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Minimum Variance Control Structure for Adaptive Optics Systems*

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Abstract: **This paper shows that the minimum variance adaptive optics controller under nearly ideal conditions is the integral controller used in adaptive optics systems. The inputs to the controller dynamics are obtained from a MAP reconstructor that uses the estimation error covariance of the wavefront error. The conditions assumed to obtain this controller are: isotropic first-order (but non-stationary) temporal atmospheric aberrations; no loop delay; and no deformable mirror dynamics.**

I. Introduction

The objective of an adaptive optics system in astronomy is to minimize the effects of atmospheric aberrations of the image of the astronomical target (c.f. [1]-[2]). At the heart of an adaptive optics system is a feedback loop: the incoming wavefront is reflected from a deformable mirror (DM), the reflected (error) wavefront is measured by a wavefront sensor (WFS), and the shape of the DM is adjusted based on the WFS measurements. When operating well, current adaptive optics systems can achieve remarkable results [3]-[4]. However, the performance can be significantly affected by the control algorithm that is used.

The WFS measures approximations to the slope or curvature[1] of the error wavefront rather than directly measuring the wavefront itself. However, performance of the overall system is determined in terms of the wavefront. Thus, most controllers first reconstruct the wavefront from the WFS measurements. The reconstruction can take place in any basis: modal bases using Zernike, Karhunen-Loève, or system modes are commonly used to obtain modal controllers. Use of the natural basis (determined by the DM actuator locations) results in what is termed a zonal control algorithm. The choice of basis can affect the overall performance of the adaptive optics system due to spatial discretization.

Various philosophies have also been used to reconstruct the wavefront (c.f. [5]). The reconstructor computes the (unweighted) pseudo-inverse of the mapping from the wavefront to the slopes. Weightings that reflect the measurement error can be incorporated in the pseudo-inverse computation. Use of prior knowledge of the wavefront second order statistics (covariance) can be incorporated in the reconstructor through Bayesian estimation. Wallner [6], and Law and Lane [7] used the open loop a priori wavefront variance to obtain the *maximum a posteriori* (MAP). Kasper [5] used the closed loop wavefront variance obtained from the modal variances presented in [8] in the reconstruction estimation.

The reconstructed wavefront serves as the input to the compensation algorithm. The usual dynamic compensation that is used in the control loop is an integral or proportional-plus integral controller, although higher order dynamics [9]-[10] have been studied. For modal controllers, the controller gain can be varied based on the signal-to-noise (SN) ratio [11]-[12].

A linear-quadratic-Gaussian (LQG) control formulation for adaptive optics systems determines the controller that minimizes the wavefront error variance. Paschall and Anderson [13] applied LQG design techniques to an adaptive optics system that modeled the first 14 Zernike modes (ignoring the piston mode) with spectra generated by with first order, independent Markov models. Looze *et. al.* [10] designed a diagonal modal controller based on an LQG model of the modes and the measured modal spectra. Le Roux *et. al.* [14] developed and simulated an LQG design for both classical and multiconjugate (c.f. [15]) adaptive optics systems. Wiberg *et. al.* [16]-[17] used LQG in the design of a static controller.

This paper computes the LQG controller for an adaptive optics system under nearly ideal conditions: isotropic first-order temporal atmospheric aberrations; no loop delay; and no DM dynamics. Under these conditions and when the time constant of the atmospheric model is zero, the minimum variance controller is the usual adaptive optics system integral controller. If the lags of the atmosphere model are non-zero, the LQG controller is a first order controller whose poles are the same lags. For any value of the lag, the signal that is fed to the dynamics is generated by a gain that is the similar to the MAP reconstructor, but with the modal (or atmospheric) variance replaced by the error covariance of the estimated atmosphere.

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[1] The term slope will be used in the subsequent discussion. It should be understood that this term can be replaced by curvature if a curvature WFS is used.

II. ADAPTIVE OPTICS CONTROL FORMULATION

A. *Overview of the Adaptive Optics Control Loop*

The basic adaptive optics control problem with the deformable mirror ((DM) is shown in Figure 1. The wavefront, surface shape of the deformable mirror, and residual error (labeled science signal) can all be regarded as being in the pupil plane (with appropriate projections). The wavefront enters the pupil and is reflected from the DM. The reflected wavefront is sensed by the wavefront sensor (WFS) which produces a discrete measurement every T_s seconds (the frame rate of the WFS). The measurement is typically either the slope or curvature [1] of the wavefront. The controller then processes the measurement to produce the DM commands, which are updated at the WFS frame rate. Application of these commands to the DM modifies the mirror surface shape with a corresponding effect on the reflected wavefront.

