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Linear Systems with Input Constraints: Stability and Optimality

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***Abstract*— For discrete-time linear systems with inputs constraints, we characterize the the generalized stability region and give an algorithm to £nd the facet of this region. we show that there exists a £nite horizon such that the in£nite horizon constrained LQR problem can turns to a £nite horizon constrained LQR problem and give an upper bound of this horizon for any given initial state, we also given an upper bound of this £nite horizon for all initial states inside the stability region.**

I. Introduction

Nowadays, the discrete-time linear systems with constraints are probably the most important class of systems in practice due to the fast development of computer and the its application in control area. It's known that global stability is not guaranteed in the presence constrained input. There has been a lot of effort to characterize the *null controllable reigon* since Desoer and Wing [2], most of this literature [4], [7], [9] focus on linear programming and projection algorithms. A simple result is given in this paper which is similar to the algorithm given by Anes Jamak [6].

Model predictive control (MPC) are popularly used to study the constrained multivariable control problems [5], [1], [11], [12]. Finite moving horizon policies were used in these literatures. But the stability of MPC is very complicated and not easy to investigate. On the other hand, if the £nite horizon tends to in£nite horizon, the stability can be ensured but solving such an in£nite constrained problem is generally impracticable. Fortunately, it has been shown that for a given initial state,thein£nite horizon constrained LQR problem can equal to a £nite horizon constrained LQR problem. We will given an upper bound of this £nite horizon for a given initial state and compute an upper bound for every state in the given null controllable region.

II. Controllable region

Consider the linear system

$$x(k+1) = Ax(k) + Bu(k) \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector and $u(k) \in \mathbb{R}^m$ is the control input which is constrained by $\|u\|_\infty \leq 1$. We assume that (A, B) is controllable. A control signal u is said to be admissible if it satis£es the above constraints.

De£nition 1 A state x_0 is said to be *K-step null-controllable* if there exists an admissible control such that the time response x of (1) with initial condition $x(0) = x_0$ satis£es $x(K) = 0$. The set of all K-step null-controllable states is called the *K-step null-controllable region* and is denoted by $\mathcal{C}_K(A, B)$.

De£nition 2 A state x_0 is said to be *null-controllable* if there exists an admissible control such that the time response x of (1) with initial condition $x(0) = x_0$ satis£es $\lim_{k\to\infty} x(k) = 0$. The set of all null-controllable states is called the *null-controllable region* and is denoted by $\mathcal{C}(A, B)$.

In more general case, denote a neighborhood of the origin by

$$\mathcal{O}_{P,\delta} = \{x : x'Px \leq \delta\}.$$

De£nition 3 A state x_0 is said to be *K-step $\mathcal{O}_{P,\delta}$-controllable* if there exists an admissible control such that the time response x of (1) with initial condition $x(0) = x_0$ satis£es $x(K) \in \mathcal{O}_{P,\delta}$. The set of all K-step $\mathcal{O}_{P,\delta}$-controllable states is called the *K-step $\mathcal{O}_{P,\delta}$-controllable region* and is denoted by $\mathcal{C}_K(A, B, \mathcal{O}_{P,\delta})$.

Proposition 1 The K-step null-controllable region is given by

$$\mathcal{C}_K(A, B) = \left\{ -\sum_{i=0}^{K-1} A^{-i-1} Bu(i) : |u(i)| \leq 1 \right\} \tag{3}$$

the null-controllable region is given by

$$\mathcal{C}(A, B) = \left\{ -\sum_{i=0}^{\infty} A^{-i-1} Bu(i) : |u(i)| \leq 1 \right\}$$

and $\mathcal{C}_1(A, B) \subseteq \mathcal{C}_2(A, B) \subseteq \cdots \subseteq \mathcal{C}_K(A, B) \subseteq \cdots \subseteq \mathcal{C}(A, B)$.

By carrying out a similarity transformation if necessary, we may assume that A and B are of the form

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \in \mathbb{R}^{(n_1+n_2)\times(n_1+n_2)}$$
$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \in \mathbb{R}^{(n_1+n_2)\times m}$$

with A_1 having all eigenvalues outside the unit circle, A_2 having all eigenvalues on or inside the unit circle. It's known that the null-controllable region is given by $\mathcal{C}(A, B) = \mathcal{C}(A_1, B_1) \oplus \mathbb{R}^{n_2}$, and $\mathcal{C}(A_1, B_1)$ is a bounded convex subset in $\mathbb{R}^{n_1}$. Therefore we will only focus on the anti-stable part and assume that A has all eigenvalues outside the unit circle.

