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Probabilistic Model Validation Problems with $\mathcal{H}_\infty$ Type Uncertainties*

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***Abstract*—A mixed deterministic/probabilistic model validation problem is investigated in this paper, which consists in an additive uncertain model with uncertainty characterized by the $\mathcal{H}_\infty$ norm, and time-domain experimental data corrupted by a Gaussian noise sequence. Our aim is to compute the probability for such an uncertain model to be invalidated by the data, and our main results are bounds on this probability that are computable based on the statistics of chi-square random variables.**

Keywords: Probabilistic model validation, uncertain model, $\mathcal{H}_\infty$ norm, Gaussian noise.

I. Introduction

Model validation construes an important step in the general engineering modelling process that seeks to assess the validity of a model based on experimental data. This step typically checks whether a theoretical model can regenerate the actual data, and it has for long been a well-accepted notion in modelling and system identification. In particular, there has been especially noteworthy effort devoted over the last decade, to problems that are concerned with the validation or invalidation of uncertain models (see, e.g., [4]). Notable results include time- and frequency-domain invalidation tests [2, 3, 5, 9–13], for uncertain models characterized either by the $\mathcal{H}_\infty$ norm [2, 3, 5, 9–13] or the ℓ_1 norm [7, 10]. For a rather broad class of models, whether with an additive or an LFT (linear fractional transform) uncertainty description, these results provide clear, definitive answers on whether a given model is invalidated, whereas under deterministic hypotheses prescribed on the model uncertainty and measurement noise, such answers are obtained by solving linear matrix inequalities.

There has also been an alternative approach that seeks to assess model validity using probabilistic measures, attempting to determine the probability for a given model to be validated or invalidated. While in such a framework it remains possible to quantify the model uncertainty via deterministic measures such as $\mathcal{H}_\infty$ and ℓ_1 norms, the measurement noise is given instead a probabilistic description. The question then concerns, rather than a decisive answer, but the likelihood that a model may exist so that it satisfies the given model structure and uncertainty bound, and fits the measurement data corrupted by random noises with the prescribed probabilistic description. It has been argued that such probabilistic measures can provide useful, complementary information on the model fidelity, and for that reason have been pursued by a number of researchers. The existing results range from *Monte Carlo* type algorithms [6] to probability bounds [8,14,15], for uncertainties measured by the $\mathcal{H}_\infty$ and ℓ_1 norm, using either time- or frequency-domain data.

This paper continues to develop probabilistic means for the validation and invalidation of uncertain models. We adopt a formulation in the same spirit as alluded above, in which the modelling uncertainty is characterized deterministically by the $\mathcal{H}_\infty$ norm, while the noise is modelled as an *i.i.d.* sequence of Gaussian random variables. We assume that the uncertain model is described by an additive structure, and the available experimental data are time-series input-output measurements. Our goal is to develop computable probabilistic bounds that can be used to predict the likelihood that the model may be invalidated by the data. On the other hand, due to the mixed nature of deterministic/probabilistic descriptions, the exact computation of the probability for invalidation appears to be rather difficult a problem and remains open.

II. Notation and Problem Formulation

Throughout this paper, let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}^n$ the space of n-dimensional real vectors, and $\mathbb{R}^{m\times n}$ the space of $m \times n$ real matrices. Similarly, let $\mathbb{C}$ denote the set of complex numbers, $\mathbb{C}^n$ the space of n-dimensional complex vectors, and $\mathbb{C}^{m\times n}$ the space of

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$m \times n$ complex matrices. For any $x \in \mathbb{C}^n$, denote by x^T its transpose. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and define the normed space

$$\mathcal{H}_\infty^{m\times n} = \Big\{F(z) : \mathbb{D} \to \mathbb{C}^{m\times n} | F(z) \text{ analytic in } \mathbb{D} \text{ and } \|F(z)\| = \sup_{z\in\mathbb{D}} \bar{\sigma}(F(z)) < \infty\Big\},$$

