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Inverse optimal constrained input-to-state stabilization of nonlinear systems

Tianshi Chen, Zhiyuan Liu, Hong Chen and Run Pei

***Abstract*— A system with both control and disturbance inputs is constrained input-to-state stabilizable if there exist feedback control laws that render the closed-loop system constrained input-to-state stable (cISS) with respect to the disturbance. Based on a control Lyapunov function, Sontag type controllers to the constrained input-to-state stabilization problem is constructed for a class of nonlinear systems, and it is shown that input-to-state stabilizability implies (arbitrary) constrained input-to-state stabilizability, but the converse is not true. Moreover, the constrained input-to-state stabilizability is both necessary and sufficient for the solvability of an inverse optimal problem. The designed controllers remain cISS against a certain class of input uncertainties, even input unmodeled dynamics.**

I. INTRODUCTION

Constrained input-to-state stability (cISS) [1] is a derivative concept from input-to-state stability (ISS) [2]. However, cISS, unlike ISS or integral-ISS (iISS) [3], is not confined to forward complete systems. cISS reflects the qualitative property of small overshot when the magnitude of disturbances is constrained below a threshold and can be seen as a generalization of small-signal $\mathcal{L}_\infty$ stability [4] in much the same way ISS generalizes $\mathcal{L}_\infty$ stability. Furthermore, cISS is a property with broad applicability. For example, it has been shown in [1] that the PD-controlled manipulator used in [3] to motivate iISS is also cISS, thus it can handle some bounded disturbance with constrained magnitude, which can't be dealt with iISS or ISS.

In this paper, some important problems about constrained input-to-state stabilization are discussed for a class of nonlinear systems. Firstly, based on a control Lyapunov function, an explicit construction of constrained input-to-state stabilizing control laws is introduced. Then the relation between constrained input-to-state stabilizability and respectively, input-to-state stabilizability and continuous stabilizability are discussed. Then we show that, for such systems, constrained input-to-state stabilizability is not only sufficient but also necessary for the solvability of a generalized *inverse optimal gain assignment* problem which was proposed in [5]. The designed controllers remain cISS against a certain class of input uncertainties, even input unmodeled dynamics, and achieve kinds of stability margin, such as disk margin [6].

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T. Chen, Z. Liu and R. Pei are with the department of Control Science and Engineering, Harbin Institute of Technology, Harbin 150001, China tianshi@hit.edu.cn, zhiyuan@public.hr.hl.cn

H. Chen is with the department of Control Science and Engineering, Jilin University, Changchun 130000, China

The rest of the paper is organized as follows: Section II introduces the fundamental results about cISS; Section III discusses some important problems about constrained input-to-state stabilization; the inverse optimal gain assignment problem is studied in IV; finally, Section V summarizes the conclusion of this paper.

II. PRELIMINARIES

A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. If in addition α is unbounded, then it is said to be of class $\mathcal{K}_\infty$. A continuous function $\gamma : [0, c) \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}_\mathcal{C}$ if it is strictly increasing in $[0, c)$ and satisfies $\gamma(0) = 0$ and $\gamma(s)$ increases to $+\infty$ as $s \to c$, where c is a positive constant. Obviously, the inverse function of any $\mathcal{K}\backslash\mathcal{K}_\infty$ function belongs to class $\mathcal{K}_\mathcal{C}$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each fixed $t \geq 0$ and $\beta(r, t)$ decreases to 0 as $t \to \infty$ for each fixed $r \geq 0$.

Consider a general system

$$\dot{x} = f(x, d) \tag{1}$$

where f is locally Lipschitz, $f(0, 0) = 0$ and d is measurable locally essentially bounded disturbance input.

Definition 1: [1] system (1) is said to be constrained input-to-state stable (cISS) with respect to d if there exist some $\beta \in \mathcal{KL}$ and $\gamma : [0, c) \to \mathbb{R}_{\geq 0} \in \mathcal{K}_\mathcal{C}$ such that for any d with $\|d\| < c$ and any initial state $x(0)$ the corresponding solution of (1) satisfies the following estimate

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma(\|d\|) \tag{2}$$

where $\|d\| := ess\sup\{|d(t)| : t \geq 0\}$.

Definition 2: A continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be a cISS-Lyapunov function for (1) if there exist functions $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and $\delta : \mathbb{R}_{\geq 0} \to [0, c) \in \mathcal{K}\backslash\mathcal{K}_\infty$, such that for all x, d, it hold that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \tag{3}$$

$$|d| \leq \delta(|x|) \Rightarrow DV(x)f(x, d) \leq -\alpha_3(|x|) \tag{4}$$

It has been shown in [1] that a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a cISS-Lyapunov function for system (1) if and only if there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and $H : [0, c) \to \mathbb{R}_{\geq 0} \in \mathcal{K}_\mathcal{C}$, such that (3) and the following inequality hold

$$DV(x)f(x, d) \leq -\alpha_3(|x|) + H(|d|) \tag{5}$$

for all x and all d, which provides a dissipation type characterization for the cISS property. The next theorem shows that the existence of a smooth cISS-Lyapunov function is necessary as well as sufficient for the system to be cISS.

Theorem 1: System (1) is cISS if and only if it admits a cISS-Lyapunov function.