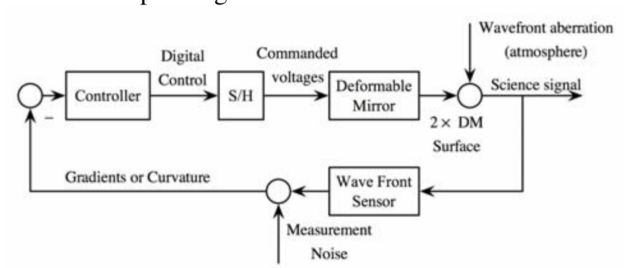

Figure 1. Adaptive optics system overview.

The wavefront aberration, deformable mirror surface shape and residual signals within the control loop are distributed spatially over the pupil. However, they will be approximated by their values at the n_a actuator positions (in the pupil plane coordinates) of the deformable mirror. It will be assumed that the active actuators are distributed within the pupil outer radius D (although this geometry of the distribution is not critical to the subsequent analysis). The actuator positions will be denoted by the set of (2-dimensional) vectors of pupil plane coordinates $\{\vec{r}_i\}_{i=1}^{n_a}$. It will be assumed that the n_g outputs of the wavefront sensor are based on the incident (residual) wavefront values at these positions. The WFS operates with a period T_s, which is also the period of the zero-order hold (ZOH). Dynamics associated with the DM and WFS will be neglected other than the integration time of the CCD in the WFS.

Since it will be assumed that all statistical processes are ergodic, the steady-state time average ($\langle \bullet \rangle$) is the same as the ensemble expectation of the process.

B. *Mirror and Camera Models*

Let twice the surface height of the DM at position $\vec{r}$ and time t be denoted by $s(\vec{r},t)$, the wavefront phase aberration by $w(\vec{r},t)$ and the wavefront residual phase input to the WFS camera by $e(\vec{r},t)$. Each component of mirror surface vector $\vec{x}_{m,k} \in \mathbb{R}^{n_a}$ at time kT_s denotes twice the height of the mirror surface at an actuator location:

$$\vec{x}_{m,k} = \begin{bmatrix} x_{m1,k} \\ \vdots \\ x_{mn_a,k} \end{bmatrix} \qquad x_{mi,k} = s(\vec{r}_i, kT_s) \tag{1}$$

Similarly, the i^{th} component of the vector $\vec{x}_{a,k} \in \mathbb{R}^{n_a}$ at time kT_s denotes the wavefront phase aberration at $\vec{r}_i$:

$$\vec{x}_{a,k} = \begin{bmatrix} x_{a1,k} \\ \vdots \\ x_{an_a,k} \end{bmatrix} \qquad x_{ai,k} = w(\vec{r}_i, kT_s); \tag{2}$$

and, the i^{th} component of the wavefront residual phase vector $\vec{e}_k \in \mathbb{R}^{n_a}$ denotes the wavefront residual input to the WFS at the corresponding actuator at $\vec{r}_i$ at time kT_s:

$$\vec{e}_k = \begin{bmatrix} e_{1,k} \\ \vdots \\ e_{n_a,k} \end{bmatrix} \qquad e_{i,k} = e(\vec{r}_i, kT_s) \tag{3}$$

The i^{th} component of the vector $\vec{u}_k \in \mathbb{R}^{n_a}$ denotes the voltage command to the i^{th} actuator. Each component of the vector $\vec{y}_k \in \mathbb{R}^{n_g}$ is a gradient or curvature produced by the WFS camera.