where $\bar{\sigma}(\cdot)$ stands for the largest singular value. We shall write $\mathcal{H}_\infty^{m\times n}$ as $\mathcal{H}_\infty$ when the dimension is clear from the context. The $\Lambda-$ transform of a sequence $\{\Delta(k)\}_{k=0}^\infty$ is defined as

$$\hat{\Delta}(\lambda) = \sum_{k=0}^{\infty} \Delta(k)\lambda^k.$$

Furthermore, for a matrix $A \in \mathbb{R}^{m\times n}$, its Frobenius norm is defined as

$$\|A\|_F = \sqrt{\text{trace}(A^T A)} = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2},$$

and S-norm as

$$\|A\|_S = \sqrt{\max_i \sum_{j=1}^{n} |a_{ij}|^2}.$$

For a square matrix $A \in \mathbb{R}^{n\times n}$, we write $A \geq 0$ if A is nonnegative definite. For a sequence $\{x(k)\}_{i=0}^{N-1}$, we use T_x to denote the lower triangular Toeplitz matrix

$$T_x = \begin{bmatrix} x(0) & 0 & \cdots & 0 \\ x(1) & x(0) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x(N-1) & x(N-2) & \cdots & x(0) \end{bmatrix}.$$

Similarly, for a causal, linear shift-invariant system whose impulse response is given by the sequence $\{M(k)\}_{k=0}^\infty$, where $M(k) \in \mathbb{R}^{m\times n}$, we define a corresponding block lower Toeplitz matrix T_M by

$$T_M = \begin{bmatrix} M(0) & 0 & \cdots & 0 \\ M(1) & M(0) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ M(N-1) & M(N-2) & \cdots & M(0) \end{bmatrix}.$$

Let π_N be the truncation operator such that

$$\pi_N x = \begin{bmatrix} x(0) & x(1) & \cdots & x(N-1) \end{bmatrix}^T.$$

The symbol $\mathcal{P}\{A\}$ denotes the probability for the event A, $\xi \sim \mathcal{N}(\mu, \Lambda)$ indicates a random vector ξ which is Gaussian distributed and its mean and covariance are respectively μ and Λ, $\zeta \sim \lambda(a, r)$ represents a gamma distributed random variable with shape parameter a and scale parameter r, and finally $\chi_N(x, \delta)$ returns the noncentral chi-square cumulative distribution function (cdf) with N degrees of freedom and noncentrality parameter δ, at the value x.

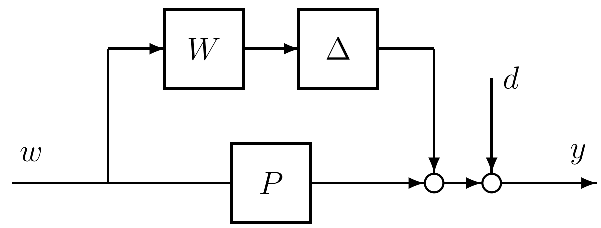

Fig. 1. *Additive uncertain model*

We consider the class of causal, BIBO stable, linear time-invariant (LTI) systems. The additive uncertain model for the system is depicted in Fig. 1 with the input-output relationship

$$y = (P + \Delta W) * w + d \tag{1}$$

where w is an input signal, y is the output, d denotes the measurement noise, P the nominal model, Δ the modeling uncertainty and $*$ is the convolution operator. We formulate a time domain mixed deterministic/probabilistic model validation problem below.

Mixed Model Validation Problem *Given time-series input-output* $\{w(k)\}_{i=0}^{N-1}$ *and* $\{y(k)\}_{i=0}^{N-1}$, *given also* P, W, *and a noise description* Θ, *what is the model validation probability for* $\Delta \in \mathbf{\Delta}$, *such that*

$$\pi_N y = \pi_N[(P + \Delta W) * w] + \pi_N d, \tag{2}$$

for some $\{d(k)\}_{i=0}^{N-1} \in \Theta$, *and for some* $\gamma > 0$,

$$\mathbf{\Delta} = \{\Delta : \hat{\Delta} \in \mathcal{H}_\infty,\ \|\hat{\Delta}\|_\infty \leq \gamma\}? \tag{3}$$

