{aligned}$$

Step 7 If the above algorithm converges, then the observer gain matrix is obtained as

$$L = (X_2^k)^{-1} P^k$$

Otherwise, set $k = k + 1$ and return to step 2.

Remark 2 The necessary iterations may be reduced by using approach proposed by [18].

Remark 3 If a feasible solution is not found for the given α after a sufficiently large number of iterations, the approximately feasible candidate obtained through the algorithm can be used as initial value for starting the next iteration with a larger α.

Remark 4 In case that probability distribution $f_\Delta(\Delta)$ of bounded uncertainty Δ is unavailable, a uniform distribution can be assumed [24].

E. Residual evaluation

To evaluate the residual based on (4), the threshold J_{th} needs to be determined. If a gain matrix L that satisfies (7) is found, then J_{th} can be set as

$$J_{th} = \alpha \left\| u \right\|_2$$

which guarantees the false alarm rate FAR defined by

$$FAR = \Pr\{\left\| r \right\|_2 > J_{th} \mid f = 0\}$$

to be zero, because $\left\| r \right\|_2 \le \left\| G_{ru}(z) \right\|_\infty \left\| u \right\|_2$. Using the approach developed by [21], the threshold J_{th} can also be selected to guarantee the false alarm rate be under a user defined level.

IV. EXAMPLE

In this section, two examples are given to illustrate the proposed design procedure.

Example 1 Consider the FD problem of a system in the form of (1) with

$$\begin{aligned} A &= \begin{bmatrix} 0.7+\theta_1 & 0 & \theta_9 & \theta_6 \\ 0 & 0.8+\theta_2\theta_3 & 0 & \theta_7 \\ 0 & 0 & 0.6+\theta_4 & \theta_8 \\ 0 & 0 & 0 & 0.5+\theta_5 \end{bmatrix} \\ B &= \begin{bmatrix} 0 & 0 \\ 0 & -3.91 \\ 0.035 & 0 \\ -2.53 & 0.31 \end{bmatrix}, \; E_f = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \\ C &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; D = O, \; F_f = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

The nominal value of the parameter vector is $\theta_1 = -0.5$, $\theta_2 = -0.55$, $\theta_3 = 0.28$, $\theta_4 = 0.086$, $\theta_5 = -0.11$, $\theta_6 = 0.1$, $\theta_7 = -0.042$, $\theta_8 = 0.601$, $\theta_9 = -0.29$. The parameter change is smaller than 10% of the nominal value and is of uniform distribution. Given $\alpha = 2.5$.

Select A_o according to (2) and set $N = 5000$, $\epsilon = 0.01$, $r = 0.001$. The proposed design procedure yields

$$\begin{aligned} X_1 &= \begin{bmatrix} 1.0075 & -0.0049 & 0.1860 & -0.0242 \\ -0.0049 & 0.1928 & 0.0099 & 0.0179 \\ 0.1860 & 0.0099 & 0.2604 & 0.0018 \\ -0.0242 & 0.0179 & 0.0018 & 0.4236 \end{bmatrix} \\ X_2 &= \begin{bmatrix} 1.3000 & 0.1582 & 0.0512 & -0.1764 \\ 0.1582 & 0.8767 & -0.1318 & -0.0777 \\ 0.0512 & -0.1318 & 1.5097 & 0.3835 \\ -0.1764 & -0.0777 & 0.3835 & 1.9179 \end{bmatrix} \end{aligned}$$

$$P = \begin{bmatrix} 0.4219 & -0.2738 & 0.1000 \\ -0.1262 & -0.1045 & -0.2821 \\ -0.0717 & 0.7187 & 0.8349 \\ 0.0523 & 0.1242 & 0.5551 \end{bmatrix}$$

Finally, an observer-based residual generator that satisfies (7) is obtained as

$$\begin{aligned} \hat{x}(k+1) &= A_o\hat{x}(k) + Bu(k) + L(y(k) - \hat{y}(k)) \\ \hat{y}(k) &= C\hat{x}(k) \\ r(k) &= y(k) - \hat{y}(k) \end{aligned}$$

with

$$\begin{aligned} A_o &= \begin{bmatrix} 0.2 & 0 & -0.29 & 0.1 \\ 0 & 0.646 & 0 & -0.042 \\ 0 & 0 & 0.686 & 0.601 \\ 0 & 0 & 0 & 0.39 \end{bmatrix} \\ L &= \begin{bmatrix} 0.3646 & -0.2372 & 0.1153 \\ -0.2180 & -0.0066 & -0.2536 \\ -0.0970 & 0.4980 & 0.4776 \\ 0.0714 & -0.0569 & 0.1943 \end{bmatrix} \end{aligned}$$

The next example shows that a residual generator which minimizes α can be found by iteratively using the proposed design procedure.

Example 2 The system under consideration is the vehicle lateral dynamics which is described by the so-called bicycle model [25]:

$$\begin{bmatrix} \dot\beta \\ \dot\gamma \end{bmatrix} = \begin{bmatrix} -\frac{C_{\alpha V}+C_{\alpha H}}{mv_{ref}} & \frac{l_H C_{\alpha H} - l_V C_{\alpha V}}{mv_{ref}^2} - 1 \\ \frac{l_H C_{\alpha H} - l_V C_{\alpha V}}{I_z} & -\frac{l_V^2 C_{\alpha V} + l_H^2 C_{\alpha H}}{I_z v_{ref}} \end{bmatrix} \begin{bmatrix} \beta \\ \gamma \end{bmatrix} + \begin{bmatrix} \frac{C_{\alpha V}}{mv_{ref}} \\ \frac{l_V C_{\alpha V}}{I_z} \end{bmatrix} \delta_L^* \tag{13}$$

where β denotes vehicle side slip angle, γ yaw rate and δ_L^* steering angle, original vehicle parameters of a car have been adopted. It is assumed that only yaw rate sensor is used.

