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Risk Analysis in Robust Control — Making the Case for Probabilistic Robust Control

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***Abstract*— This paper offers a critical view of the "worst-case" approach that is the cornerstone of robust control design. It is our contention that a blind acceptance of worst-case scenarios may lead to designs that are actually more dangerous than designs based on probabilistic techniques with a built-in risk factor. The real issue is one of modeling. If one accepts that no mathematical model of uncertainties is perfect then a probabilistic approach can lead to more reliable control even if it cannot guarantee stability for all possible cases. Our presentation is based on case analysis. We first establish that worst-case is not necessarily "all-encompassing." In fact, we show that for some uncertain control problems to have a conventional robust control solution it is necessary to make assumptions that leave out some feasible cases. Once we establish that point, we argue that it is not uncommon for the risk of unaccounted cases in worst-case design to be greater than that of the accepted risk in a probabilistic approach. With an example, we quantify the risks and show that worst-case can be significantly more risky. Finally, we argue that the deterministic worst-case analysis is not necessary more reliable than the probabilistic analysis.**

1. INTRODUCTION

In recent years, a number of researchers have proposed probabilistic control methods as alternatives for overcoming the computational complexity and conservatism of deterministic worst-case robust control framework (e.g., [1]–[17], [20]–[24] and the references therein). The philosophy of probabilistic control theory is to sacrifice extreme cases of uncertainty. Such paradigm has lead to the novel concepts of probabilistic robustness margin and confidence degradation function (e.g., [1]). Despite the claimed advantages of probabilistic approach, the deterministic worst-case approach remains dominating for design and analysis purposes. It is a common contention that a probabilistic design is more risky than a worst-case design. Such a contention may have been the main obstacle preventing the wide acceptance of the probabilistic paradigm, especially in the development of highly reliable systems. When referring to the probabilistic approach, a cautious warning is usually attached. Statements like "if one is willing to accept a small risk of performance violation" can be found in a number of robust control papers. A typical argument is that the worst-case method takes every case of "uncertainty" into account and is certainly the most safe, while the probabilistic method considers only most of the instances of "uncertainty" and, hence, is more risky.

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We illustrate with two very simple cases that a worst case design may not necessarily consider all possible cases. Our purpose here is to make the point that the basic issue is one of modeling and as such it is never perfect. Worst-case scenario may mean "the worst case that we can imagine," or "the worst-case that we can afford to consider to have a robust solution."

In practice, the coefficients of a linear model are complex functions of physical parameters. Even if the physical parameters are bounded in a narrow interval, the variations of the coefficients can be fairly large. A simple example is provided by the process of discharge of a cylindrical tank. The nonlinear model is of the form

$$A\frac{dH}{dt} + \rho\sqrt{H} = Q_i$$

where A is the tank cross section, H the height of liquid inside the tank, Q_i the volumetric input flow rate and ρ the hydraulic resistance in the discharge. Linearizing in the neighborhood of a steady state operating point, $(\overline{H}, \overline{Q}_i)$, satisfying $\rho\sqrt{\overline{H}} = \overline{Q_i}$ one obtains the linear model

$$\frac{dh}{dt} + \frac{\rho}{2A\sqrt{\overline{H}}}h = \frac{q_i}{A}$$

with $h = H - \overline{H}, q_i = Q_i - \overline{Q}_i$. Clearly, the parameter $a = \frac{\rho}{2A\sqrt{\overline{H}}}$ takes values in the interval $[0, \infty)$. Hence, any design assuming bounded uncertainties for the parameter, a, cannot include all possible heights for the liquid.

For the second case consider a first order system of the form $G(s) = \frac{q}{s-p}$ with uncertain parameters q and p. Assuming a unity feedback and controller of the form $C(s) = \frac{K}{s+a},\ K > 0,\ a > 0$, one can show that the controller will robustly stabilize the plant if $p < a,\ q > \frac{ap}{K}$. It is clear that for any finite controller gain, K, there exists a range of values of q where the closed-loop system is unstable. The designer of a worst-case controller would be faced with the choice of selecting a different controller structure or assuming, based on other considerations, that a neighborhood of $q = 0$ can be excluded from the design. With the next result we develop this point in a more general form.