Proof: The proof can be found in [1]. ■

III. Constrained input-to-state stabilization

In this paper, we consider the following system

$$\dot{x} = f(x) + g_1(x)d + g_2(x)u \tag{6}$$

where $x \in \mathbb{R}^n$ is the state, $d \in \mathbb{R}^p$ is a disturbance, $u \in \mathbb{R}^m$ is a control input, $f : \mathbb{R}^n \to \mathbb{R}^n$, $g_1 : \mathbb{R}^n \to \mathbb{R}^{n\times p}$, $g_2 : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are locally Lipschitz functions and $f(0) = 0$. Firstly, we introduce some definitions. The system (6) is continuously stabilizable if there exists a continuous map $k : \mathbb{R}^n \to \mathbb{R}^m$ with $k(0) = 0$ such that system (6) with $u = k(x)$ is globally asymptotically stable (GAS) when $d = 0$. It's input-to-state stabilizable if there exists such a k so that system (6) becomes ISS. It's constrained input-to-state stabilizable if there exist such a k and a constant c such that system (6) becomes cISS when $\|d\| \le c$; and if in addition c can be chosen arbitrarily, then we say that system (6) is arbitrarily constrained input-to-state stabilizable. Now we introduce the concept of cISS-control Lyapunov function (cISS-clf), whose existence leads to an explicit construction of constrained input-to-state stabilizing control laws, then the relation between input-to-state stabilizability and constrained input-to-state stabilizability is analyzed.

Definition 3: A continuously differentiable and radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ is said to be a cISS-clf for system (6) if there exists a class $\mathcal{K}\backslash\mathcal{K}_\infty$ function δ such that for all $x \ne 0$ and all d we have

$$|d| \le \delta(|x|) \Rightarrow \inf_u \{L_f V + L_{g_1} V d + L_{g_2} V u\} < 0 \tag{7}$$

It is easy to show that a necessary and sufficient condition for the above V and δ to satisfy (7) is

$$L_{g_2} V = 0 \Rightarrow L_f V + |L_{g_1} V|\delta(|x|) < 0 \ \forall x \ne 0. \tag{8}$$

This result can be used to select an appropriate cISS-clf from the candidate functions. In addition, we say that a clf V satisfies the *small control property* if there exists a control law $\alpha_c(x)$ continuous on $\mathbb{R}^n$ such that $L_f V + L_{g_1} V \delta(|x|) + L_{g_2} V \alpha_c(x) < 0$ for $x \ne 0$.

Theorem 2: For system (6), if there exists a cISS-clf V with small control property, then the following Sontag type control law $u = k_s(x)$ defined as

$$k_s(x) = \begin{cases} -\frac{w(x)+\sqrt{w^2+|L_{g_2}V|^4}}{|L_{g_2}V|^2}(L_{g_2}V)^T, & L_{g_2}V \ne 0 \\ 0, & L_{g_2}V = 0 \end{cases} \tag{9}$$

where $w(x) = L_f V + |L_{g_1} V|\delta(|x|)$, constrained input-to-state stabilize (6) with gain margin $(1/2, +\infty)$. On the other hand, if (6) is constrained input-to-state stabilizable, there exists a cISS-clf V with small control property.

Proof: We only prove here that $k_s(x)$ is constrained input-to-state stabilizing and achieves a gain margin $(1/2, +\infty)$ (see Definition 5 in Section IV). Let the input uncertainty $\triangle$ be of the form $diag\{\kappa_1, ..., \kappa_m\}$ with constants $\kappa_i \in (a, b)$, $i = 1, ..., m$ and $\kappa = \min\{\kappa_1, ..., \kappa_m\}$. Then the perturbed system is expressed as

$$\dot{x} = f(x) + g_1(x)d + g_2(x)diag\{\kappa_1, ..., \kappa_m\}u. \tag{10}$$

Assume V is an appropriate cISS-clf for system (6), thus the time derivative of V along the trajectories of (10) with k_s is $\dot{V} = L_f V + L_{g_1} V d + L_{g_2} V diag\{\kappa_1, ..., \kappa_m\} k_s(x)$. Since $L_{g_2} V diag\{\kappa_1, ..., \kappa_m\}(L_{g_2} V)^T \ge \kappa |L_{g_2} V|^2$ and

$$-(w + \sqrt{w^2 + |L_{g_2} V|^4}) \le 0,$$

$$\dot{V} \le L_f V + |L_{g_1} V|\delta(|x|) - \kappa w(x) - \kappa\sqrt{w^2 + |L_{g_2} V|^4}$$

$$\le (-\kappa + \frac{1}{2})(w + \sqrt{w^2 + |L_{g_2} V|^4}) + \frac{1}{2}(w - \sqrt{w^2 + |L_{g_2} V|^4})$$

which is obviously negative definite when $\kappa \ge 1/2$, because $L_{g_2} V = 0$ implies $w < 0$ for $x \in \mathbb{R}^n \backslash \{0\}$. ■

Theorem 3: For system (6), input-to-state stabilizability implies arbitrary constrained input-to-state stabilizability.

Proof: From the converse theorem [7], we know that if system (6) is input-to-state stabilizable, there exists a ISS-control Lyapunov function with small control property, that is, there exist a continuously differentiable and radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ and a class $\mathcal{K}_\infty$ function ρ such that for all $x \ne 0$ and all d, we have $|d| \le \rho(|x|) \Rightarrow \inf_u \{L_f V + L_{g_1} V d + L_{g_2} V u\} < 0$. Obviously, for any $c > 0$, there exists a class $\mathcal{K}\backslash\mathcal{K}_\infty$ function $\delta : \mathbb{R}_{\ge 0} \to [0, c)$ such that $\delta(r) \le \rho(r)$ for $r \ge 0$. Then it's easy to verify that V satisfies (7) with δ and V is a cISS-clf function with small control property. By Theorem 2, system (6) is constrained input-to-state stabilizable. ■

Corollary 1: For system (6), if $g_1 = g_2 = g$, continuous stabilizability implies arbitrary constrained input-to-state stabilizability.