We shall assume that the noise sequence $\{d(k)\}_{i=0}^{N-1}$ is a *i.i.d.* sequence of standard normal random variables, that is,

$$\Theta = \{d : d \sim \mathcal{N}(0, I)\}. \tag{4}$$

We denote the model validation probability as $\phi_M(\gamma)$:

$$\phi_M(\gamma) = \mathcal{P}\big\{\Delta \in \mathbf{\Delta} : \pi_N y = \pi_N[(P + \Delta W) * w] + \pi_N d,\ d \sim \Theta\big\}. \tag{5}$$

Remark 1 The assumption (4) on the noise d does not compromise the generality of Gaussian distributed noises. Indeed, consider the general Gaussian distribution $d \sim \mathcal{N}(\mu, \Sigma^T\Sigma)$. Let $d' = (\Sigma^T\Sigma)^{-\frac{1}{2}}(d - \mu)$. Then $d' \sim \mathcal{N}(0, I)$. Now substitute d by d' in (1), we obtain

$$y' = (P + \Delta W) * w' + d',$$

with $y' = (\Sigma^T\Sigma)^{-\frac{1}{2}}(y - \mu)$ and $w' = (\Sigma^T\Sigma)^{-\frac{1}{2}}w$.

III. Main Results

From analytic function interpolation theory [1], it is known that for a given sequence $\{\Delta(k)\}_{k=0}^{N-1}$ of matrices $\Delta(k) \in \mathbb{R}^{m\times n}$, there exists a $\hat{\Delta} \in \mathcal{H}_\infty$ such that $\|\hat{\Delta}\|_\infty \leq \gamma$ and

$$\hat{\Delta}(z) = \sum_{k=0}^{N-1} \Delta(k) z^k + \mathcal{O}(z^N)$$

if and only if

$$T_\Delta^T T_\Delta \leq \gamma^2 I. \tag{6}$$

This result, known as the Carathéodory-Fejér condition, has been used widely in deterministic model invalidation tests, and will form the basis of our development in the present paper. Indeed, for a given $\gamma > 0$, define $U(\gamma) = \{\Delta : T_\Delta^T T_\Delta \leq \gamma^2 I\}$. Then if Δ is given a probabilistic description, the probability for such a $\hat{\Delta}$ to exist is $\mathcal{P}\{\Delta \in U(\gamma)\}$, which in principle can be computed by integrating the joint density function of the random variables $\Delta(k), k = 0, \cdots, N-1$, over the set defined by (6). It is not difficult to see that the model validation probability $\phi_M(\gamma)$ amounts to a similar computation. Unfortunately, this computation poses rather formidable a task.

In light of the aforementioned computational difficulty, in this paper we seek bounds on $\phi_M(\gamma)$. Define the sets $P(\gamma) = \{\Delta : \|T_\Delta\|_F \leq \gamma\}$ and $Q(\gamma) = \{\Delta : \|T_\Delta\|_S \leq \gamma\}$. One can immediately see that

$$P(\gamma) \subseteq U(\gamma) \subseteq Q(\gamma).$$

Accordingly, it follows that

$$\mathcal{P}\{\Delta \in P(\gamma)\} \leq \mathcal{P}\{\Delta \in U(\gamma)\} \leq \mathcal{P}\{\Delta \in Q(\gamma)\}. \tag{7}$$

The idea then is to compute $\mathcal{P}\{\Delta \in P(\gamma)\}$ and $\mathcal{P}\{\Delta \in Q(\gamma)\}$. For this purpose, we need to characterize the density functions defined by the multivariate functions $\|T_\Delta\|_F$ and $\|T_\Delta\|_S$.

A. A Weighted Chi-Square Problem

We show that with $\{\Delta(k)\}_{k=0}^{N-1}$ as an *i.i.d.* sequence of Gaussian random variables, the function $\|T_\Delta\|_F$ defines a weighted chi-square random variable. We derive the distribution function of this random variable by solving the following problem.