A. Uncertainties in Modeling Uncertainties

In many practical situations of worst-case design one models uncertainties as bounded random variables. The issue of selecting the bounds is not trivial and is, oftentimes, not addressed in detail. The following theorem shows that, regardless of the assumed size of the uncertainty set, a worst-case robust controller actually can always fail. Hence, if there are "uncertainties in modeling the uncertainties" it may be better to model them as random variables varying from $-\infty$ to ∞ in order to pursue "worst-case" in a strict sense.

Theorem 1: Let the transfer function of the uncertain plant be

$$G(s) = \frac{\sum_{i=0}^{\ell} \beta_i s^{\ell-i}}{\sum_{i=0}^{\kappa} \alpha_i s^{\kappa-i}}, \qquad \alpha_0 = 1, \qquad \ell \leq \kappa.$$

Assume that for a given finite uncertainty range in the parameters $\alpha_i,\ i = 1, \cdots, \kappa$ and $\beta_j,\ j = 0, 1, \cdots, \ell$, there exists a controller of the form

$$C(s) = \frac{\sum_{i=0}^{m} b_i s^{m-i}}{\sum_{i=0}^{n} a_i s^{n-i}}, \quad a_0 = 1, \quad b_0 \neq 0, \quad m \leq n$$

which robustly stabilizes the system. Then, there always exists a value of parameter α_i or β_j, outside the assumed uncertainty range and which will make the closed-loop system unstable.

Remark 1: A proof for an equivalent result for a multi-variable plant may be feasible. However, the following general argument conveys the idea about the limitations in worst case design. Let E^n be the set of all instances of uncertainty. Let $G(s, q),\ q \in \mathcal{B} \subset E^n$ be the model for an uncertain plant. Assume that there exists a controller C_w that satisfies the robustness requirements for all $q \in \mathcal{B}$. Define now as $\mathcal{D}$ the set of all values of the parameter q where the controller C_w satisfies the robustness requirements. Clearly $\mathcal{D}_w \supset \mathcal{B}$ but unless $\mathcal{D}_w = E^n$ there always exist values of the parameter q where the controller does not perform. The worst-case design ignores these cases as impossible. Our contention is that the modeling of uncertainties (the set $\mathcal{B}$) may not include all cases that could occur and it may be better to accept a risk from the onset of the design.

2. Accepting Risk Can Be Less Risky

The previous result makes, very strongly, the point that worst-case modeling is not "all-encompassing" and therefore it has some risk associated to it. In this section we offer first a more formal description of the problem and argue that a probabilistic approach may easily lead to more reliable designs. The next section uses a case study to quantify the actual risks of both approaches.

A. Designing with Uncertain Uncertainties

We incorporate the fact that modeling is never exact by postulating an uncertainty set, $\mathcal{U}$ and a bounding set, $\mathcal{B}$, that models the uncertainties. The actual relationship between the two sets is not known. The worst-case design finds a controller C_w to guarantee every uncertainty instance $q \in \mathcal{B}$. The probabilistic design seeks a controller C_p to guarantee most uncertainty instances $q \in \mathcal{B}$. Formally we can define the following relevant subsets

$$\begin{aligned} \mathcal{M} &= \mathcal{U} \cap \mathcal{B}, &(1)\\ \mathcal{N} &= \overline{\mathcal{U}} \cap \mathcal{B}, &(2)\\ \mathcal{E} &= \mathcal{U} \cap \overline{\mathcal{B}}. &(3) \end{aligned}$$

Here $\overline{X}$ denotes the complementary set of X. Clearly, $\mathcal{M}$ contains those uncertainties that are modeled while the set $\mathcal{N}$ describes modeled uncertainties that never occur and $\mathcal{E}$ describes the unmodeled uncertainties. The existence of these last two sets creates either inefficiencies or risks in the worst case design. To see this, consider the extreme situation where the designed robust controller guarantees the robustness requirement only for instances in $\mathcal{B}$. The controller is over-designed because it deals with situations that cannot occur and it has the risk of failure if an instance in the set $\mathcal{E}$ arises.

