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Continuous Time Constrained Linear Quadratic Regulator – Convex Duality Approach

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***Abstract*— A continuous time infinite horizon linear quadratic regulator with input constraints is studied. On the theoretical side, optimality conditions, both in the open loop and feedback form, are shown together with smoothness of the value function and local Lipschitz continuity of the optimal feedback. Arguments are self-contained, use basic ideas of convex conjugacy, and in particular, use a dual optimal control problem. A method of calculating the optimal and stabilizing feedback without relying on discrete optimization is outlined.**

I. INTRODUCTION

The discrete time infinite horizon constrained linear quadratic regulator has received much attention in the literature, with the purpose of computing stabilizing feedbacks for constrained linear systems. See Rawlings and Muske [22], Chmielewski and Manousiouthakis [10], Scokaert and Rawlings [29], Bemporad, Morari, Dua and Pistikopoulos [4], Grieder, Borelli, Torrisi and Morari [17], and the references therein. The resulting computations all require numerically solving finite horizon optimal control problems, either online for the receeding horizon control, or offline and leading to explicit piecewise affine feedbacks. This is also the feature of suboptimal feedback construction for continuous time regulator in Kojima and Morari [20].

Here we show how for a continuous time infinite horizon linear quadratic regulator with input constraints ($\mathcal{CLQR}$), the stabilizing optimal feedback can be computed without solving any discrete time control problems, and often, without any optimization at all. Computation is done offline and leads to a lookup table; one should not expect an explicit formula, as the piecewise structure is not present. What makes this possible is that the gradient of the value function, a key ingredient of the optimal feedback, can be propagated backwards from the solution to the Riccati equation through the Hamiltonian differential system.

The properties of the value function and the optimal feedback can be studied by reducing the problem, in the spirit of the Optimality Principle, to a finite horizon one with a terminal cost. This was suggested for the discrete time regulator in [10], [29]. Then, for discrete problems with both input and state constraints, finite dimensional parametric optimization tools and piecewise linear-quadratic function technology, see Fiacco [11] and Rockafellar and Wets [27], can be applied; this was done for example in [10] and [4]. In continuous time, the piecewise structure is not present, and the control problems are infinite dimensional. Some tools for analyzing the value function for such problems were given by Goebel [14]. Here, for $\mathcal{CLQR}$, we propose a direct and largely self-contained approach.

The feature of $\mathcal{CLQR}$ that makes this approach possible is convexity. We work in the framework of convex duality as suggested by Rockafellar [25], [26]; in particular we consider a control problem dual to $\mathcal{CLQR}$. For finite-horizon problems, convex duality was utilized by Rockafellar and Wolenski [28] and [14]. To a certain extent, convex analysis has been used directly in the study of $\mathcal{CLQR}$ by Di Blasio [7], and Heemels, Eijndhoven, and Stoorvogel [19], but duality has not been fully taken advantage of. We attempt to do it here, working directly with $\mathcal{CLQR}$. Alternatively, the properties needed for the algorithm could be deduced from the general results by Goebel [12] (see also [15]), which in turn rely on finite-horizon results of [28]. This was the approach by Goebel [13], where similar properties are used to show the existence of continuous stabilizing feedbacks for linear systems with general saturation nonlinearities, extending a results by Sontag, Sussmann, Yang [31]. We add that it is convexity that guarantees that the mentioned propagation of the value function through the Hamiltonian system works; this is related to the method of characteristics for the Hamilton-Jacobi equation working globally even for nonsmooth but convex problems, as in [28], but not for nonconvex but smooth ones; see Byrnes [8].

To our knowledge, open-loop and feedback optimality conditions for $\mathcal{CLQR}$ and results on smoothness of its value function, as we give here, have not been previously stated. The extensive literature on infinite horizon optimal control problems in theoretical economics, in particular Benveniste and Scheinkman [5], [6], Araujo and Scheinkman [2], Gota and Montrucchio [16], often uses assumptions not compatible with $\mathcal{CLQR}$. Our necessary open-loop optimality condition could be derived from the general results by Seierstad [30], but without reference to feedback. Results of [25] (and their extension by Barbu [3]) lead only to local results and do not give sufficient feedback optimality conditions. For other results in continuous time, a comprehensive reference is Carlson, Haurie, and Leizarowitz [9]. Finally, we note that the treatment of state constraints for $\mathcal{CLQR}$ with the tools we present requires further research. Such constraints result in the adjoint arcs, or the optimal arcs for the dual problem, being potentially discontinuous; see Rockafellar [24] and Hartl, Sethi, and Vickson [18].

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II. Preliminaries and the Dual Problem

The continuous time infinite horizon linear quadratic regulator with control constraints ($\mathcal{CLQR}$) is as follows: minimize

$$\frac{1}{2}\int_0^{+\infty} y(t)^T Q y(t) + u(t)^T R u(t)\, dt \tag{1}$$

subject to linear dynamics and an initial condition

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t),\ x(0) = x_0, \\ y(t) = Cx(t), \end{cases} \tag{2}$$

and a constraint on the input

$$u(t) \in U \quad \text{for almost all } t \in [0, +\infty). \tag{3}$$

The following are assumed throughout the paper.

Assumption 2.1: (standing assumption)

(i) *Matrices Q and R are symmetric and positive definite.*

(ii) *The pair (A, B) is controllable. The pair (A, C) is observable.*

(iii) *The set U is closed, convex, and $0 \in \operatorname{int} U$.*

The optimal value function $V : \mathbb{R}^n \mapsto [0, +\infty]$ is the infimum of (1) subject to (2), (3), parameterized by the initial condition x_0. The infimum is taken over all locally integrable control functions $u : [0, +\infty) \mapsto \mathbb{R}^k$.