Assuming no loop delay, the discrete-time models of the DM and WFS (including the integration time of the CCD) can be represented by:

$$\begin{aligned} \vec{x}_{m,k+1} &= M\vec{u}_k \\ \vec{y}_k &= H\vec{w}_k + \vec{\theta}_k \end{aligned} \tag{4}$$

where

$M \triangleq n_a \times n_a$ matrix that coverts control commands (voltages) to surface positions

$H \triangleq n_g \times n_a$ matrix that generates gradient measurements from the residual phase

and the measurements are assumed to include an additive, stationary, Gaussian, zero-mean, stochastic white noise process $\vec{\theta}_k \in \mathbb{R}^{n_g}$ with autocorrelation:

$$R_{\xi}(k) = \Theta\delta_k \tag{5}$$

Let $P_m(z)$ denote the transfer function from commanded voltages $\vec{u}_k$ to the mirror surface $\vec{x}_{m,k}$:

$$P_m(z) = \frac{1}{z}M \qquad \tilde{x}_m(z) = P_m(z)\tilde{u}(z) \tag{6}$$

where z is the Z-transform variable, $\tilde{x}_m(z)$ is the Z-transform of $\vec{x}_{m,k}$, and $\tilde{u}(z)$ is the Z-transform of $\vec{u}_k$. It will be assumed that M is invertible.

C. *Atmospheric Aberration Model*

The atmospheric wavefront phase aberration at the actuator positions will be modeled using a "coloring" filter, which consists of a unit power spectral density (PSD) white input to a linear system (see Figure 2).

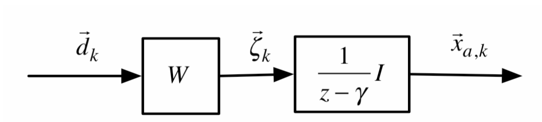

Figure 2. Atmospheric phase aberrations modeled using a colored stochastic process.

The input signal to the model in Figure 2 is a vector-valued stationary, Gaussian, zero mean, stochastic white noise process with autocorrelation:

$$R_d\left(k\right) = I\delta_k \tag{7}$$

where I is the identity matrix and δ_k is the Kronecker delta function. The PSD of d_k is the discrete Fourier transform of the autocorrelation function (7):

$$\Gamma_d\left(f\right) = \boldsymbol{F}\left\{R_d\left(k\right)\right\} = I \tag{8}$$

The intermediate signal $\vec{\zeta}_k$ is a stationary, Gaussian, zero-mean, stochastic white noise process with autocorrelation:

$$R_\xi\left(k\right) = WW^T\delta_k \qquad \Gamma_\xi\left(f\right) = WW^T \triangleq \Xi \tag{9}$$

It will be assumed that W is invertible.

The wavefront aberration $\vec{x}_{a,k}$ is also a stochastic process, but has a time correlation as well as the component (spatial) correlation given by (9). Denote the transfer function of the first order system of Figure 2 by:

$$P_a\left(z\right) = \frac{1}{z-\gamma}I \tag{10}$$

Then the PSD of the wavefront phase aberration is:

$$\begin{aligned}\Gamma_{x_a}\left(f\right) &= P_a\left(z\right)\Xi P_a^T\left(1/z\right)\Big|_{z=e^{j2\pi T_s f}} \\ &= \frac{1}{1+\gamma^2-2\gamma\cos\left(2\pi T_s f\right)}\Xi\end{aligned} \tag{11}$$

The matrix gain W represents the spatial correlation and strength of the aberration. The parameter γ is computed from the temporal correlation τ of the PSD as:

$$\gamma = e^{-\tau/T_s} \tag{12}$$

where the sample period is T_s. Then the atmosphere aberration can be modeled in discrete time as:

$$\begin{aligned}\vec{x}_{a,k+1} &= \gamma\vec{x}_{a,k} + W\vec{d}_k \\ \vec{e}_k &= \vec{x}_{m,k} + \vec{x}_{a,k}\end{aligned} \tag{13}$$

D. *Objective*

A common measure of adaptive optics performance is the Strehl ratio S [1] of the image produced by the system. Several approximations to the Strehl ratio are often used. For example, if the variance of the wavefront error σ_w^2 is small, it can be shown [1] that

$$S \approx \exp\left(-\left(\frac{2\pi}{\lambda}\right)^2\sigma_w^2\right) \approx 1-\left(\frac{2\pi}{\lambda}\right)^2\sigma_w^2, \tag{14}$$

where λ is the wavelength of the incoming beam. The important property illustrated by (14) is that the Strehl ratio is a strictly decreasing function of the error wavefront variance. Hence the objective of minimizing the error wavefront variance is equivalent to maximizing the Strehl ratio.