Having established the fact that a robust control design can be risky, we now argue that probabilistic design can actually be less risky. As an added benefit, it is known that many worst-case synthesis problems are either not tractable, or have known solutions which are unduly conservative and expensive to implement. But when using a probabilistic method, the previously intractable problems may become solvable, the conservatism may be overcame, and high performance controller with simple structure may be obtained.

For brevity, we use notation C^V to represent the statement that "the robustness requirement is violated for the system with controller C". The subindex w will refer to worst-case design while p will refer to probabilistic design. Note that the risk of a probabilistic design is

$$\begin{aligned} P_e^p = {} & \Pr\{C_p^V \mid q \in \mathcal{M}\} \Pr\{q \in \mathcal{M}\} \\ & + \Pr\{C_p^V \mid q \in \mathcal{E}\} \Pr\{q \in \mathcal{E}\}. \end{aligned}$$

While the risk of a worst-case design is

$$P_e^w = \Pr\{C_w^V \mid q \in \mathcal{E}\} \Pr\{q \in \mathcal{E}\}.$$

Hence the ratio of risks will be

$$\frac{P_e^p}{P_e^w} = \frac{\Pr\{C_p^V \mid q \in \mathcal{E}\}}{\Pr\{C_w^V \mid q \in \mathcal{E}\}} + \frac{\Pr\{C_p^V \mid q \in \mathcal{M}\} \Pr\{q \in \mathcal{M}\}}{\Pr\{C_w^V \mid q \in \mathcal{E}\} \Pr\{q \in \mathcal{E}\}}. \quad (4)$$

The first term is related to the performance of both types of controllers outside the design region. The behavior of the worst-case design in this region is of no concern to the designer, after all it "never gets there." All the design effort is placed in assuring performance over the set $\mathcal{B}$. Hence we can reasonably expect $\Pr\{C_w^V \mid q \in \mathcal{E}\}$ to be high.

In fact, if the set $\mathcal{N}$, of impossible situations included in the design, is large then $\Pr\{C_w^V \mid q \in \mathcal{E}\}$ could be close to one and the first term in the right-hand side of (4) can easily be less than some number $\lambda \in (0,1)$. The second term contains the factor $\Pr\{C_p^V \mid q \in \mathcal{M}\}$ which is under the probabilistic designer and is a measure of the accepted risk. It is reasonable to expect that this risk is less than $\Pr\{q \in \mathcal{E}\}$ so that

$$\begin{aligned} &\frac{\Pr\{C_p^V \mid q \in \mathcal{M}\}\Pr\{q \in \mathcal{M}\}}{\Pr\{q \in \mathcal{E}\}} \\ &< (1-\lambda)\ \Pr\{C_w^V \mid q \in \mathcal{E}\} \end{aligned}$$

and consequently $\frac{P_e^p}{P_e^w} < 1$. Factoring in a poor performance for the worst-case design outside of the modeled region one can see that the risk of the probabilistic design can become smaller than the risk of a worst-case design with unmodeled uncertainties.

From a different point of view, many experiments of performance degradation of probabilistic designs indicate a fairly flat characteristic. If the unmodeled uncertainty set $\mathcal{E}$ is relatively small then one could argue that

$$\Pr\{C_p^V \mid q \in \mathcal{E}\} \approx \Pr\{C_p^V \mid q \in \mathcal{M}\}$$

and the ratio of risks is approximately given by

$$\frac{P_e^p}{P_e^w} \approx \frac{\Pr\{C_p^V \mid q \in \mathcal{M}\}}{\Pr\{C_w^V \mid q \in \mathcal{E}\}\Pr\{q \in \mathcal{E}\}}.$$

The numerator is under the control of the designer in a probabilistic approach while the denominator has not even been considered as existing in a worst-case design.

3. Comparing Worst-Case and Probabilistic Designs

The problem of quantifying the differences in performance between a worst-case design and a probabilistic design is extremely difficult in general. In this section we use a case study to quantify the risks and demonstrate that it is *not uncommon* for a probabilistic controller to be (highly) more reliable than a worst-case controller. We postulate that if the result holds for simple systems then it is also likely for more complex situations.