Weighted Chi-square Problem *Let $x = Nx_1^2 + (N-1)x_2^2 + \cdots + x_N^2$, where $\{x_i\}_{i=1}^N$ are independent and Gaussian distributed, with $x_i \sim \mathcal{N}(0,1)$. What is the probability of $\mathcal{P}\{x \leq \gamma\}$?*

For $a > 0$, denote

$$\Gamma(a) = \int_0^\infty t^{a-1} e^{-t}\ \mathrm{d}t.$$

Lemma 1 *For any $a > 0$, $b > 0$,*

$$\int_0^x \tau^{a-1}(x-\tau)^{b-1}\ \mathrm{d}\tau = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} x^{a+b-1}.$$

Note that for a gamma distributed variable x, $x \sim \lambda(a,r)$, $a > 0, r > 0$, the probability density function (pdf) of x is

$$f_{\lambda(a,r)}(x) = \frac{r^a}{\Gamma(a)} x^{a-1} e^{-rx}$$

and its cumulative distribution function

$$F_{\lambda(a,r)}(\gamma) = \int_0^\gamma \frac{r^a}{\Gamma(a)} x^{a-1} e^{-rx}\ \mathrm{d}x.$$

Also note that if $x = cx_1^2$, where $c > 0$ and $x_1 \sim \mathcal{N}(0,\sigma^2)$, then $x \sim \lambda(\frac{1}{2}, \frac{1}{2c\sigma^2})$.

Lemma 2 *If $x = x_1 + x_2$, where $x_1 \sim \lambda(a_1, r_1)$, $x_2 \sim \lambda(a_2, r_2)$, with $a_1 + a_2 \geq 1$ and $r_2 \geq r_1$, then the pdf of x is*

$$f(x) = (\frac{r_1}{r_2})^{a_1} \sum_{k=0}^\infty f_{\lambda(a_1+a_2+k, r_2)}(x)(1-\frac{r_1}{r_2})^k \frac{\Gamma(a_1+k)}{\Gamma(a_1)k!}. \tag{8}$$

Remark 2 In the case of $r_1 = r_2 = r$, $f(x) = f_{\lambda(a_1+a_2,r)}(x)$. More specifically, if $x = x_1^2 + x_2^2 + \cdots + x_N^2$, where $\{x_i\}_{i=1}^N$ are independent and $x_i \sim \mathcal{N}(0,1)$, we have $x_i^2 \sim \lambda(\frac{1}{2}, \frac{1}{2})$. Then by induction,

$$f(x) = f_{\lambda(\frac{N}{2},\frac{1}{2})}(x),$$

which implies that x is chi-square distributed with N degrees of freedom.

Lemma 3 *The series $\sum_{k=0}^\infty (\frac{r_1}{r_2})^{a_1} f_{\lambda(a_1+a_2+k,r_2)}(x)(1-\frac{r_1}{r_2})^k \frac{\Gamma(a_1+k)}{\Gamma(a_1)k!}$ is uniformly convergent to $f(x)$.*

Lemma 4 *For $a_1 > 0$ and $r_2 \geq r_1 > 0$,*

$$\sum_{k=0}^\infty (1-\frac{r_1}{r_2})^k \frac{\Gamma(a_1+k)}{\Gamma(a_1)k!} = (\frac{r_1}{r_2})^{-a_1}. \tag{9}$$

Lemma 5 *If $x = x_1 + x_2 + \cdots + x_N$, where $x_i \sim \lambda(a_i, r_i)$, $i = 1, \cdots, N$, with $a_i > 0$ and $r_N \geq \cdots \geq r_1$, then the pdf of x is*

$$\begin{aligned} f\,(x) = (\frac{r_1}{r_2})^{a_1} \sum_{k_1=0}^\infty \Bigg(\cdots \Bigg((\frac{r_{N-1}}{r_N})^{a_1+\cdots+a_{N-1}+S_{N-2}} \\ \times \sum_{k_{N-1}=0}^\infty f_{\lambda(a_1+\cdots+a_N+S_{N-1}, r_N)}(x)(1-\frac{r_{N-1}}{r_N})^{k_{N-1}} \\ \times \frac{\Gamma(a_1+\cdots+a_{N-1}+S_{N-1})}{\Gamma(a_1+\cdots+a_{N-1}+S_{N-2})k_{N-1}!} \Bigg) \cdots \Bigg) \\ \times (1-\frac{r_1}{r_2})^{k_1} \frac{\Gamma(a_1+k_1)}{\Gamma(a_1)k_1!}, \end{aligned}$$

where $S_j = \sum_{i=1}^{j} k_i$.