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Denote $\mathcal{O}_{P,\delta}(K) = \{x : x'(A')^K P A^K x \le \delta\}$. Then $\mathcal{O}_{P,\delta}(K) = A^{-K}\mathcal{O}_{P,\delta}$.

Proposition 2 The K-step $\mathcal{O}_{P,\delta}$-controllable region is given by

$$\mathcal{C}_K(A,B,\mathcal{O}_{P,\delta}) = \mathcal{O}_{P,\delta}(K) + \mathcal{C}_K(A,B).$$

and

$$\lim_{K\to\infty} \mathcal{C}_K(A,B,\mathcal{O}_{P,\delta}) = \mathcal{C}(A,B).$$

Proof: Let $x_0 \in \mathcal{C}_K(A,B,\mathcal{O}_{P,\delta})$. Then there exists $x(K) \in \mathcal{O}_{P,\delta}$ such that

$$x(K) = A^K x_0 + \sum_{i=0}^{K-1} A^{K-i-1}Bu(i)$$

for some $|u(i)| \le 1$. It is equivalent to

$$x_0 = A^{-K}x(K) - \sum_{i=0}^{K-1} A^{-i-1}Bu(i).$$

Hence

$$\begin{aligned}\mathcal{C}_K(A,B,\mathcal{O}_{P,\delta}) &= A^{-K}\mathcal{O}_{P,\delta} + \mathcal{C}_K(A,B)\\ &= \mathcal{O}_{P,\delta}(K) + \mathcal{C}_K(A,B).\end{aligned}$$

Also note that $\lim_{K\to\infty} A^{-K} = 0$, so we have

$$\begin{aligned}&\lim_{K\to\infty} \mathcal{C}_K(A,B,\mathcal{O}_{P,\delta})\\ =\ &\lim_{K\to\infty} A^{-K}\mathcal{O}_{P,\delta} + \lim_{K\to\infty}\mathcal{C}_K(A,B)\\ =\ &\mathcal{C}(A,B).\end{aligned}$$

Since A has all its eigenvalues outside the unit circle, $C_K(A,B)$ exponentially converges to $C(A,B)$, and for practical application, there always has a 'large enough' K such that $C_K(A,B)$ is a good approximation of $C(A,B)$ and in general, K will between 10 and 30. From proposition 2, $C_K(A,B)$ acts the important role in characterizing $\mathcal{C}_K(A,B,\mathcal{O}_{P,\delta})$ and it is also a good approximation of $\mathcal{C}_K(A,B,\mathcal{O}_{P,\delta})$ for 'large enough' K. So, we will mainly focus on how to describe $C_K(A,B)$.

From (3),we get that

$$C_K(A,B) = \{T_K V_K\} = T_K \mathcal{V}_K$$

Where

$$\begin{aligned}T_K &= -[\ \textstyle\sum_{i=1}^K A^{-i}B \quad \sum_{i=2}^K A^{-i}B \quad \cdots \quad A^{-K}B\ ]\\ &= [\ t_1 \quad t_2 \quad \cdots \quad t_{mK}\ ]\\ V_K &= \begin{bmatrix}\triangle u(0)\\ \triangle u(1)\\ \vdots\\ \triangle u(K-1)\end{bmatrix} = \begin{bmatrix}v_1\\ v_2\\ \vdots\\ v_{mK}\end{bmatrix}, |v_i| \le 1.\end{aligned}$$

It's easy to see that $\mathcal{V}_K$ is an mK-dimensional polytope. $\mathcal{C}_K(A,B)$ can be represented as a projection of $\mathcal{V}_K$ and $\mathcal{C}_K(A,B)$ is a $n-$polytope in $\mathbb{R}^n$.

A. *Vertex Representation of Null-Controllable region*

Proposition 2 Every polytope is the convex hull of its vertices: $\mathcal{P} = conv(vert(\mathcal{P}))$, every polytope is the intersection of its facet-de£ning half-spaces: $\mathcal{P} = \mathcal{P}(F,z) = \{x \in \mathbb{R}^n : Fx \le z\}$ with $F = [\ f_1 \quad f_2 \quad \cdots \quad f_k\ ]^T \in \mathbb{R}^{k\times n}$, $z = [\ z_1 \quad z_2 \quad \cdots \quad z_k\ ]^T \in \mathbb{R}^k$ and each inequality represents one facet of $\mathcal{P}$.