Consider a feedback system as follows.

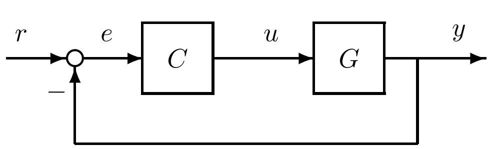

Fig. 1. Standard Feedback Configuration

The transfer function of the plant is $G(s) = \frac{q}{s-p}$ where p and q are uncertain parameters. These parameters are assumed as independent Gaussian random variables with density $\mathcal{N}(q_0, \sigma_q)$ and $\mathcal{N}(p_0, \sigma_p)$ respectively and $p_0 < 0,\ q_0 > 0$. For the worst case design it is assumed that $(q,p) \in \mathcal{B}(r)$ where $\mathcal{B}(r) = \{(x,y) \mid \ |x-q_0| \le r,\ |y-p_0| \le r\}$ is the uncertainty bounding set with radius $r > 0$. We use P^{Box} to denote $\Pr\{(q,p) \in \mathcal{B}(r)\}$, i.e., the coverage probability of the uncertainty bounding set. It can be shown that

$$P^{\text{Box}} = \text{erf}\left(\frac{r}{\sqrt{2}\,\sigma_p}\right)\text{erf}\left(\frac{r}{\sqrt{2}\,\sigma_q}\right)$$

where

$$\text{erf}(x) \stackrel{\text{def}}{=} \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt.$$

Hence, by increasing the radius, r, one can reduce the uncertainty modeling error.

Consider two controllers $C_A = \frac{K_A}{s+a}, \quad a > 0$ and $C_B = K_B$. We assume further that $1 < K_B < \frac{K_A}{a}$.

When using controller C_A, the deterministic stability margin of the system is given by

$$\rho_A = \min\left(\frac{K_A\,q_0 - a\,p_0}{K_A + a},\ a - p_0\right).$$

When using controller C_B, the deterministic stability margin of the system is given by

$$\rho_B \quad = \quad \frac{K_B\,q_0 - p_0}{K_B + 1}.$$

For uncertainty bounding set with radius $r \in (\rho_B,\ \rho_A)$, controller B may make the system unstable while controller A robustly stabilizes the system. More specifically, controller B can only stabilize a proportion of the family of uncertain plants. Such a proportion, denoted by $\mathbb{P}(r)$, is referred to as *proportion of stability*, which is computed as the ratio of the volume of the set of parameters making the system stable to the total volume of the uncertainty box, i.e.,

$$\begin{aligned} &\mathbb{P}(r) \\ = \quad &\frac{\text{vol}\{(q,p) \in \mathcal{B}(r) \mid \text{System is stable for } (q,p)\}}{\text{vol}\{\mathcal{B}(r)\}}. \end{aligned}$$

Here "vol" denotes the Lebesgue measure. We have shown that the proportion of instability for controller B is given by

$$\mathbb{P}^B(r) = \begin{cases} 1 & \text{for} \quad 0 < r < \rho_B; \\ 1 - \frac{K_B\left(r + \frac{p_0 + r}{K_B} - q_0\right)^2}{8r^2} & \text{for} \quad \rho_B \le r \le \rho_B^*; \\ \frac{1}{2} - \frac{\frac{p_0}{K_B} - q_0}{2r} & \text{for} \quad r > \rho_B^* \end{cases}$$

with $\rho_B^* = \frac{K_B\ q_0 - p_0}{K_B - 1}$. For an uncertainty bounding set with radius $r \in (\rho_B,\ \rho_A)$, controller B is actually a probabilistic controller because its proportion of stability is strictly less than 1. Obviously, controller A is a worst-case controller and is naturally considered to be more reliable than the probabilistic controller B. However, the following exact computation of probabilities of instability for both controllers reveals that the worst-case controller can actually be substantially more risky than the probabilistic controller.