Proof: This result can be seen as a combination of Theorem 1 in [2] and Theorem 3. However, it can be proved in a direct way with another construction instead of Sontag type control, which leads to an interesting result and is sketched below as a remark. ■

Remark 3.1: Without loss of generality, we can assume $\dot{x} = f(x)$ is GAS. Thus we only need to construct a continuous map k with $k(0) = 0$ such that k renders the closed-loop system (6) arbitrary cISS. Firstly, we show that, if $\dot{x} = f(x)$ is GAS, there exist a Lyapunov function V and some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\alpha_3 : \mathbb{R}_{\ge 0} \to [0, c) \in \mathcal{K}\backslash\mathcal{K}_\infty$ where c can be chosen arbitrarily such that (3) and $L_f V(x) \le -\alpha_3(|x|)$ are satisfied (The proof is similar to [2] and is omitted here due to limitation of space). Now construct the feedback control law $u = k(x)$ as

$$k(x) = -\frac{\alpha_3(|x|)}{2m}(L_g V)^T \tag{11}$$

Then following the same procedures in [2, p. 441], we can prove that $k(x)$ render the closed-loop system (6) cISS by Theorem 1. $k(x)$ is continuous everywhere, because $\alpha_3(|x|)$, L_gV are continuous. Also notice that c is an arbitrarily chosen positive constant, therefore $k(x)$ is arbitrarily constrained input-to-state stabilizing. Now denote (11) as $k_{cISS}(x) = -a_{cISS}(x)(L_gV)^T/2m$ with $a_{cISS}(x) = \alpha_3(|x|)$ and the corresponding ISS control law (32) in [2] as $k_{ISS}(x) = -a_{ISS}(x)(L_gV)^T/2m$, then it's easy to show

$$\lim_{|x|\to\infty} \frac{k_{ISS}(x)}{k_{cISS}(x)} = \frac{a_{ISS}}{a_{cISS}} = +\infty \tag{12}$$

which implies that for system (6) with $g_1 = g_2$, if the magnitude of the disturbance is finite and constrained below a predefined threshold and if system (6) moves in a large region, the designed constrained input-to-state stabilizing control law is more efficient and saves much more energy. Moreover, for system (6), Sontag type control (9) can also have such benefit (see Example 2).

IV. Inverse optimal gain assignment problem

In the following, we show that, if system (6) is constrained input-to-state stabilizable, then an inverse optimal gain assignment problem is solvable.

Definition 4: The inverse optimal gain assignment problem for system (6) is solvable if: 1. there exist a class $\mathcal{K_C}$ function $H : [0, c) \to \mathbb{R}_{\geq 0}$ whose derivative H' is also a class $\mathcal{K_C}$ function, a matrix-valued function $R(x)$ such that $R(x) = R^T(x) > 0$ for all x, positive definite, radially unbounded functions $L(x)$ and $E(x)$; 2. there exists a continuous feedback control law $u = k(x)$ with $k(0) = 0$, which minimizes the cost function $J(u) =$

$$\sup_{|d|\leq c}\{\lim_{t\to\infty}[E(x(t)) + \int_0^t (L(x) + u^TRu - H(|d|))d\tau]\}$$

Before we start our developments, let us introduce the notation and also some properties of *Legendre-Fenchel* transform for class $\mathcal{K_C}$ functions. For a class $\mathcal{K_C}$ function H whose derivative H' exists and is also a class $\mathcal{K_C}$ function, let ℓH denotes the *Legendre-Fenchel* transform

$$\ell H(h) = h(H')^{-1}(h) - H((H')^{-1}(h)), \tag{13}$$

where $(H')^{-1}$ stands for the inverse function of H'.

Lemma 4.1: If $H, H' : [0, c) \to \mathbb{R}_{\geq 0} \in \mathcal{K_C}$, then the Legendre-Fenchel transform satisfies the following properties: (a) $\ell H(h) = \int_0^h (H')^{-1}(s)ds$; (b) $\ell\ell H(h) = H(h)$, for $h < c$; (c) $\ell H \in \mathcal{K}_\infty$; (d) $\ell H(H'(h)) = h(H')(h) - H(h)$, for $h < c$.

Lemma 4.2: (Young's Inequality [8, Th. 156]): For any vectors x that satisfies $|x| < c$ and y , the following inequality holds $x^Ty \leq H(|x|) + \ell H(|y|)$, and the equality is achieved if and only if $y = H'(|x|)x/|x|\ \forall\ |x| < c$, that is, for $x = (H')^{-1}(|y|)y/|y|$.

The next theorem provides a sufficient condition for the solvability of the inverse optimal gain assignment problem.