For any vector $f \in \mathbb{R}^n$, de£ne $v_i(f) = sign(f^T t_i)$ for all $i = 1,...,mK$ and $X(f) = \sum_{i=1}^{mK} t_i v_i(f) \doteq T_K V_K(f)$.

Theorem 2.1: (Vertex Representation) The K-step null-controllable region satis£es

$$\mathcal{C}_K(A,B) = conv\left\{X(f) : f \in \mathbb{R}^n, \|f\|_2 = 1\right\}.$$

Proof : For any normal vector $f \in \mathbb{R}^n$, $H = \{x \in \mathbb{R}^n : f^T x = a\}$ is a hyperplane in $\mathbb{R}^n$. Let

$$M = \max\{f^T x : x \in \mathcal{C}_K(A,B)\},$$

then $\mathcal{F} = \{x \in \mathcal{C}_K(A,B) : f^T x = M\}$ is a face of $\mathcal{C}_K(A,B)$. Let $\dim(\mathcal{F}) = k$ then $\mathcal{F}$ is a *vertex* for $k = 0$, an *edge* for $k = 1$ and a *facet* for $k = n-1$.

For each point $x \in \mathcal{C}_K(A,B)$, we have $x = T_K V_K$ for some appropriate choice of V_K so we can get

$$\begin{aligned}M &= \max\{f^T x : x \in \mathcal{C}_K(A,B)\} = \max_{V_K} f^T T_K V_K\\ &= \max_{v_i} \sum_{i=1}^{mK} (f^T t_i) v_i.\end{aligned}$$

The maximum is achieved by letting $v_i = sign(f^T t_i)$ for all $i = 1,...,mK$. Then $X(f) = \sum_{i=1}^{mK} t_i v_i$ satis£es $X(f) \in \mathcal{C}_K(A,B)$ and $f^T X(f) = M$, so $X(f) \in \mathcal{F}$. In fact, $X(f)$ is a vertex of $\mathcal{C}_K(A,B)$ almost for all normal vector f except that f is orthogonal to some face of $\mathcal{C}_K(A,B)$, in this case $X(f)$ is one point on that face and f satis£es: $f^T t_i = 0$ for some $i \in [1,...,mK]$.

By proposition 3,

$$\mathcal{C}_K(A,B) = conv\left\{X(f) : f \in \mathbb{R}^n, \|f\|_2 = 1\right\}.$$

Algorithm to £nd all vertices :
Theorem 2.1 gives us a way to £nd all the vertices of $\mathcal{C}_K(A,B)$. Let $\mathcal{V}_{\mathcal{P}} = vert(\mathcal{C}_K(A,B))$ be the collection of all vertices of $\mathcal{C}_K(A,B)$, we will give the set $\mathcal{P}$ in the following algorithm using pseudo code.

Pseudo code function: $\mathcal{V}_{\mathcal{P}} = vert(T_K)$

1. Set $\mathcal{V}_{\mathcal{P}} = \emptyset$.

2. for all normal vector f

    If $f^T t_i \ne 0$ for all $i = 1,...,mK$

        Let $v_i = sign(f^T t_i)$

        Let $X(f) = \sum_{i=1}^{mK} t_i v_i$;

    If $X(f) \notin \mathcal{V}_{\mathcal{P}}$ then add $X(f)$ to $\mathcal{V}_\mathcal{P}$;

End if;

End for.

B. Facial Representation of Null-Controllable region

Although we can get the vertex presentation of $\mathcal{C}_K(A,B)$, it is a very hard task to calculate for all normal vectors especially in high dimensional space and not very easy to test whether a given x_0 belongs to $\mathcal{C}_K(A,B)$ or not. On the other hand, by proposition 3, if we can get the facial presentation $\mathcal{C}_K(A,B) = \mathcal{P}(F,z) = \{x \in \mathbb{R}^n : Fx \leq z\}$, it is very simple to characterize $\mathcal{C}_K(A,B)$ by inequalities.