The wavefront variance can be approximated by an appropriate weighted sum of the wavefront values at the actuator positions:

$$\sigma_w^2\left(kT_s\right) \approx \frac{4l_a^2}{\pi D^2}\sum_{i=1}^{n_a}\left(x_{ai,k} + x_{mi,k}\right)^2 \tag{15}$$

where l_a is the spacing between actuators arranged in a rectangular grid. Other actuator geometries can be accommodated with minor adjustments to (15). Equations (14)-(15) indicate that the Strehl ratio can be maximized by minimizing a quadratic function of the atmosphere wavefront and the DM surface.

The wavefront variance can be normalized to eliminate the leading constant and express the normalized variance in terms of a quadratic form of the performance signal $\vec{e}_k \in \mathbb{R}^{n_a}$:

$$\sigma_{wn}^2\left(kT_s\right) = \vec{e}_k^T\vec{e}_k \qquad \vec{e}_k = x_{ai,k} + x_{mi,k} \tag{16}$$

E. *Complete Model*

The adaptive optics system corresponding to the assumptions and definitions of Sections II.A-II.D is shown in Figure 3.

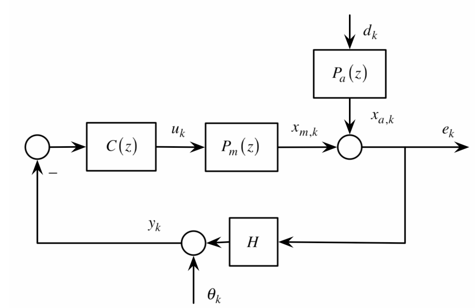

Figure 3. Discrete-time, discrete-space adaptive optics system model.

A state realization for the overall plant model for the system shown in Figure 3 (minus the compensator) can be constructed from the individual element models:

$$\begin{aligned}\mathbf{x}_{k+1} &= \mathbf{A}_{ao}\mathbf{x}_k + \mathbf{B}_{ao}u_k + \mathbf{W}_{ao}d_k \\ \vec{y}_k &= \mathbf{C}_{ao}\mathbf{x}_k + \theta_k \\ \vec{e}_k &= \mathbf{C}_e\mathbf{x}_k\end{aligned} \tag{17}$$

where

$$\mathbf{A}_{ao} = \begin{bmatrix}0_{n_a\times n_a} & 0_{n_a\times n_a} \\ 0_{n_a\times n_a} & \gamma I_{n_a}\end{bmatrix}$$

$$\begin{aligned}\mathbf{B}_{ao} &= \begin{bmatrix}M \\ 0_{n_a\times n_a}\end{bmatrix} \qquad \mathbf{W}_{ao} = \begin{bmatrix}0_{n_a\times n_a} \\ W\end{bmatrix} \\ \mathbf{C}_{ao} &= \begin{bmatrix}H & H\end{bmatrix} \qquad \mathbf{C}_e = \begin{bmatrix}I_{n_a} & I_{n_a}\end{bmatrix}\end{aligned} \tag{18}$$

III. Linear-Quadratic-Gaussian Formulation

This subsection presents the discrete-time Linear-Quadratic-Gaussian (LQG) control problem, its solution and the compensator that is obtained. Assume that the system is linear and time invariant (LTI) with state $\mathbf{x}_k$, input $\mathbf{u}_k$ and measurement $\mathbf{y}_k$ given by the state variable model

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k + \mathbf{W}\xi_k \\ \mathbf{y}_k &= \mathbf{C}\mathbf{x}_k + \eta_k \end{aligned}, \tag{19}$$

with ξ_k and η_k zero-mean, Gaussian, white noise sequences with covariances I and the symmetric, positive definite matrix $\mathbf{\Theta}$, respectively. The objective is to select $\mathbf{u}_k$ to minimize the functional

$$J = \lim_{k \to \infty} \frac{1}{2} \left\langle \mathbf{x}_k^T \mathbf{Q} \mathbf{x}_k \right\rangle, \tag{20}$$

where $\mathbf{Q}$ is a symmetric, positive semi-definite matrix, $(\mathbf{A}, \mathbf{B})$ and $(\mathbf{A}, \mathbf{W})$ are stabilizable, and $(\mathbf{A}, \mathbf{C})$ and $(\mathbf{A}, \mathbf{Q})$ are detectable [19]-[20]. Although not needed in this paper, quadratic penalties on the control variable are standard. Equations (19)-(20) define the LQG problem.