Theorem 4: Consider the auxiliary system of (6):

$$\dot{x} = f(x) + g_1(x)\ell H(2|L_{g_1}V|)\frac{(L_{g_1}V)^T}{|L_{g_1}V|^2} + g_2(x)u \tag{14}$$

where $V(x)$ is a Lyapunov function candidate for (14), $H, H' : [0, c) \to \mathbb{R}_{\geq 0} \in \mathcal{K_C}$ and ℓH denotes the Legendre-Fenchel transform of H. Suppose that there exists a matrix-valued function $R(x) = R^T(x) > 0$ for all x such that

$$u = k(x) = -R(x)^{-1}(L_{g_2}V)^T \tag{15}$$

globally stabilizes (14) with respect to $V(x)$, then the control law $u = k^*(x) = \beta k(x) = -\beta R(x)^{-1}(L_{g_2}V)^T$ with any $\beta \geq 2$ solves the inverse optimal gain assignment problem for (6) by minimizing the cost function $J(u) =$

$$\sup_{|d|\leq\lambda c}\{\lim_{t\to\infty}[2\beta V(x(t)) + \int_0^t [L(x) + u^TRu - \beta\lambda H(\frac{|d|}{\lambda})]d\tau]\} \tag{16}$$

for any $\lambda \in [1, 2]$, where

$$L(x) = -2\beta[L_fV + \ell H(2|L_{g_1}V|) - L_{g_2}VR^{-1}(L_{g_2}V)^T] + \beta(2-\lambda)\ell H(2|L_{g_1}V|) + \beta(\beta-2)L_{g_2}VR^{-1}(L_{g_2}V)^T$$

Proof: It has been shown in Lemma 4.1 and 4.2 that properties of the Legendre-Fenchel transform mentioned in the Appendix of [5] for class $\mathcal{K}_\infty$ functions also applies to class $\mathcal{K_C}$ functions, thus the proof of this theorem is essentially the same with [5, Th. 3.1]. ■

Theorem 5: If system (6) is constrained input-to-state stabilizable, the inverse optimal gain assignment problem is solvable.

Proof: By Theorem 2, there exist a cISS-clf $V(x)$ and $\delta \in \mathcal{K}\backslash\mathcal{K}_\infty$ such that (7) and the small control property are satisfied. Then we can construct a Sontag type control $u = k_s(x)$, which can be rewritten as (15). Since we have shown in Theorem 2 that $k_s(x)$ can constrained input-to-state stabilize system (6) and achieves a gain margin $(1/2, +\infty)$, if we can find $H, H' : [0, c) \to \mathbb{R}_{\geq 0} \in \mathcal{K_C}$ and show that $u = k_s(x)/2$ can globally stabilize the auxiliary system (14), then the proof is complete.

Assume the range of δ is $[0, \lambda c)$, where $\lambda \in [1, 2]$. Since $|L_{g_1}V(x)| = 0$ vanishes at the origin $x = 0$, there exists a class $\mathcal{K}_\infty$ function π such that

$$|L_{g_1}V| \leq \pi(|x|) \tag{17}$$

and $\int_0^h \pi\circ\delta^{-1}(\lambda s)ds \to +\infty$ as $h \to c$. Since $\delta \in \mathcal{K}\backslash\mathcal{K}_\infty$, $\delta'(r) > 0$, for $r > 0$, it is easy to prove that there always exists $\pi \in \mathcal{K}_\infty$ such that $\lim_{r\to+\infty}\pi(r)\delta'(r) = +\infty$. Let α be any $\mathcal{K}_\infty$ function so that $\alpha(r) \leq \pi(r)\delta'(r)$ for $r \geq 0$, then $\int_0^h \pi\circ\delta^{-1}(\lambda s)ds = \frac{1}{\lambda}\int_0^{\delta^{-1}(\lambda h)}\pi(t)\delta'(t)dt \geq \frac{1}{\lambda}\int_0^{\delta^{-1}(\lambda h)}\alpha(t)dt$ with $t = \delta^{-1}(\lambda s)$. Obviously, the last integral increases to $+\infty$ as h goes to c. Thus such a $\pi \in \mathcal{K}_\infty$ always exists. Notice that $\delta\circ\pi^{-1}(r) \in \mathcal{K}\backslash\mathcal{K}_\infty$, $\int_0^r \delta\circ\pi^{-1}(s)ds \in \mathcal{K}_\infty$, then define

$$\xi'(2r) = \frac{1}{\lambda}\delta\circ\pi^{-1}(r),\ H'(h) = (\xi')^{-1}(h). \tag{18}$$

It is easy to check $H, H' \in \mathcal{K}_{\mathcal{C}}$. From Lemma 4.1 and the definition of H, it follows that $\ell H(h) = \int_0^h (H')^{-1}(s)ds = \xi(h)$, which implies that

$$\ell H(2h) = \xi(2h) = \frac{1}{\lambda}\int_0^h \delta\circ\pi^{-1}(s)ds \leq \frac{1}{\lambda}h\delta\circ\pi^{-1}(h).$$

Then by Theorem 2, the time derivative of V along the trajectories of (14) with $u = k_s(x)/2$ for all $x \neq 0$ is

$$\begin{aligned}\dot{V} &= L_fV + \frac{1}{2}L_{g_2}Vk_s(x) + \ell H(2|L_{g_1}V|)\\ &\leq L_fV + \frac{1}{2}L_{g_2}Vk_s(x) + \frac{1}{\lambda}|L_{g_1}V|\delta\circ\pi^{-1}(|L_{g_1}V|)\end{aligned}$$