Note that for any facet $\mathcal{F}$ of $\mathcal{C}_K(A,B)$ with outer normal f, let $M = \max\{f^T x : x \in \mathcal{C}_K(A,B)\}$, it satis£es

$$\mathcal{F} = \{x \in \mathcal{C}_K(A,B) : f^T x = M\}, \dim(\mathcal{F}) = n-1.$$

De£ne

$$v_i = \begin{cases} 1, & f^T t_i > 0 \\ -1, & f^T t_i < 0 \\ \alpha_i, \quad \forall |\alpha_i| \leq 1, & f^T t_i = 0 \end{cases}$$

and let $n_\perp$ be the number of t_i such that $f^T t_i = 0$, then

$$\begin{aligned} M &= \sum_{i=1}^{mK} f^T t_i v_i \\ \mathcal{F} &= \left\{ x : \sum_{i=0}^{mK-n_\perp} t_i v_i + \sum_{j=1}^{n\perp} \alpha_j t_j \right\} \end{aligned}$$

Since $\dim(\mathcal{F}) = n-1$, the outer normal f is orthogonal to at lest $n-1$ vectors from T_K and $n_\perp \geq n-1$. This gives us a way to £rst £nd all candidate outer normal and then to check which candidate vector is really an outer normal of some facet of $\mathcal{C}_K(A,B)$. A similar algorithm is given by Anes Jamak in his master thesis, which is a little bit complex and mainly focus on single input case.

Algorithm to £nd facet :
Let $\mathcal{F} = \mathcal{F}(F,z) = \{x \in \mathbb{R}^n : Fx \leq z\}$ be the collection of all facets of $\mathcal{C}_K(A,B)$, we will give the matrix pair (F,z) in the following algorithm using pseudo code.

Pseudo code function: $(F,z) = facet(T_K)$

1. Set $F = [\ ]$; $z = [\ ]$.
2. Generate all combinations of $n-1$ vectors from the set T_K, note that there are C_{mK}^{n-1} combinations.
Denote these combinations as

$$W_i = \begin{bmatrix} t_{i1} & t_{i2} & \cdots & t_{in} \end{bmatrix},\ i = 1, 2, \cdots, C_{mK}^{n-1}.$$

3. For $i = 1$ to C_{mK}^{n-1}
    If $rank(W_i) = n-1$ then
        Find a vector f which is orthogonal to W_i and normalize f;
        Let $M = \sum_{i=0}^{mK} sign(f^T t_i) f^T t_i$;
        If f is not already included as a row in the matrix F then
            Set $F = \begin{bmatrix} F \\ f^T \end{bmatrix}, z = \begin{bmatrix} z \\ M \end{bmatrix}$;
        End If;
    End If;
  End For.
4. Set $F = \begin{bmatrix} F \\ -F \end{bmatrix}, z = \begin{bmatrix} z \\ z \end{bmatrix}$

III. Constrained LQR controller design

For any initial point $x_0 \in \mathcal{C}_K(A,B)$, we want to £nd a control sequence u which solves the constrained linear quadratic regulation (CLQR)problem

$$\begin{aligned} V_\infty(x_0) &= \min_{u(0),\dots} \left\{ \sum_{i=0}^{\infty} [x^T(i)Qx(i) + u^T(i)Ru(i)] \right\}, \\ x(0) &= x_0, \\ x(i+1) &= Ax(i) + Bu(i), \\ |u(i)| &\leq 1. \end{aligned}$$

The weighting matrices Q, R are symmetric with $R > 0, Q \geq 0$. It's well known that the unconstrained linear quadratic regulation problem has solution $K = K_{LQ}$ with

$$K_{LQ} = -(R + B^T PB)^{-1} B^T PA$$

$$P = A^T PA + Q - (A^T PB)(R + B^T PB)^{-1}(B^T PA)$$

and the minimum cost is given by $x_0^T P x_0$. But in general, the in£nite horizon CLQR problem is not directly solvable. Note that

$$\begin{aligned} V_\infty(x_0) &= \min_{u(0),\dots} \left\{ \sum_{i=0}^{\infty} [x^T(i)Qx(i) + u^T(i)Ru(i)] \right\} \\ &= \min_{u(0),\dots} \left\{ \sum_{i=0}^{N-1} [x^T(i)Qx(i) + u^T(i)Ru(i)] \right. \\ &+ \left. \sum_{i=N}^{\infty} [x^T(i)Qx(i) + u^T(i)Ru(i)] \right\} \end{aligned}$$