Theorem 1 *Let $F_x(\gamma) = \mathcal{P}\{x \leq \gamma\}$, where x is defined in the weighted chi-square problem. Then we have*

$$\begin{aligned} F_x(\gamma) = (\frac{N-1}{N})^{\frac{1}{2}} \sum_{k_1=0}^{\infty} \Bigg(\cdots \Bigg((\frac{1}{2})^{\frac{N-1}{2}+S_{N-2}} \\ \times \sum_{k_{N-1}=0}^{\infty} F_{\lambda(\frac{N}{2}+S_{N-1},\frac{1}{2})}(\gamma)(\frac{1}{2})^{k_{N-1}} \\ \times \frac{\Gamma(\frac{N-1}{2}+S_{N-1})}{\Gamma(\frac{N-1}{2}+S_{N-2})k_{N-1}!} \Bigg) \cdots \Bigg) \\ \times (\frac{1}{N})^{k_1} \frac{\Gamma(\frac{1}{2}+k_1)}{\Gamma(\frac{1}{2})k_1!}. \end{aligned} \tag{10}$$

Proof. Let $a_i = \frac{1}{2}, r_i = \frac{1}{2(N+1-i)}, i = 1, \cdots, N$. ∎

Lemma 6 *$F_{\lambda(a,r)}(\gamma)$ is nonincreasing with respect to a. Furthermore, as $a \to \infty$, $F_{\lambda(a,r)}(\gamma) \to 0$.*

The above lemma suggests that $F_x(\gamma)$ in Theorem 1 can be approximated, and hence computed, to arbitrary accuracy. The following result provides a quantitative bound for this approximation.

Theorem 2 *Let $F_x(\gamma) = \mathcal{P}\{x \leq \gamma\}$, where x is defined in the weighted chi-square problem. For any integer M, $M > 0$, if $F_{\lambda(\frac{N}{2}+M,\frac{1}{2})}(\gamma) \leq \epsilon$, then*

$$0 \leq F_x(\gamma) - F_x^M(\gamma) \leq (N-1)\epsilon,$$

where

$$\begin{aligned} F_x^M(\gamma) = (\frac{N-1}{N})^{\frac{1}{2}} \sum_{k_1=0}^{M} \Bigg(\cdots \Bigg((\frac{1}{2})^{\frac{N-1}{2}+S_{N-2}} \\ \times \sum_{k_{N-1}=0}^{M} F_{\lambda(\frac{N}{2}+S_{N-1},\frac{1}{2})}(\gamma)(\frac{1}{2})^{k_{N-1}} \\ \times \frac{\Gamma(\frac{N-1}{2}+S_{N-1})}{\Gamma(\frac{N-1}{2}+S_{N-2})k_{N-1}!} \Bigg) \cdots \Bigg) \\ \times (\frac{1}{N})^{k_1} \frac{\Gamma(\frac{1}{2}+k_1)}{\Gamma(\frac{1}{2})k_1!}. \end{aligned}$$

B. SISO Models

We first consider a SISO model with a simple input-output relationship

$$T_v = T_\Delta T_u + T_d$$

where u and v are known. Our task is to calculate the probability $\phi(\gamma)$, defined as

$$\phi(\gamma) = \mathcal{P}\{\Delta \in U(\gamma) : T_d = T_v - T_\Delta T_u, d \sim \mathcal{N}(0, I)\}.$$

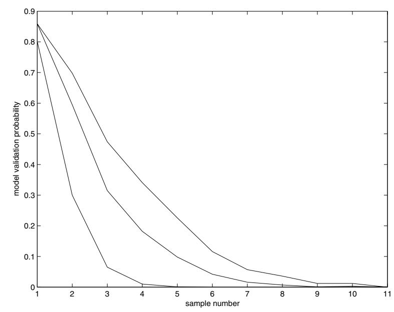

Fig. 2. *Model validation probability and bounds*

Lemma 7 *Suppose that $u(0) \neq 0$. Then*

$$\phi(\gamma) \geq F_x(\bar{\gamma}),$$

with $\bar{\gamma} = \max\{0, \frac{\gamma - \|T_v T_u^{-1}\|_F}{\|T_u^{-1}\|}\}$.