It is well-known that among the parameters in model (13) the front cornering stiffness $C_{\alpha V}$ and the rear cornering stiffness $C_{\alpha H}$ may vary over a large range, depending on the road condition and driving maneuvers [25]. This causes a strong model uncertainty in the bicycle model (13). It is assumed that $C_{\alpha H} = kC_{\alpha V}, k = 1.7278$ and $C_{\alpha V} = C_{\alpha V}^o + \Delta C_{\alpha V}$, $C_{\alpha V}^o = 103600N/rad$, $\Delta C_{\alpha V} \in [-b_1, 0]$ is a random number with uniform distribution with b_1 representing the maximal size of parameter change.

Model (13) can be re-written into the form

$$\begin{aligned} \dot{x} &= (A + \Delta A)x + (B + \Delta B)u, \\ y &= \begin{bmatrix} 0 & 1 \end{bmatrix} x \end{aligned} \tag{14}$$

with $x = \left[\begin{array}{cc} \beta & \gamma \end{array} \right]'$, $u = \delta_L^*$, $y = \gamma$ and

$$
\begin{aligned}
A &= \left[\begin{array}{cc} -\frac{(1+k)C_{\alpha V}^o}{mv_{ref}} & \frac{(kl_H - l_V)C_{\alpha V}^o}{mv_{ref}^2} - 1 \\ \frac{(kl_H - l_V)C_{\alpha V}^o}{I_z} & -\frac{(l_V^2 + kl_H^2)C_{\alpha V}^o}{I_z v_{ref}} \end{array} \right] \\
\Delta A &= \left[\begin{array}{cc} -\frac{1+k}{mv_{ref}} & \frac{kl_H - l_V}{mv_{ref}^2} \\ \frac{kl_H - l_V}{I_z} & -\frac{l_V^2 + kl_H^2}{I_z v_{ref}} \end{array} \right] \Delta C_{\alpha V} \\
B &= \left[\begin{array}{c} \frac{C_{\alpha V}^o}{mv_{ref}} \\ \frac{l_V C_{\alpha V}^o}{I_z} \end{array} \right], \Delta B = \left[\begin{array}{c} \frac{1}{mv_{ref}} \\ \frac{l_V}{I_z} \end{array} \right] \Delta C_{\alpha V}
\end{aligned}
$$

Because the sampling period of the system is $T = 0.01$ second, the discretized model is

$$
\begin{aligned}
x(k+1) &= (A_d + F_a(\Delta C_{\alpha V}))x(k) \\
&\quad + (B_d + F_b(\Delta C_{\alpha V}))u(k), \\
y(k) &= \left[\begin{array}{cc} 0 & 1 \end{array} \right] x(k)
\end{aligned}
\tag{15}
$$

where

$$
A_d = e^{AT}, \;\; B_d = \int_0^T e^{At} B dt
$$

$$
F_a(\Delta C_{\alpha V}) = e^{(A+\Delta A)T} - e^{AT}
$$

$$
F_b(\Delta C_{\alpha V}) = \int_0^T e^{(A+\Delta A)t}(B + \Delta B)dt - \int_0^T e^{At} B dt
$$

Although $\Delta A, \Delta B$ in continuous-time model (14) depend linearly on the uncertain parameter $\Delta C_{\alpha V}$, the model uncertainties F_a, F_b in the discretized model (15) depend on $\Delta C_{\alpha V}$ nonlinearly.

For the purpose of residual generation, the following observer is used

$$
\begin{aligned}
\begin{array}{c} \hat{\beta}(k+1) \\ \hat{\gamma}(k+1) \end{array} &= A_d \begin{array}{c} \hat{\beta}(k) \\ \hat{\gamma}(k) \end{array} + B_d \delta_L^*(k) + L(\gamma - \hat{\gamma}) \\
r &= \gamma - \hat{\gamma}
\end{aligned}
$$

with L as design parameter.

Assume that $\epsilon = 0.01$, $r = 0.001$, $N = 1000$. For $b_1 = 30000$, the minimal achievable α is 0.22 and the resulting $L = \left[\begin{array}{c} 0.3241 \\ 0.8874 \end{array} \right]$, which has been verified by 30000 random samples of $\Delta C_{\alpha V}$ uniformly distributed in $[-30000, 0]$ after the design. The design procedure is also carried out under other values of b_1. The results are omitted due to the limitation of space. We would like to emphasize that since the selection of ε, r, N will influence the convergence rate [16], the achieved minimal α is only sub-optimal.

V. CONCLUSION

This paper studies the fault detection problem of uncertain linear systems with arbitrary uncertainty structure. With the aid of probabilistic robustness technique, an algorithm is developed to determine the parameter of observer-based residual generators. The results can be extended to handle systems with both multiplicative uncertainty and unknown disturbances. Future study will be focused on the multi-objective design of observer-based fault detection systems directly guaranteeing specified false alarm rate and miss detection rate.

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