By the optimal principle, if we can guarantee that the input do not violate the constraints for all $u(k), k \geq N$, then the controller is

$$u(i) = K_{LQ} x(i), i \geq N$$

and

$$\begin{aligned} V_\infty(x(N)) &= \min_{u(N)} \left\{ \sum_{i=N}^{\infty} [x^T(i)Qx(i) + u^T(i)Ru(i)] \right\} \\ &= x^T(N)Px(N), \\ V_\infty(x_0) &= \min_{u(0),\dots,u(N-1)} \left\{ \sum_{i=0}^{N-1} [x^T(i)Qx(i) \right. \\ &+ \left. u^T(i)Ru(i)] + x^T(N)Px(N) \right\}. \end{aligned}$$

If this is the case, then the in£nite horizon CLQR problem turns to a £nite horizon CLQR problem which is solvable.

In fact, for every given $x_0 \in \mathcal{C}_K(A,B)$, there exits a £nite number N such that the above equation holds. This result was proved by Chmielewski and Manousiouthakis (1996),Scokaert and Rawlings(1998). But they only gave a regrussive algorithm to £nd such number by computing the £nite horizon value function for increasing values of N until it satis£es certain conditions.

We want to £nd an upper bound of the number N for any given initial state x_0 and £nd an upper bound on N for the null-controllable region $\mathcal{C}_K(A,B)$.

In order to satisfy the constraints under the controller $u(i) = K_{LQ}x(i)$ for $i \geq N$, we must have $|u(i)| = |K_{LQ}x(i)| \leq 1, i \geq N$.

Let $\mathcal{X} = \{x \in \mathbb{R}^n : |K_{LQ}x| \leq 1\}$ be the set of states such that the unconstrained LQ gain K_{LQ} is feasible. It's obvious that $\mathcal{X}$ is a polytope de£ned by a set of linear inequalities of the form $K_i x \leq \beta_i, i = 1,\ldots,\bar{n}$.

In order to guarantee that the input do not violate the constraints for all $u(i), i \geq N$ under the unconstrained LQ controller $u(i) = K_{LQ}x(i), i \geq N$, we need to £nd an invariant subset of $\mathcal{X}$.

Recall that $\mathcal{O}_{P,\delta} = \left\{x \in \mathbb{R}^n : x^T P x \leq \delta\right\}$, let

$$\delta_{max} = \max\left\{\delta : \mathcal{O}_{P,\delta} \subset \mathcal{X}\right\},$$

then $\mathcal{O}_{P,\delta_{max}}$ is the largest invariant subset of $\mathcal{X}$, δ_{max} can be determined analytically as

$$\delta_{max} = \min_{i=1,\ldots,\bar{n}} \frac{|\beta_i|^2}{K_i P^{-1} K_i^T}. \tag{12}$$

Lemma 1 Let $\gamma = \{x'Qx : x'Px = \delta_{max}\}$ then the minimum value of γ is given by $\gamma_{min} = \delta_{max}\underline{\sigma}(Q,P)$, where $\underline{\sigma}(Q,P)$ is the minimal generalized eigenvalue of Q with respect to P.

Proof: Let

$$\begin{aligned} \gamma_{min} &= \min_x x'Qx \\ s.t.\ \ x'Px &= \delta_{max}. \end{aligned}$$

By Kuhn-Tucker Theorem, at the optimal point x_o, we have

$$\left.\frac{d(x'Qx)}{dx}\right|_{x_o} - \lambda \left.\frac{d(x'Px)}{x}\right|_{x_o} = 0.$$

Which gives

$$Qx_o - \lambda P x_o = 0.$$

So, λ and x_o are the generalized eigenvalue and corresponding eigenvector of (Q,P).

Let $\underline{\sigma}(Q,P)$ denote the minimum generalized eigenvalue of (Q,P), note that $Q > 0, P > 0$, so does $\underline{\sigma}(Q,P)$.

Then, we can get the minimum value of γ as:

$$\gamma_{min} = x_o'Qx_o = \underline{\sigma}(Q,P)x_o'Px_o = \delta_{max}\underline{\sigma}(Q,P)$$

For any given initial state $x_0 \in \mathcal{C}_K(A,B)$, we £rst solve the constrained £nite horizon problem

$$\begin{aligned} V_K(x_0) &= \min_{u(0),\ldots,u(K-1)} \left\{\sum_{i=0}^{K-1}[x^T(i)Qx(i) \right. \\ &+ \left. u^T(i)Ru(i)] + x^T(K)Px(K)\right\}. \end{aligned}$$

Note that we can control x_0 to the origin in K steps, this problem is always meaningful and $V_\infty(x_0) \leq V_K(x_0)$.