We use P^{C_A} to denote

$$\Pr\{\text{Controller } C_A \text{ de-stabilizes the system}\}.$$

We have derived an exact expression as

$$\begin{aligned} & P^{C_A} \\ = \; & \frac{1}{2\pi}\left[\int_{\theta=0}^{\theta^*} \exp\left(-\frac{u^2}{2\sin^2\theta}\right) d\theta \right. \\ & \left. + \int_{\theta=\theta^*-\arctan(k)}^{\pi} \exp\left(-\frac{w^2}{2\sin^2\theta}\right) d\theta\right] \end{aligned}$$

where

$$u = \frac{a-p_0}{\sigma_p} > 0, \qquad v = \frac{\frac{K_A}{a} q_0 - p_0}{\sigma_p} > 0,$$

$$k = \frac{K_A}{a}\frac{\sigma_q}{\sigma_p}, \qquad w = \frac{v}{\sqrt{1+k^2}}$$

and $\theta^* = \arctan\left(\frac{ku}{u-v}\right) \in (0, \pi)$.

We use P^{C_B} to denote

$$\Pr\{\text{Controller } C_B \text{ de-stabilizes the system}\}.$$

We have derived an exact expression as

$$P^{C_B} = \frac{1}{2} - \frac{1}{2}\,\mathrm{erf}\left(\frac{K_B q_0 - p_0}{\sqrt{2(\sigma_p^2 + K_B^2 \sigma_q^2)}}\right).$$

A sufficient (but not necessary) condition for the worst-case controller to be more risky than the probabilistic controller (i.e., $P^{C_A} > P^{C_B}$) is

$$1 + \left(\frac{K_B\;\sigma_q}{\sigma_p}\right)^2 < \left(\frac{K_B q_0 - p_0}{a - p_0}\right)^2,$$

which can be easily satisfied.

The boundary of stability is shown in Figure 2.

In Table 1 we compute the ratio of probabilities of being unstable for both the worst-case and probabilistic designs for several situations. The results show that a worst-case controller can be thousands of times more risky than a probabilistic controller. Granted that this is a simple first order system but our contention is that if it happens even for this simple case then the situation can (easily) happen for highly complicated systems.

It has been the consensus of the field that *guarantees with certainty are often required for stability and performance in control, while the probabilistic design is a viable approach when only probabilistic guarantees are required* (see, e.g, lines 16-25, page 4652 of [18]). From our general arguments and the concrete example, we can see that, in a strict sense, "guarantees with certainty" are only possible within the uncertainty bounding set. Such worst-case guarantees may

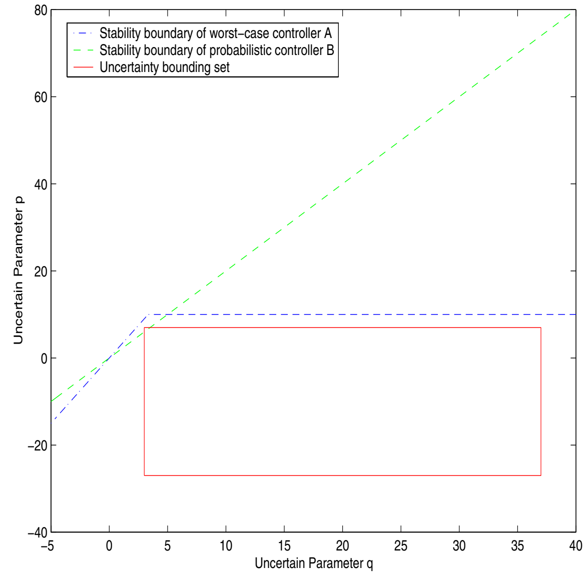

Fig. 2. Comparison of worst-case controller and probabilistic controller ($r = 17,\ a = 10,\ p_0 = -10,\ q_0 = 20,\ K_A = 30,\ K_B = 2,\ \sigma_p = 10,\ \sigma_q = 5$)

not implies better robustness than probabilistic guarantees because of the fact the uncertainty bounding set does not include all instances of uncertainty.