Notice that $\lambda \in [1, 2]$ and (17), then we have

$$\dot{V} \leq L_fV + |L_{g_1}V|\delta(|x|) + \frac{1}{2}L_{g_2}Vk_s(x) \qquad (19)$$

which is negative definite, because $u = k_s(x)$ render the closed-loop system (6) cISS with gain margin $(1/2, +\infty)$ by Theorem 2. Thus the auxiliary system (14) is asymptotically stabilized by the control law $k_s/2$. The same technique as [5] can be used to achieve radial unboundedness of $L(x)$ if $L(x)$ is only positive definite. Then by Theorem 4, k_s solves the inverse optimal gain assignment problem. ■

It is shown in the following that constrained input-to-state stabilizability is not only sufficient but also necessary for the solvability of the inverse optimal gain assignment. Moreover, from [6], we know that if a controller $u = k(x)$ for system (6) without d is optimal with respect to

$$J(u) = \int_0^\infty (l(x) + u^TR(x)u)dt,$$

then the controller remains stabilizing in the presence of some input uncertainties. Now, such property is generalized to the inverse optimal gain assignment problem. Before the statements, some concepts are first introduced.

Definition 5: The nonlinear feedback system (P, k) is said to have a sector margin (a, b) if the perturbed closed-loop system $(P, k, \triangle)$ (see Fig.1) is cISS for any $\triangle$ which is of the form $diag\{\varphi_1(\cdot), ..., \varphi_m(\cdot)\}$ where $\varphi_i(\cdot)$ are locally Lipschtiz static nonlinearities which belong to the sector (a, b), that is, $as^2 < s\varphi_i(s) \leq bs^2, \forall s \in \mathbb{R}$; If $\varphi_i(s) = s$ $\forall s \in \mathbb{R}$, then the sector margin reduces to the gain margin.

Definition 6: System $\triangle$

$$(\triangle)\ \left\{\begin{array}{ll}\dot{z} = \hat{f}(z, \hat{u}) & \hat{u} \in \mathbb{R}^m\\ \hat{y} = \hat{h}(z, \hat{u}) & \hat{y} \in \mathbb{R}^m\end{array}\right. \qquad (20)$$

is said to be: strictly passive if $\dot{S} \leq -\alpha(|z|) + \hat{u}^T\hat{y}$, $\alpha \in \mathcal{K}$; output feedback passive (OFP(ρ)) if $\dot{S} \leq \hat{u}^T\hat{y} - \rho\hat{y}^T\hat{y}$; input feedforward passive (IFP(v)) if $\dot{S} \leq \hat{u}^T\hat{y} - v\hat{u}^T\hat{u}$, where $S(x)$ is the storage function of system $\triangle$ and is assumed to be continuously differentiable and radially unbounded.

Definition 7: The nonlinear feedback system (P, k) is said to have a disk margin $D(a)$ if the perturbed closed-loop system $(P, k, \triangle)$ is cISS for any $\triangle$ which is strictly IFP(v), $v > a$, with a radially unbounded storage function.

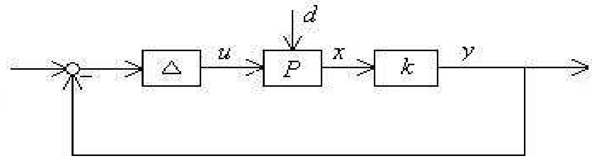

Fig. 1. Nonlinear feedback loop with plant P, control law k and input uncertainty $\triangle$

Theorem 6: If the inverse optimal gain assignment problem is solvable for system (6), $u = k(x) = -\beta R(x)^{-1}(L_{g_2}V)^T$ constrained input-to-state stabilizes system (6) with gain margin $(1/2, +\infty)$. If in addition, $R(x) = diag\{r_1(x), ..., r_m(x)\}$, $u = -\beta R(x)^{-1}(L_{g_2}V)^T$ achieves a sector margin $(1/2, +\infty)$.

Proof: We only prove the second part. The time derivative of V along trajectories of (6) with $\varphi(k(x))$ is

$$\dot{V} = L_fV + L_{g_1}Vd + \frac{1}{2}L_{g_2}Vk(x) + L_{g_2}V(\varphi(k(x)) - \frac{1}{2}k(x))$$

Since $R(x) = diag\{r_1(x), ...r_m(x)\}$,

$$\begin{aligned}\dot{V} &= L_fV + L_{g_1}Vd + \frac{1}{2}L_{g_2}Vk(x)\\ &-\frac{1}{\beta}\sum_{i=1}^m r_i(x)[k_i(x)\varphi_i(k_i(x)) - \frac{1}{2}k_i^2(x)]\end{aligned}$$

when $s\varphi_i(s) \geq s^2/2$, $\dot{V} \leq L_fV + L_{g_1}Vd + L_{g_2}Vk(x)/2$. Since $u = k(x)$ solves the inverse optimal gain assignment problem for system (6), the following equation is satisfied:

$$L_fV - \frac{\beta}{2}L_{g_2}VR^{-1}(L_{g_2}V)^T + \frac{\lambda}{2}\ell H(2|L_{g_1}V|) = -\frac{1}{2\beta}L(x) \qquad (21)$$

then $\quad \dot{V} \leq -\frac{1}{2\beta}L(x) - \frac{\lambda}{2}\ell H(2|L_{g_1}V|) + \frac{\lambda}{2}2|L_{g_1}V|\frac{|d|}{\lambda},$

by Lemma 4.2, we have $\quad \dot{V} \leq -\frac{1}{2\beta}L(x) + \frac{\lambda}{2}H(\frac{|d|}{\lambda}).$