Theorem 3.1: For a given x_0, the upper bound of N such that $V_\infty(x_0) = V_N(x_0)$ is given by

$$\bar{N} < \frac{V_K(x_0)}{\gamma_{min}}.$$

Proof: Assume we already get the optimal solution of the constrained in£nite horizon problem and let the optimal state trajectory be $\left\{\ x_o(1)\quad x_o(2)\quad x_o(3)\quad \ldots\ \right\}$.

Denote the step that the optimal trajectory £rst come into $\mathcal{O}_{P,\delta_{max}}$ by $\overline{N}$, then we have

$$V_{\bar{N}}(x_0) = V_\infty(x_0) \leq V_K(x_0).$$

Note that

$$\begin{aligned} & V_{\bar{N}}(x_0) \\ = & \sum_{i=0}^{\bar{N}-1}[x_o^T(i)Qx_o(i) + u_o^T(i)Ru_o(i)] + x_o^T(\bar{N})Px_o(\bar{N}) \\ > & \sum_{i=0}^{\bar{N}-1} x_o^T(i)Qx_o(i) > \bar{N}\gamma_{min}. \end{aligned}$$

So, we can get

$$\begin{aligned} \bar{N}\gamma_{min} &< V_K(x_0) \\ \text{or } \bar{N} &< \frac{V_K(x_0)}{\gamma_{min}}. \end{aligned}$$

Note that for different given $x_0 \in \mathcal{C}_K(A,B)$, there exists different £nite number N such that the in£nite horizon CLQR problem turns to a £nite horizon CLQR problem which is solvable. We next give an algorithm to compute an upper-bound $\bar{N}$ for every point $x_0 \in \mathcal{C}_K(A,B)$. Assume we already know all vertices (say N_v) of $\mathcal{C}_K(A,B)$.

Algorithm to compute the upper-bound $\bar{N}$

step 1. For each vertex $x_i \in \mathcal{P}$

1.1 Solve CLQR problem (16) and get $V_K(x_i)$.

1.2 Set $\bar{N}_i = \frac{V_K(x_i)}{\gamma_{min}}$.

end for.

step 2. Let $\bar{N} = \max\left\{\bar{N}_i, i = 1,\ldots,N_v\right\}$ be the upper-bound.

Theorem 3.2: $\bar{N}$ given by the above algorithm is an upper-bound for every $x_0 \in \mathcal{C}_K(A,B)$ such that the in£nite horizon CLQR problem (9) turns to a £nite horizon CLQR problem (11) which is solvable.

Proof : Let x be a point on some facet $\mathcal{F}$ of $\mathcal{C}_K(A,B)$ and $x_1, x_2, \ldots, x_{n_{\mathcal{F}}}$ are the vertices of the facet $\mathcal{F}$, then

$$x = \sum_{i=1}^{n_{\mathcal{F}}} \alpha_i x_i$$

for some $0 \le \alpha_i \le 1, i = 1, \ldots, n_{\mathcal{F}}$ and $\sum_{i=1}^{n_{\mathcal{F}}} \alpha_i = 1$. By the above algorithm, for each x_i, there exists N_i and feasible control sequence $u_i(0), u_i(1), \ldots, u_i(N_i - 1)$ which solve CLQR problem (11) and $x_i(N_i) \in \mathcal{Z}_{\gamma_{max}}$.

Let $\bar{N}_{\mathcal{F}} = \max\{N_i, i = 1, \ldots, n_{\mathcal{F}}\}$, set

$$\begin{aligned} u_i(k) &= \begin{cases} u_i(k), & 0 \le k < N_i \\ K_{LQ} x_i(k), & k \ge N_i \end{cases} \qquad (18) \\ u_x(k) &= \sum_{i=1}^{n_{\mathcal{F}}} \alpha_i u_i(k). \end{aligned}$$