Proof. By (7),

$$\begin{aligned} \phi(\gamma) &\geq \mathcal{P}\{\Delta \in P(\gamma) : T_d = T_v - T_\Delta T_u, d \sim \mathcal{N}(0, I)\} \\ &= \mathcal{P}\{\|(T_v - T_d)T_u^{-1}\|_F \leq \gamma : d \sim \mathcal{N}(0, I)\}. \end{aligned}$$

Meanwhile,

$$\|(T_v - T_d)T_u^{-1}\|_F \leq \|T_v T_u^{-1}\|_F + \|T_u^{-1}\|\|T_d\|_F$$

then

$$\begin{aligned} &\{\|(T_v - T_d)T_u^{-1}\|_F \leq \gamma : d \sim \mathcal{N}(0, I)\} \\ \supseteq\ &\{\|T_v T_u^{-1}\|_F + \|T_u^{-1}\|\|T_d\|_F \leq \gamma : d \sim \mathcal{N}(0, I)\} \\ =\ &\{\|T_d\|_F \leq \frac{\gamma - \|T_v T_u^{-1}\|_F}{\|T_u^{-1}\|} : d \sim \mathcal{N}(0, I)\}. \end{aligned}$$

Let $\bar{\gamma} = \max\{0, \frac{\gamma - \|T_v T_u^{-1}\|_F}{\|T_u^{-1}\|}\}$, and $\phi_P(\bar{\gamma}) = \mathcal{P}\{\|T_d\|_F \leq \bar{\gamma} : d \sim \mathcal{N}(0, I)\}$. It is obvious $\phi(\gamma) \geq \phi_P(\bar{\gamma})$. Since

$$\|T_d\|_F = Nd^2(0) + (N-1)d^2(1) + \cdots + d^2(N-1),$$

$\phi_P(\bar{\gamma}) = F_x(\bar{\gamma})$ where $F_x(\cdot)$ is defined in (10). ∎

Lemma 8 *Suppose that $u(0) \neq 0$. Then*

$$\phi(\gamma) \leq \chi_N(\tilde{\gamma}^2, \|T_v\|_S^2),$$

with $\tilde{\gamma} = \gamma\|T_u\|_S$.

Proof. As in Lemma 7, we also have

$$\begin{aligned} \phi(\gamma) &\leq \mathcal{P}\{\Delta \in Q(\gamma) : T_d = T_v - T_\Delta T_u, d \sim \mathcal{N}(0, I)\} \\ &= \mathcal{P}\{(\|T_v - T_d)T_u^{-1}\|_S \leq \gamma : d \sim \mathcal{N}(0, I)\} \\ &\leq \mathcal{P}\{\frac{\|T_v - T_d\|_S}{\|T_u\|_S} \leq \gamma : d \sim \mathcal{N}(0, I)\} \\ &= \mathcal{P}\{\|T_v - T_d\|_S \leq \gamma\|T_u\|_S : d \sim \mathcal{N}(0, I)\}. \end{aligned}$$

Let $\tilde{\gamma} = \gamma\|T_u\|_S$. From the definition of $\|\cdot\|_S$,

$$\|T_v - T_d\|_S^2 = \sum_{k=0}^{N-1}(v(k) - d(k))^2.$$

It is not difficult to see $\sum_{k=1}^{N}(v(k) - d(k))^2$ is non-central chi-square distributed with N degrees of freedom and non-centrality parameter $\|T_v\|_S^2$. Hence, $\phi(\gamma) \leq \chi_N(\tilde{\gamma}^2, \|T_v\|_S^2)$. ■

Example Based on the above two lemmas, we provide a numerical example to demostrate the result. In this example, we choose $\gamma = 1$, and randomly generate some input-output data u and v. The upper and lower bounds, shown in Fig.2, are calculated for from 1 to 11 samples. The middle curve is the real model validation probability $\phi(\gamma)$, which is approximated by *Monte Carlo* method.