The compensator using the current measurement that solves the LQG problem is obtained in feedback form by solving two algebraic Riccati equations (ARE):

$$\begin{aligned} \mathbf{u}_k &= -\mathbf{G}\left(\hat{\mathbf{x}}_k + \mathbf{L}_f\left(\mathbf{y}_k - \mathbf{C}\hat{\mathbf{x}}_k\right)\right) \\ \hat{\mathbf{x}}_{k+1} &= \mathbf{A}\hat{\mathbf{x}}_k + \mathbf{B}\mathbf{u}_k + \mathbf{L}\left(\mathbf{y}_k - \mathbf{C}\hat{\mathbf{x}}_k\right) \end{aligned}. \tag{21}$$

The state feedback gain $\mathbf{G}$ is found in terms of the solution of the control ARE:

$$\begin{aligned} \mathbf{K} &= \mathbf{A}^T\mathbf{K}\mathbf{A} + \mathbf{Q} - \mathbf{A}^T\mathbf{K}\mathbf{B}\left(\mathbf{B}^T\mathbf{K}\mathbf{B}\right)^{-1}\mathbf{B}^T\mathbf{K}\mathbf{A} \\ \mathbf{G} &= \left(\mathbf{B}^T\mathbf{K}\mathbf{B}\right)^{-1}\mathbf{B}^T\mathbf{K}\mathbf{A} \end{aligned}, \tag{22}$$

where $\mathbf{K}$ is the unique, positive-definite, symmetric, stabilizing (i.e., $(\mathbf{A} - \mathbf{B}\mathbf{G})$ is stable) solution of the ARE in (22). The filter gain $\mathbf{L}$ and filter feed-through gain (the innovations gain) $\mathbf{L}_f$ are found in terms of the solution of the filtering ARE (which is dual to (22)):

$$\begin{aligned} \mathbf{\Sigma} &= \mathbf{A}\mathbf{\Sigma}\mathbf{A}^T + \mathbf{W}\mathbf{W}^T \\ &\quad - \mathbf{A}\mathbf{\Sigma}\mathbf{C}^T\left(\mathbf{C}\mathbf{\Sigma}\mathbf{C}^T + \mathbf{\Theta}\right)^{-1}\mathbf{C}\mathbf{\Sigma}\mathbf{A}^T \\ \mathbf{L} &= \mathbf{A}\mathbf{\Sigma}\mathbf{C}^T\left(\mathbf{C}\mathbf{\Sigma}\mathbf{C}^T + \mathbf{\Theta}\right)^{-1} \\ \mathbf{L}_f &= \mathbf{\Sigma}\mathbf{C}^T\left(\mathbf{C}\mathbf{\Sigma}\mathbf{C}^T + \mathbf{\Theta}\right)^{-1} \end{aligned}, \tag{23}$$

where $\mathbf{\Sigma}$ is the unique, positive-definite, symmetric, stabilizing (i.e., $(\mathbf{A} - \mathbf{L}\mathbf{C})$ is stable) solution of the ARE in (23). Software that solves the ARE and computes the LQG controller is widely available [21].

Note that $\mathbf{\Sigma}$ is the steady-state one-step prediction estimation error:

$$\begin{aligned} \mathbf{\Sigma} &\triangleq \left\langle \left(\mathbf{x}_k - \hat{\mathbf{x}}_k\right)\left(\mathbf{x}_k - \hat{\mathbf{x}}_k\right)^T \right\rangle \\ &\equiv \left\langle \left(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1}\right)\left(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1}\right)^T \right\rangle \end{aligned} \tag{24}$$

where the subscript in the second expression in (24) denotes the use of measurements through time $(k-1)T_s$ to construct the estimate of the state[2] at time kT_s. This notation emphasizes that the current measurement is not used to determine the state estimate $\hat{\mathbf{x}}_k$. The estimation error $\mathbf{\Sigma}_e$ is the covariance between the state and its estimate using all measurements through the current time:

$$\mathbf{\Sigma}_e \triangleq \left\langle \left(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k}\right)\left(\mathbf{x}_{a,k} - \hat{\mathbf{x}}_{k|k}\right)^T \right\rangle \tag{25}$$