4. Risks in Robustness Analysis

For an analysis problem with robustness requirement $\mathbf{P}$ and uncertainty set $\mathcal{B}$, the deterministic worst-case approach is to determine whether $\mathbf{P}$ is guaranteed for every $q \in \mathcal{B}$. It is well known that computational complexity and conservatism are the main difficulties of robustness analysis within this traditional framework. From a probabilistic perspective, by sampling uniformly from $\mathcal{B}$, one can perform N i.i.d Monte Carlo simulations and estimate the probability of violating the robustness requirement. It has been remarked by Kargonekar and Tikku that *if one is willing to draw conclusions with a high degree of confidence, then the computational complexity decreases dramatically* (see, page 3470 of [10]). More formally, let $\epsilon, \delta \in (0,1)$ and define a system to be *ϵ-non-robust* if

$$\frac{\mathrm{vol}(\{q \in \mathcal{B} \mid \mathbf{P} \text{ is violated for } q\})}{\mathrm{vol}(\mathcal{B})} \geq \epsilon.$$

Then one can detect any ϵ-non-robust system with probability greater than $1-\delta$ based on $N > \frac{\ln\frac{1}{\delta}}{\ln\frac{1}{1-\epsilon}}$ i.i.d. simulations. This sample size has been discovered in [10], [22]. A fundamental problem of probabilistic robustness analysis is to make this type of detection with high confidence. Obviously, the main obstacle for one to equally accept the probabilistic analysis as the deterministic analysis is the introduction of risk parameters $\epsilon,\ \ \delta$. However, our analysis indicates that there is not sufficient reason for a discrimination between the probabilistic analysis and the deterministic analysis. Our argument proceeds as follows.

TABLE I

COMPARISON OF RISKS ($\rho_B < r < \rho_A$)

a	r	p_0	q_0	σ_p	σ_q	K_A	K_B	P^{Box}	$\mathbb{P}^B(r)$	P^{C_A}	$\frac{P^{C_A}}{P^{C_B}}$
10	17	-10	20	10	5	30	2	0.91	0.9998	2.3×10^{-2}	112
40	49	-10	50	20	10	4000	10	0.99	0.9956	6.2×10^{-3}	2.2×10^{4}

First, with modern computational power, one can detect any ϵ-non-robust system with a probability extremely close to 1. To be specific, consider $\delta = 10^{-10}$ and a small ϵ, the sample size is $N \approx \frac{23}{\epsilon}$, which indicates a low complexity for many situations. Since we can exponentially reduce δ with a linearly growing complexity, the contribution of δ to the risk of probabilistic analysis is negligible. Second, from our previous investigation, a system with

$$\frac{\text{vol}(\{q \in \mathcal{B} \mid \mathbf{P} \text{ is violated for } q\})}{\text{vol}(\mathcal{B})} < \epsilon.$$

is not necessarily more risky than a robust system in a worst-case sense.

It should be noted that the above argument is also applicable to the robustness margin problem. Recently, Barmish et. al. pointed out that *if one is willing to accept some small risk probability of performance violation, it often possible to expand the radius of allowable uncertainty by a considerable amount beyond that provided by the classical robustness theory* (see, page 853 of [1]). Our contention is that, the probabilistic robustness margin, derived from a tolerance of performance violation, is as credible as the deterministic robustness margin. Since system robustness depends on sets of uncertainty with radius less than and beyond the margin, both types of margins are not risk-free.

It has been commonly acknowledged that *probabilistic methods overcome the computational complexity and conservatism of worst-case approach at the expense of a probabilistic risk* (see., [5], [21]). Based on our previous analysis, we would remark that *the expense of risk is not necessarily extra as comparing to that of deterministic worst-case approach.*

5. CONCLUSION

In this paper, we demonstrate that the deterministic worst-case robust control design does not imply a risk free solution and that, in fact, it can be more risky than a probabilistic controller. In the final analysis, every design and analysis is subject to a level of risk. The goal of design should be to make the risk acceptable, instead of assuming that it can make it vanish. Therefore, the probabilistic method should not be discriminated a priori as a more risky alternative.

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