We may now state the following bounds for the model validation probability $\phi_M(\gamma)$.

Theorem 3 *Assume that $w(0) \neq 0$ and $W(0) \neq 0$. Then, the model validation probability $\phi_M(\gamma)$ satisfies the following bounds*

$$F_x(\bar{\gamma}) \leq \phi_M(\gamma) \leq \chi_N(\tilde{\gamma}^2, \|T_y - T_P T_w\|_S^2),$$

with $\bar{\gamma} = \max\{0, \frac{\gamma - \|(T_y - T_P T_w)T_w^{-1}T_W^{-1}\|_F}{\|T_w^{-1}T_W^{-1}\|}\}$ and $\tilde{\gamma} = \gamma\|T_W T_w\|_S$.

Proof. It follows by setting $u = T_W w$ and $v = y - T_P w$. ■

For given data, the upper bound in Theorem 3 can be calculated immediately. On the other hand, a lower bound can be computed in light of Theorem 2.

C. MIMO Models

For an MIMO system, we also consider first the simplified model

$$T_v = T_\Delta T_u + T_d,$$

where $v(k) \in \mathbb{R}^m$, $u(k) \in \mathbb{R}^n$, $\Delta(k) \in \mathbb{R}^{m\times n}$ and $d(k) \in \mathbb{R}^m$, $k = 0, \cdots, N-1$.

We assume $d \sim \mathcal{N}(0, I)$, *i.e.*, $\{d(k)\}_{k=0}^{N-1}$ are independent and $d(k)$ is standard normal distributed. Suppose that n *i.i.d* experiments have been carried out for this system. As a result, for jth experiment,

$$T_{v^j} = T_\Delta T_{u^j} + T_{d^j}, \text{ for } j = 1, \cdots, n.$$

Write it in a compact form as

$$\tilde{V} = T_\Delta \tilde{U} + \tilde{D} \tag{11}$$

where $\tilde{V} = [T_{v^1} \ \cdots \ T_{v^n}]$, $\tilde{U} = [T_{u^1} \ \cdots \ T_{u^n}]$ and $\tilde{D} = [T_{d^1} \ \cdots \ T_{d^n}]$. Define the matrix $E_{nN} = [r_1 \ \cdots \ r_{nN}]$, where $r_k \in \mathbb{R}^{nN}$, $k = 1, \cdots, nN$, are constructed as

$$r_k = \begin{cases} e_{(k-1)N+1} & k = 1, \ \cdots, \ n \\ e_{(k-n-1)N+2} & k = n+1, \ \cdots, \ 2n \\ \cdots & \\ e_{(k-n(N-1)-1)N+N} & k = n(N-1)+1, \ \cdots, \ Nn \end{cases}$$

with e_i being the ith Euclidean coordinate of $\mathbb{R}^{nN}$. Multiplied by E_{nN} in both sides, equation (11) is then

$$T_V = T_\Delta T_U + T_D,$$

with $V = [v^1 \ \cdots \ v^n]$, $U = [u^1 \ \cdots \ u^n]$ and $D = [d^1 \ \cdots \ d^n]$.

It is shown in [7] that the matrix T_U is of full rank if and only if the $n \times n$ matrix $\begin{bmatrix} u_1^1(0) & \cdots & u_1^n(0) \\ \vdots & \ddots & \vdots \\ u_n^1(0) & \cdots & u_n^n(0) \end{bmatrix}$ is of full rank.