Therefore $u = k(x)$ renders system (6) cISS by Theorem 1, and achieves sector margin $(1/2, +\infty)$. ■

Example 1: Consider the following system

$$\begin{aligned}\dot{x}_1 &= -x_1 + (x_1 - \cos x_1)d + u\\ \dot{x}_2 &= -x_2 + (x_2 + \cos x_1)d - u\end{aligned} \qquad (22)$$

No matter what control law u is applied, $d \equiv 2$ gives $d(x_1 + x_2)/dt = x_1 + x_2$. This means that system (22) is not input-to-state stabilizable. On the other hand, let $x = [x_1, x_2]^T$, $V(x) = \log(1 + x_1^2) + \log(1 + x_2^2)$ is a cISS-clf with small control property, since for all x, d

$$L_fV + L_{g_1}Vd \leq -\frac{2|x|^2}{1 + |x|^2} + 6|d|.$$

Therefore, (22) is constrained input-to-state stabilizable. Let $\delta(r) = r^2/3(1 + r^2)$ and $\pi(r) = 4r + r^2$, it is easy to check $|L_{g_1}V| \leq \pi(|x|)$ and compute that $\ell H(2r) = \xi(2r) = (r - \ln(r - 4\sqrt{4 + r} + 9) - 4\arctan(\sqrt{4 + r} - 2))/6$,

$\xi'(2r) = (8-4\sqrt{4+r}+r)/6(9-4\sqrt{4+r}+r)$, and $H(h) = 4(-\sqrt{6h(1-6h)} + \arcsin\sqrt{6h})/3 - 2h - \ln(1-6h)/3$. Then the Sontag type control $k_s(x) = -2R(x)^{-1}(L_{g_2}V)^T$, where $R(x) = 2|L_{g_2}V|^2/(w(x) + \sqrt{w^2 + |L_{g_2}V|^4})$ can solve the inverse optimal control problem (16) with V, H, R as right above, $\lambda = \beta = 2$ and $L(x) := -4(L_fV + \ell H(2|L_{g_1}V|) - L_{g_2}VR^{-1}(L_{g_2}V)^T)$. Fig. 2 shows the simulation results for a constant disturbance $d(t) = 0.3$ and initial conditions $x_1(0) = -x_2(0) = 2$ with respectively $u = k_s(x)$ and $u = k_s(x)/2$, which illustrates that k_s can constrained input-to-state stabilize (22) with certain robustness to the disturbance on the input gain.

Remark 4.1: From Theorem 3, we know that input-to-state stabilizability implies (arbitrary) constrained input-to-state stabilizability. However, from Example 1 we find that the converse is not true. Therefore, to achieve constrained input-to-state stabilization is, to some extent, easier than input-to-state stabilization.

Theorem 7: If the inverse optimal gain assignment problem is solvable for system (6) with $R(x) = I$, then $k(x) = \beta(L_{g_2}V)^T$ constrained input-to-state stabilizes the nonlinear feedback system (P, k) with disk margin $D(1/2)$. More specifically, if $\triangle$ is strictly IFP(v), $v > 1/2$, then $k(x)$ constrained input-to-state stabilize the following system

$$(P)\ \dot{x} = f(x) + g_1(x)d + g_2(x)u,\ u = \hat{y},\ y = k(x)$$
$$(\triangle)\ \dot{z} = \hat{f}(z, \hat{u}),\ \hat{u} = -y,\ \hat{y} = h(z, \hat{u})$$

Proof: Since system $\triangle$ is strictly IFP(v), there exists a Lyapunov type function $\hat{V}$ for $\triangle$ such that $\dot{\hat{V}} \leq -\alpha(|z|) + \hat{u}^T\hat{y} - v\hat{u}^T\hat{u}$. Since $k(x)$ solves the inverse optimal gain assignment problem, (21) is satisfied with $R(x) = I$. Now consider $V_c(x, z) = V(x) + \hat{V}/\beta$ as the composite cISS-clf.

Then
$$\dot{V}_c \leq -\frac{1}{2\beta}L(x) + \frac{\beta}{2}|L_{g_2}V|^2 - \frac{\lambda}{2}\ell H(2|L_{g_1}V|) + L_{g_1}Vd + L_{g_2}Vu + \frac{1}{\beta}(-\alpha(|z|) + \hat{u}^T\hat{y} - v\hat{u}^T\hat{u})$$

substitute $u = -\hat{y}$, $\hat{u} = y$ and $y = k(x)$ in to the above equation, and also notice Lemma 4.2, then we have

$$\dot{V}_c \leq -\frac{1}{2\beta}L(x) - \frac{1}{\beta}\alpha(|z|) + \frac{\lambda}{2}H(\frac{|d|}{\lambda}) + \beta(\frac{1}{2} - v)|L_{g_2}V|^2$$

when $v \geq 1/2$,
$$\dot{V}_c \leq -\frac{1}{2\beta}L(x) - \frac{1}{\beta}\alpha(|z|) + \frac{\lambda}{2}H(\frac{|d|}{\lambda})$$

Therefore by Theorem 1, we conclude that $k(x) = \beta(L_{g_2}V)^T$ constrained input-to-state stabilizes the nonlinear feedback system (P, k) with disk margin $D(1/2)$. ■

Corollary 2: If $u = -k(x) = -\beta(L_{g_2}V)^T$ solves the inverse optimal gain assignment problem for system (6) with $R(x) = I$, the system $\dot{x} = f(x) + g_2(x)u,\ y = k(x)$ is strictly OFP($-1/2$) with βV as the storage function.