It's easy to see that $x_i(\bar{N}_{\mathcal{F}}) \in \mathcal{Z}_{\gamma_{max}}$ and $\{u_x(k), k = 0, \ldots, \bar{N}_{\mathcal{F}} - 1\}$ is also a feasible control sequence. Apply this control sequence to x, we can get

$$\begin{aligned} & x(\bar{N}_{\mathcal{F}}) \\ =\ & A^{\bar{N}_{\mathcal{F}}-1} x + \sum_{k=0}^{\bar{N}_{\mathcal{F}}-1} A^{\bar{N}_{\mathcal{F}}-k-1} B u_x(k) \\ =\ & A^{\bar{N}_{\mathcal{F}}-1} \sum_{i=1}^{n_{\mathcal{F}}} \alpha_i x_i + \sum_{k=0}^{\bar{N}_{\mathcal{F}}-1} A^{\bar{N}_{\mathcal{F}}-k-1} B \sum_{i=1}^{n_{\mathcal{F}}} \alpha_i u_i(k) \\ =\ & \sum_{i=1}^{n_{\mathcal{F}}} \alpha_i \left\{ A^{\bar{N}_{\mathcal{F}}-1} x_i + \sum_{k=0}^{\bar{N}_{\mathcal{F}}-1} A^{\bar{N}_{\mathcal{F}}-k-1} B u_i(k) \right\} \\ =\ & \sum_{i=1}^{n_{\mathcal{F}}} \alpha_i x_i(\bar{N}_{\mathcal{F}}) \\ \in\ & \mathcal{Z}_{\gamma_{max}}. \end{aligned}$$

So, $\bar{N}_{\mathcal{F}} = \max\{N_i, i = 1, \ldots, n_{\mathcal{F}}\}$ is an upper-bound of all x on the facet $\mathcal{F}$. Note that for any $x_0 \in \mathcal{C}_K(A,B)$, there exists some ρ with $0 \le \rho \le 1$ such that $\rho^{-1} x_0$ is on some facet $\mathcal{F}$ of $\mathcal{C}_K(A,B)$ and $\rho^{-1} x_0 = \sum_{i=1}^{n_{\mathcal{F}}} \alpha_i x_i$. Set

$$\begin{aligned} \bar{N} &= \max\{N_i, i = 1, \ldots, N_v\} \\ u_{x_0}(k) &= \rho \sum_{i=1}^{n_{\mathcal{F}}} \alpha_i u_i(k), \qquad 0 \le k < \bar{N}, \end{aligned}$$

where $u_i(k)$ is de£ned in (18). We can get

$$\begin{aligned} & x_0(\bar{N}) \\ =\ & A^{\bar{N}-1} x_0 + \sum_{k=0}^{\bar{N}-1} A^{\bar{N}-k-1} B u_{x_0}(k) \\ =\ & \rho \left\{ A^{\bar{N}-1}(\rho^{-1} x_0) + \sum_{k=0}^{\bar{N}-1} A^{\bar{N}-k-1} B \sum_{i=1}^{n_{\mathcal{F}}} \alpha_i u_i(k)) \right\} \\ =\ & \rho \sum_{i=1}^{n_{\mathcal{F}}} \alpha_i x_i(\bar{N}) \\ \in\ & \rho \mathcal{Z}_{\gamma_{max}}. \end{aligned}$$

So, $\bar{N} = \max\{N_i, i = 1, \ldots, N_v\}$ is an upper-bound for every $x_0 \in \mathcal{C}_K(A,B)$ such that the in£nite horizon CLQR problem (9) turns to a £nite horizon CLQR problem (11) which is solvable. ■

IV. Example

In this section, we consider the antistable second order system

$$x(k+1) = \begin{bmatrix} 1.5 & 0 \\ 0 & 2 \end{bmatrix} x(k) + \begin{bmatrix} 1 & 1 \\ 0.5 & 1 \end{bmatrix} u(k).$$

with $x_0 = [\ 3.5 \quad 1.3\ ]^T$, $\|u\|_\infty \le 1$. Set

$$Q = \begin{bmatrix} 0.1 & 0.05 \\ 0.05 & 0.1 \end{bmatrix}, R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

we can get

$$K_{LQ} = \begin{bmatrix} 1.08 & -1.26 \\ -0.97 & 2.95 \end{bmatrix}, P = \begin{bmatrix} 4.80 & -8.21 \\ -8.21 & 16.96 \end{bmatrix}$$

The controllable region $\mathcal{C}_5, \mathcal{C}_{10}, \mathcal{C}_{20}, \mathcal{C}_{30}$ are given in Fig 1, it can be seen that $\mathcal{C}_{20}$ is a good approximation of $\mathcal{C}(A,B)$.