The prediction estimation error and the estimation error are related by the dynamic model:

$$\mathbf{\Sigma} = \mathbf{A}\mathbf{\Sigma}_e\mathbf{A}^T + WW^T \tag{26}$$

IV. Application of LQG to Adaptive Optics Model

A. LQG Setup

The LQG solution (21)-(23) will be applied to the adaptive optics system modeled in Section II. The LQG system matrices are:

$$\begin{aligned} &\mathbf{A} = \mathbf{A}_{ao} \quad \mathbf{B} = \mathbf{B}_{ao} \quad \mathbf{W} = \mathbf{W}_{ao} \quad \mathbf{C} = \mathbf{C}_{ao} \\ &\mathbf{Q} = \mathbf{C}_e^T\mathbf{C}_e \quad \mathbf{\Theta} = H\Theta H^T \end{aligned} \tag{27}$$

The measurements and controls are:

$$\mathbf{u}_k = \vec{u}_k \qquad \mathbf{y}_k = \vec{y}_k \tag{28}$$

The objective for the LQG problem is the normalized error wavefront variance (16).

For the presentation of the subsequent sections, partition $\mathbf{K}$ and $\mathbf{\Sigma}$ conformally with $\mathbf{A}_{ao}$:

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix} \qquad \mathbf{\Sigma} = \begin{bmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} \end{bmatrix} \tag{29}$$

B. Control ARE

The control ARE (22) for the adaptive optics LQG problem can be simplified to obtain solutions for the individual blocks of $\mathbf{K}$:

$$\mathbf{K}_{11} = \mathbf{K}_{12} = \mathbf{K}_{21}^T = \mathbf{K}_{22} = I_{n_a} \tag{30}$$

The control gain is then

$$\begin{aligned} \mathbf{G} &= \left(M^T M\right)^{-1}\begin{bmatrix} 0 & M^T \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & \gamma I \end{bmatrix} \\ &= \begin{bmatrix} 0_{n_a \times n_a} & \gamma M^{-1} \end{bmatrix} \end{aligned} \tag{31}$$

That is, the LQG gain simply inverts the voltage-to-mirror surface gain of the system and normalizes by the

[2] The notation $\hat{\mathbf{x}}_{k|l}$ denotes the estimate of $\mathbf{x}$ at time kT_s using measurements through time lT_s.

atmospheric time constant.

C. Estimation ARE

The estimation ARE (23) can be simplified to obtain solutions for the individual blocks of $\mathbf{\Sigma}$:

$$\mathbf{\Sigma}_{11} = 0_{n_a \times n_a} \quad \mathbf{\Sigma}_{12} = \mathbf{\Sigma}_{21}^T = 0_{n_a \times n_a} \tag{32}$$

Define $\Sigma = \Sigma_{22}$. Then:

$$\begin{aligned} 0 = &\left(1 - \gamma^2\right)\Sigma - WW^T \\ &+\gamma^2 \Sigma H^T \left(H\Sigma H^T + \Theta\right)^{-1} H\Sigma \end{aligned} \tag{33}$$

The matrix $\mathbf{\Sigma}$ is the unique positive-semidefinite solution of (33) that stabilizes

$$\mathbf{A}_F = \gamma I - \gamma \mathbf{L}_a H \tag{34}$$

where:

$$\mathbf{L}_a = \Sigma H^T \left(H\Sigma H^T + \Theta\right)^{-1} \tag{35}$$

The gains are:

$$\mathbf{L}_f = \begin{bmatrix} 0_{n_a \times n_a} \\ \mathbf{L}_a \end{bmatrix} \quad \mathbf{L} = \begin{bmatrix} 0_{n_a \times n_a} \\ \gamma \mathbf{L}_a \end{bmatrix} = \gamma \mathbf{L}_f \tag{36}$$

D. Controller

The controller (21) can be written as:

$$\begin{aligned} \vec{u}_k &= -\mathbf{G}\left(I - \mathbf{L}_f \mathbf{C}\right)\hat{\mathbf{x}}_k - \mathbf{G}\mathbf{L}_f \vec{y}_k \\ \hat{\mathbf{x}}_{k+1} &= \left(\mathbf{A} - \mathbf{B}\mathbf{G} - \mathbf{L}\mathbf{C} + \mathbf{B}\mathbf{G}\mathbf{L}_f \mathbf{C}\right)\hat{\mathbf{x}}_k \\ &\quad + \left(\mathbf{L} - \mathbf{B}\mathbf{G}\mathbf{L}_f\right)\vec{y}_k \end{aligned} \tag{37}$$