Proof: Let $S(x) = \beta V(x)$ and also notice (21), then $L_{g_2}S = y^T$ and $L_fS \leq -\frac{1}{2}L(x) + \frac{\beta^2}{2}|L_{g_2}V|^2$. Then the time derivative of S along the solutions of the system is

$$\dot{S} = L_fS + L_{g_2}Su \leq -\frac{1}{2}L(x) + y^Tu + \frac{1}{2}y^Ty$$

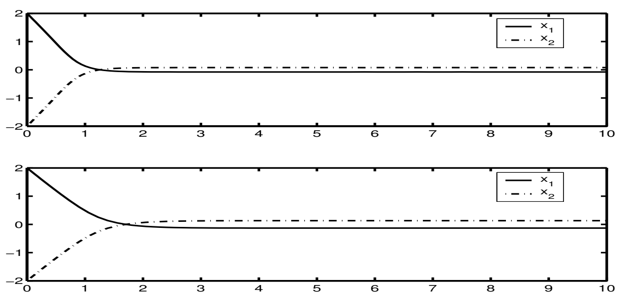

Fig. 2. Trajectories of states with $u = k_s$ and $u = k_s/2$

Since $L(x)$ is positive definite and radially unbounded, we can find a class $\mathcal{K}$ function α such that $\alpha(|x|) \leq L(x)$ for $x \in \mathbb{R}^n$. Therefore, we complete the proof. ■

Remark 4.2: Notice that the form and the assumption of perturbation $\triangle$ here are obviously more general than that of [5], moreover, Theorem 7 and Corollary 2 generalizes and recovers respectively the result for general optimal control in [6], thus our result is more general and formal.

In order to achieve disk margin $D(1/2)$, we shall find ways to make $R(x) = I$. The next theorem shows that $R(x) = I$ can be achieved for systems that are constrained input-to-state stabilizable.

Theorem 8: If a cISS-clf V has small control property, that is, there exists a continuous control $\alpha_c(x)$ such that $L_fV + L_{g_1}V\delta(|x|) + L_{g_2}V\alpha_c(x) < 0, \forall x \neq 0$. If in addition,

$$\lim_{\varepsilon \to 0} \max_{|x|=\varepsilon} \frac{\alpha_c(x)}{|L_{g_2}V(x)|} < +\infty \tag{23}$$

then the inverse optimal gain assignment problem is solvable with $R(x) = I$.

Proof: Since idea of the proof is similar to [5], we only sketch here. Firstly from (19), there exists a positive definite function $W(x)$ such that

$$L_fV - \frac{\beta}{2}L_{g_2}VR^{-1}(L_{g_2}V)^T + |L_{g_1}V|\delta(|x|) = -W(x) \tag{24}$$

Since (23) implies the existence of a continuous positive function $\varrho(V(x))$ such that $R^{-1}(x) \leq \varrho(V(x))I$ for each $x \in \mathbb{R}^n$, multiply (24) by $\varrho(V)$ and let $\tilde{V}(x) = \int_0^{V(x)} \varrho(s)ds$, which is obviously a continuously differentiable Lyapunov type function, then

$$L_f\tilde{V} - \frac{\beta}{2}|L_{g_2}\tilde{V}|^2 + |L_{g_1}\tilde{V}|\delta(|x|) \leq -\varrho(V)W(x) + \frac{\beta}{2}L_{g_2}V(R^{-1} - \varrho(V)I)(L_{g_2}V)^T\varrho(V) \leq -\varrho(V)W(x) \tag{25}$$

Since $\varrho(V)L_{g_1}V$ is continuous and vanishes at $x = 0$, there exists $\tilde{\pi} \in \mathcal{K}_\infty$ such that $|\varrho(V)L_{g_1}V| \leq \tilde{\pi}(|x|)$. Similar to Theorem 5, we can define $\tilde{H}(h) = \int_0^h(\tilde{\xi}')^{-1}(s)ds$, where $\tilde{\xi} \in \mathcal{K}_\infty$ whose derivative $\xi' \in \mathcal{K}\backslash\mathcal{K}_\infty$, such that $\ell\tilde{H}(2h) \leq h\delta \circ \tilde{\pi}^{-1}(h)/\lambda$, where $\lambda \in [1, 2]$. The time derivative of $\tilde{V}$ along the solutions of (14) with $u = -\beta(L_{g_2}\tilde{V})^T/2$ is

$$\dot{\tilde{V}} = L_f\tilde{V} - \frac{\beta}{2}|L_{g_2}\tilde{V}|^2 + \ell H(2|L_{g_1}\tilde{V}|)$$

substitute (25) into above equation, we get

$$\dot{\tilde{V}} \leq -\varrho(V)W(x) + \ell H(2|L_{g_1}\tilde{V}|) - |L_{g_1}\tilde{V}|\delta(|x|)$$

which is negative definite. Thus $u = -\beta(L_{g_2}\tilde{V})^T/2$ stabilizes the auxiliary system (14), and solves the inverse optimal gain assignment problem with $\tilde{R}(x) = I$. ■