Lemma 9 *Let T_U be of full rank. Then,*

$$\phi(\gamma) \geq F_x(\bar{\gamma})$$

where $\bar{\gamma} = \max\{0, \frac{\gamma - \|T_V T_U^{-1}\|_F}{\|T_U^{-1}\|}\}$ and

$$F_x(\gamma) = (\frac{N-1}{N})^{\frac{n^2}{2}} \sum_{k_1=0}^{\infty}\Bigg(\cdots\Bigg((\frac{1}{2})^{\frac{N-1}{2}n^2 + S_{N-2}} \times \sum_{k_{N-1}=0}^{\infty} F_{\lambda(\frac{N}{2}n^2 + S_{N-1}, \frac{1}{2})}(\gamma)(\frac{1}{2})^{k_{N-1}} \times \frac{\Gamma(\frac{N-1}{2}n^2 + S_{N-1})}{\Gamma(\frac{N-1}{2}n^2 + S_{N-2})k_{N-1}!}\Bigg)\cdots\Bigg) \times (\frac{1}{N})^{k_1}\frac{\Gamma(\frac{n^2}{2} + k_1)}{\Gamma(\frac{n^2}{2})k_1!}.$$

Proof. The Frobenius norm of T_D is given by

$$\|T_D\|_F = \sum_{k=0}^{N-1}\left((N-k)\sum_{i=1}^{n}\sum_{j=1}^{n}(d_i^j(k))^2\right).$$

Since $d \sim \mathcal{N}(0, I)$ and

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(d_i^j(k))^2 \sim \lambda(\frac{n^2}{2}, \frac{1}{2}),$$

the proof is then completed as in that for Lemma 7. ■

Lemma 10 *Let T_U be of full rank. Then,*

$$\phi(\gamma) \leq \prod_{j=1}^{n} \chi_{nN}(\tilde{\gamma}^2, \|T_{v^j}\|^2)$$

with $\tilde{\gamma} = \gamma \|T_U\|_S$.

Proof. Let $\tilde{\gamma} = \gamma \|T_U\|_S$. Due to the fact

$$\|T_D\|_S = \max_j \sum_{i=1}^{n} \sum_{k=0}^{N-1} (v_i^j(k) - d_i^j(k))^2$$

and $d \sim \mathcal{N}(0, I)$,

$$\begin{aligned}\phi(\gamma) &\le \mathcal{P}\{\|T_V - T_D\|_S \le \gamma \|T_U\|_S : D \sim \mathcal{N}(0, I)\} \\ &= \mathcal{P}\{\max_j \sum_{i=1}^{n} \sum_{k=0}^{N-1} (v_i^j(k) - d_i^j(k))^2 \le \tilde{\gamma} : \\ &\qquad\qquad D \sim \mathcal{N}(0, I)\} \\ &= \prod_{j=1}^{n} \chi_{nN}(\tilde{\gamma}^2, \|T_{v^j}\|^2).\end{aligned}$$

■

In a similar manner, we may establish the following bounds for $\phi_M(\gamma)$.

Theorem 4 *If* $W(0) \ne 0$ *and* $w^j(0) \ne 0$, $j = 1, \cdots, n$, *then the model validation probability* $\Phi_M(\gamma)$ *satisfies the following bounds*

$$F_x(\bar{\gamma}) \le \phi_M(\gamma) \le \prod_{j=1}^{n} \chi_{nN}(\tilde{\gamma}^2, \|T_{v^j}\|^2),$$

with $\bar{\gamma} = \max\{0, \frac{\gamma - \|T_V T_U^{-1}\|_F}{\|T_U^{-1}\|}\}$ *and* $\tilde{\gamma} = \gamma \|T_U\|_S$, *where* $U = \begin{bmatrix} T_W w^1 & \cdots & T_W w^n \end{bmatrix}$ *and* $V = \begin{bmatrix} y^1 - T_P w^1 & \cdots & y^n - T_P w^n \end{bmatrix}$.

Proof. It follows by setting $u^j = T_W w^j$ and $v^j = y^j - T_P w^j$. ■

IV. Conclusion

In this paper, we have studied a mixed deterministic/probabilistic model validation problem . We are interested in an additive model structure with the modelling uncertainty characterized by $\mathcal{H}_\infty$ norm, and we assume that the time-series input-output data are available for validation, subject to noisy observations characterized by a *i.i.d.* Gaussian sequence. We derived computable lower and upper bounds that determine the probability for the uncertain model t explain the data, under the prescribed uncertainty bound and noise description.

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