That is:

$$\begin{aligned} \hat{x}_{k+1} &= \mathbf{A}_c \hat{x}_k + \mathbf{B}_c y_k \\ u_k &= -\mathbf{C}_c \hat{x}_k - \mathbf{D}_c y_k \end{aligned} \tag{38}$$

The system matrix is:

$$\begin{aligned} \mathbf{A}_c &= \mathbf{A} - \mathbf{B}\mathbf{G} - \mathbf{L}\mathbf{C} + \mathbf{B}\mathbf{G}\mathbf{L}_f \mathbf{C} \\ &= \begin{bmatrix} \gamma \mathbf{L}_a H & -\gamma\left(I - \mathbf{L}_a H\right) \\ -\gamma \mathbf{L}_a H & \gamma\left(I - \mathbf{L}_a H\right) \end{bmatrix} \end{aligned} \tag{39}$$

The controller input matrix is:

$$\mathbf{B}_c = \mathbf{L} - \mathbf{B}\mathbf{G}\mathbf{L}_f = \begin{bmatrix} -\gamma \mathbf{L}_a \\ \gamma \mathbf{L}_a \end{bmatrix} \tag{40}$$

The controller output matrix is:

$$\mathbf{C}_c = \mathbf{G}\left(I - \mathbf{L}_f \mathbf{C}\right) = \gamma M^{-1}\left[-\mathbf{L}_a H \quad \left(I - \mathbf{L}_a H\right)\right] \tag{41}$$

The feed-through matrix is:

$$\mathbf{D}_c = \mathbf{G}\mathbf{L}_f = M^{-1}\mathbf{L}_a \tag{42}$$

The controller transfer function can be simplified by a transformation of the estimated state. Let the transformed state be denoted by $\tilde{x}_k$:

$$\tilde{x}_k = T\hat{x}_k \quad T = \begin{bmatrix} I & 0 \\ I & I \end{bmatrix} \quad T^{-1} = \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix} \tag{43}$$

The transformed controller realization is:

$$\begin{aligned} \tilde{x}_k &= \mathbf{A}_{ct} \tilde{x}_k + \mathbf{B}_{ct} \vec{y}_k \\ \vec{u}_k &= -\mathbf{C}_{ct} \tilde{x}_k - \mathbf{D}_{ct} \vec{y}_x \end{aligned} \tag{44}$$

where

$$\begin{aligned} \mathbf{A}_{ct} &= T\mathbf{A}_c T^{-1} \quad & \mathbf{B}_{ct} &= T\mathbf{B}_c \\ \mathbf{C}_{ct} &= \mathbf{C}T^{-1} \quad & \mathbf{D}_{ct} &= \mathbf{D}_c \end{aligned} \tag{45}$$

Then:

$$\begin{aligned} \mathbf{A}_{ct} &= \begin{bmatrix} \gamma I & -\gamma\left(I - \mathbf{L}_a H\right) \\ 0 & 0 \end{bmatrix} \quad \mathbf{B}_{ct} = \begin{bmatrix} -\gamma \mathbf{L}_a \\ 0 \end{bmatrix} \\ \mathbf{C}_{ct} &= \gamma M^{-1}\left[-I \quad I - \mathbf{L}_a H\right] \end{aligned} \tag{46}$$

The transfer function of the controller is:

$$\mathbf{T}_c(z) = \frac{\gamma z}{z - \gamma} M^{-1}\mathbf{L}_a \tag{47}$$

where $\mathbf{L}_a$ is given in (35) and Σ is the solution to the ARE (33) (repeated here):

$$\begin{aligned} \mathbf{L}_a &= \Sigma H^T \left(H\Sigma H^T + \Theta\right)^{-1} \\ 0 &= \left(1 - \gamma^2\right)\Sigma - WW^T \\ &\quad + \gamma^2 \Sigma H^T \left(H\Sigma H^T + \Theta\right)^{-1} H\Sigma = 0 \end{aligned} \tag{48}$$

The matrix $\mathbf{L}_a$ can also be written in terms of the inverses of the covariances of the measurement noise and the prediction state estimation error:

$$\mathbf{L}_a = \left(H^T \Theta^{-1} H + \Sigma^{-1}\right)^{-1} H^T \Theta^{-1} \tag{49}$$

This gain has the same form as the (MAP) reconstructor (see [5]-[7]).