Example 2: Consider the same example used in [5]

$$\dot{x} = u + x^2 d \tag{26}$$

Since (26) is input-to-state stabilizable, (26) is arbitrarily constrained input-to-state stabilizable by Theorem 3. Take $V(x) = x^2/2$ and get $L_{g_1}V = x^3$, $L_{g_2}V = x$. If we assume $\delta \in \mathcal{K}\backslash\mathcal{K}_\infty$ as $\delta(r) = 10r/(1+r)$, $w(x) = |L_{g_1}V|\delta(|x|) = 10x^4/(1+|x|)$. Then the Sontag type control $u = k_s(x) = -2R^{-1}(L_{g_2}V)^T$, where

$$R(x) = 2[\frac{10x^2}{1+|x|} + \sqrt{\frac{100x^4}{(1+|x|)^2} + 1}]^{-1}.$$

Now choose the $\mathcal{K}_\infty$ function π as $\pi(r) = r^3$. From (18), we can take $\xi(2r) = 5\int_0^r s^{\frac{1}{3}}/(1+s^{\frac{1}{3}})ds$, then $H(h)$ and $\ell H(2h)$ can be computed as $H(h) = 25 + 30\ln 5 - 2h - 30\ln(5-h) - 150/(5-h) + 125/(5-h)^2$ and $\ell H(2h) = \xi(2h) = 5h - 7.5h^{\frac{2}{3}} + 15h^{\frac{1}{3}} - 15\ln(1+h^{\frac{1}{3}})$. Since the time derivative of the Lyapunov function along the solutions of the auxiliary system (14) of (26) with $u = k_s(x)/2$ is $\dot{V} = \ell H(2|L_{g_1}V|) - L_{g_2}VR^{-1}(L_{g_2}V)^T \leq (w - \sqrt{w^2 + |L_{g_1}V|^4})/2 < 0$ for $x \neq 0$, $u = k_s(x)/2$ is stabilizing. Then by Theorem 4, with $\beta = \lambda = 2$, $u = k_s(x)$ solves the inverse optimal gain assignment problem with V, H, R as right above and $L(x) := -4(\ell H(2|L_{g_1}V|) - L_{g_2}VR^{-1}(L_{g_2}V)^T$, which is obviously positive definite and radially unbounded, therefore the inverse optimal gain assignment problem is well defined.

Besides, it's easy to check $V(x) = x^2/2$ satisfies the assumption of Theorem 8, therefore system (26) can be redesigned to achieve a disk margin $D(1/2)$. By choosing

$$\varrho(r) = \frac{1}{2}[\frac{20r}{1+\sqrt{2r}} + \sqrt{\frac{400r^2}{(1+\sqrt{2r})^2} + 1}] \tag{27}$$

it is easy to show that $R(x)^{-1} \leq \varrho(V(x))I$. Consider $\tilde{V}(x) = \int_0^{V(x)} \varrho(r)dr$, then by Theorem 7, $k_{cISS}(x) = 2(L_{g_2}\tilde{V})^T = 2R(x)^{-1}x$ can constrained input-to-state stabilize feedback system (P, k_{cISS}) with disk margin $D(1/2)$. Now consider the perturbation $\triangle$ as $\dot{z} = -z + z^2\tilde{u}$, $\tilde{y} = z^3 + \tilde{u}/2$, which is not ISS but obviously strictly IFP(1/2) with respect to the storage function $S(z) = z^2/2$. Then from Theorem 8, the perturbed closed-loop system $(P, k_{cISS}, \triangle)$ is still cISS. Fig. 3 shows the simulation results for an almost periodic disturbance $d(t) = (3\sin 9t + 5\sin\sqrt{13t} + 7\cos 15t + 9\cos 19t)/3$, $\beta = \lambda = 2$, and initial conditions $x(0) = 2, z(0) = -2$ with respectively $k_{cISS}(x)$ and $k_{ISS}(x)$, which is the ISS Sontag type control (59) in [5]. It illustrates that k_{cISS} is a more satisficing control than k_{ISS}, since it saves more energy and the corresponding dynamic response is also better than the ISS case.

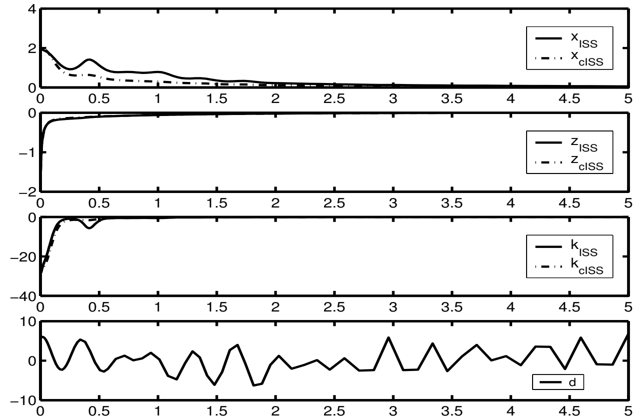

Fig. 3. Trajectories of states with control and disturbance

V. Conclusion

Some important problems about constrained input-to-state stabilization for a class of nonlinear systems are discussed. Although input-to-state stabilizability implies arbitrary constrained input-to-state stabilizability, there exist systems which can't be input-to-state stabilizable but can be constrained input-to-state stabilizable. Therefore, to achieve constrained input-to-state stabilization is, to some extent, easier than input-to-state stabilization. Furthermore, it is shown that input-to-state stabilizability is necessary as well as sufficient for the solvability of an inverse optimal problem, which is a generalization of [5] to cISS but still applies to ISS, and which can't be achieved under the assumption of iISS. The designed constructive controllers remain cISS against a certain class of input uncertainties, even input unmodeled dynamics, and achieve kinds of stability margin, such as disk margin.

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