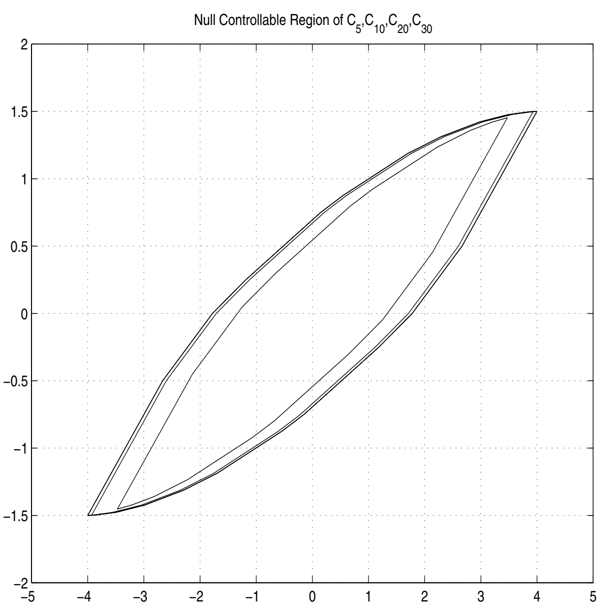

Fig. 1. Null Controllable Region of $\mathcal{C}_5, \mathcal{C}_{10}, \mathcal{C}_{20}, \mathcal{C}_{30}$

The saturated region under the unconstrained LQR controller K_{LQ} and the maximum invariant set are given in Fig 2. The states trajectory and control signals are given in Fig 3.

V. Conclusion

For discrete-time linear systems with inputs amplitude constraints, we characterize the the generalized stability region and give an algorithm to £nd the facet of this region. we show that there exists a £nite horizon such that the in£nite horizon constrained LQR problem can turns to the £nite horizon constrained LQR problem and give an upper

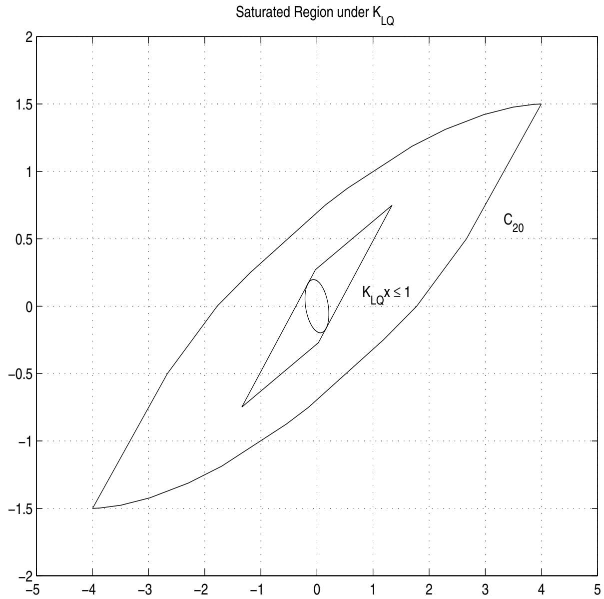

Fig. 2. Saturated Region under K_{LQ} and the maximum invariant set

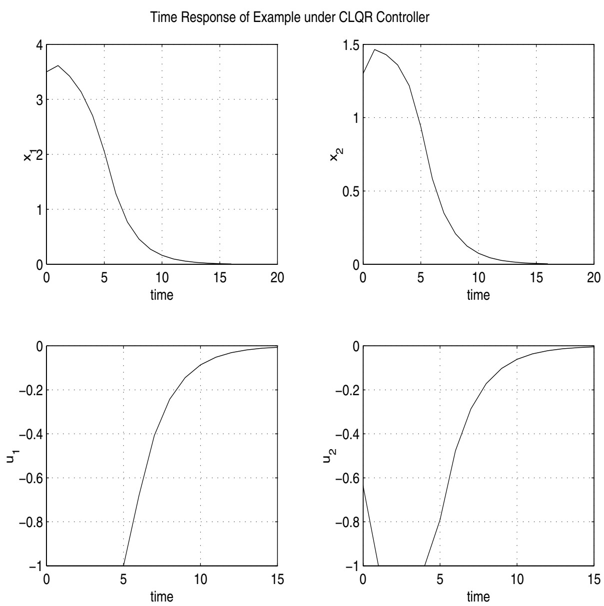

Fig. 3. The states trajectory and control signals

bound of this horizon for any given initial state, we also given an upper bound of this £nite horizon for all initial states inside the stability region.

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