The controller is a multivariable first order controller. The time constants of the n_a first order lags are the constants γ of the atmospheric aberration power spectrum. The controller has n_a zeros at he origin. The factor M^{-1} removes the effect of possibly different voltage gains and cross-coupling between actuators induced by the influence functions. The signal to be integrated is reconstructed by a MAP reconstructor with the *a posteriori* covariance of the disturbance replaced by the covariance of the state estimation error.

The zero-frequency (DC) gain is:

$$\mathbf{T}_c(1) = \frac{\gamma}{1 - \gamma} M^{-1}\Sigma H^T \left(H\Sigma H^T + \Theta\right)^{-1} \tag{50}$$

E. Asymptotic Analysis

As the time constant of the disturbance becomes vanishingly small ($\tau \to 0$) the digital constant γ approaches 1 from the left along the real axis. This limit ($\gamma = 1$) has the interpretation that the power spectrum of the wavefront aberrations behaves proportional to f^{-2} at all temporal frequencies. In particular, the wavefront aberrations have infinite energy at zero temporal frequency.

For each of the non-limiting values of γ, the LQG problem is well-defined and has the solution developed in

Section IV.D above. Although the LQG problem formulation becomes ill-defined at $\gamma = 1$, the limiting (for $\gamma = 1$) controller is still well-defined and stabilizes the adaptive optics feedback system. The limiting controller has the simple, commonly use form of an integral control law:

$$\mathbf{T}_c(z) = \frac{z}{z-1} M^{-1} \Sigma H^T \left(H \Sigma H^T + \mathbf{\Theta} \right)^{-1} \qquad (51)$$

The overall gain of the controller is determined by the prediction estimation error covariance. For bright guidestars (relative to the camera noise), the estimation error becomes small. As the estimation error becomes negligible, the prediction estimation error approaches the open loop covariance of the noise process that drives the atmospheric aberration dynamics (see (26)). This results in a maximal gain, with gain matrix

$$M^{-1}\mathbf{L}_a = M^{-1} W W^T H^T \left(H W W^T H^T \right)^{-1} \qquad (52)$$

Thus, the resulting gain is the weighted pseudo-inverse zonal reconstructor.

Conversely, for dim guidestars the estimation error becomes large. A large estimation error causes a commensurately large prediction estimation error, which become approximately equal. The large estimation error is caused by a relative increase in the camera noise covariance. The same effect can be analyzed by holding the wavefront variance constant, and scaling the camera noise covariance by a parameter α that becomes large. Let the nominal camera noise variance be $\bar{\mathbf{\Theta}}$, and let $\mathbf{\Theta} = \alpha\bar{\mathbf{\Theta}}$. The prediction estimation covariance increases proportional to the square root of α:

$$\Sigma = \sqrt{\alpha}\,\bar{\Sigma} \qquad (53)$$

The two variances are related by equation (33) with $\gamma = 1$:

$$\begin{aligned} 0 &= -\frac{1}{\alpha} W W^T + \mathbf{L}_a \left(\frac{1}{\sqrt{\alpha}} H \tilde{\mathbf{\Sigma}} H^T + \tilde{\mathbf{\Theta}} \right) \mathbf{L}_a^T \\ &\approx -\frac{1}{\alpha} W W^T + \mathbf{L}_a \tilde{\mathbf{\Theta}} \mathbf{L}_a^T \end{aligned} \qquad (54)$$

The gain for large α is approximately:

$$M^{-1}\mathbf{L}_a = \frac{1}{\sqrt{\alpha}} M^{-1} \bar{\Sigma} H^T \bar{\mathbf{\Theta}}^{-1} \qquad (55)$$

The controller gain given by (55) becomes vanishingly small.

V. Conclusion

This paper has shown that the minimum variance adaptive optics controller under nearly ideal conditions is the usual integral controller. The inputs to the controller dynamics are obtained from a MAP reconstructor that uses the estimation error covariance of the wavefront error. The conditions assumed to obtain this controller are: isotropic first-order temporal atmospheric aberrations; no loop delay; and no DM dynamics.

VI. References

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