For the unconstrained problem (1), (2) the value function is quadratic: $\frac{1}{2}x_0 \cdot Px_0$, the matrix P solves the Riccati equation, and the optimal feedback is linear: given the state x, the optimal control is $-R^{-1}B^T Px$. See Anderson and Moore [1] or Kwakernaak and Sivan [21]. In presence of (3), V is a convex function, is positive definite, and may have infinite values: $V(x_0) = +\infty$ if no feasible process (a pair $x(\cdot)$, $u(\cdot)$ satisfying (2), (3) and such that (1) is finite) exists. For any x_0 with $V(x_0) < +\infty$, an optimal process $x(\cdot)$, $u(\cdot)$ exists and satisfies $x(t) \to 0$ (this in fact holds for any feasible process), and V is lower semicontinuous.

The (maximized) Hamiltonian associated with $\mathcal{CLQR}$ is

$$H(x,p) = p^T Ax - \frac{1}{2}x^T C^T QCx + \rho(B^T p), \tag{4}$$

where the function ρ is

$$\rho(q) = \sup_{u \in U}\left\{q^T u - \frac{1}{2}u^T Ru\right\}. \tag{5}$$

This function is convex, nonnegative, bounded above by $q \mapsto \frac{1}{2}q^T R^{-1} q$, and equal to this quadratic on a neighborhood of 0. It is also differentiable, with $\nabla\rho$ Lipschitz continuous. Also, if U is polyhedral, ρ is piecewise linear-quadratic. The Hamiltonian differential system is

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B\nabla\rho(B^T p(t)), \\ \dot{p}(t) &= -A^T p(t) + C^T QCx(t). \end{aligned} \tag{6}$$

As the function ρ is convex and differentiable, we have that $u(t) = \nabla\rho(B^T p(t))$ if and only if

$$u(t) \text{ maximizes } u^T B^T p(t) - \frac{1}{2}u^T Ru \text{ over } u \in U, \tag{7}$$

and thus the first equation of (6) is exactly what the Maximum Principle suggests. Almost symmetrically, we have $w(t) = -QCx(t)$ if and only if

$$w(t) \text{ maximizes } -w^T Cx(t) - \frac{1}{2}w^T Q^{-1} w. \tag{8}$$

Following [25], [26], by the dual problem to $\mathcal{CLQR}$ we understand the following optimal control problem: minimize

$$\int_0^{+\infty} \rho(q(t)) + \frac{1}{2}w^T(t) Q^{-1} w(t)\, dt \tag{9}$$

subject to

$$\begin{cases} \dot{p}(t) = -A^T p(t) - C^T w(t),\ p(0) = p_0, \\ q(t) = B^T p(t). \end{cases} \tag{10}$$

Arc $p(\cdot)$ describes the dual state, $q(\cdot)$ is the dual output, while $w(\cdot)$ is the dual input/control. Motivation for considering such a problem should become clear after the initial arguments of Section III. Note that (10) is essentially a linear system dual to (2), subject to time reversal.

The optimal value function $W : \mathbb{R}^n \mapsto [0, +\infty)$ for the dual problem is the infimum of (9) subject to (10), parameterized by the initial condition p_0. The infimum is taken over all locally integrable control functions $w : [0, +\infty) \mapsto \mathbb{R}^k$. The value function W is a positive definite, finite everywhere, and convex function (and thus continuous). For each dual feasible process (a pair $p(\cdot)$, $w(\cdot)$ satisfying (10) and such that (9) is finite) we have $p(t) \to 0$, and for any p_0, an optimal process for the dual problem exists.

For a proper, lower semicontinuous and convex function $f : \mathbb{R}^n \mapsto (-\infty, +\infty]$, its *convex conjugate* $f^* : \mathbb{R}^n \mapsto (-\infty, +\infty]$ is defined by

$$f^*(p) = \sup_{x \in \mathbb{R}^n}\{p \cdot x - f(x)\}. \tag{11}$$

It is a proper, lower semicontinuous and convex function itself, and $(f^*)^* = f$. The last equality implies that conjugacy gives a one to one correspondence between convex functions and their conjugates. Simple examples are:

(i) Let $f(x) = \frac{1}{2}x^T Mx$ for a symmetric and positive definite matrix M. A direct calculation yields $f^*(p) = \frac{1}{2}p^T M^{-1} p$.

(ii) The function ρ in (5) is the conjugate of g given by $g(u) = \frac{1}{2}u^T Ru$ if $u \in U$ and $g(u) = +\infty$ if $u \notin U$.

There are several pairs of properties dual to each other with respect to convex conjugacy, not unlike what is seen for dual linear systems. Standard reference for these and other elements of convex analysis we use is Rockafellar [23].

Example 2.2: (duality in unconstrained case) The value function for the unconstrained linear quadratic regulator (1), (2) is $V_u(x_0) = \frac{1}{2}x_0^T Px_0$, where P is the unique symmetric and positive definite solution of the Riccati equation

$$PA + A^T P - C^T QC + PBR^{-1}B^T P = 0. \tag{12}$$

As P is invertible, the equation above is equivalent to

$$-P^{-1}A^T - AP^{-1} - BR^{-1}B^T + P^{-1}C^T QCP^{-1} = 0.$$

Just as (12) corresponds to the problem (1), (2), the second equation corresponds to a dual linear quadratic regulator (9), (10) with $\rho(z) = \frac{1}{2}z^T R^{-1} z$. The function $W_u(p_0) = \frac{1}{2}p_0^T P^{-1} p_0$ is the value function for this problem. Indeed, as (10) is stabilizable and detectable, the matrix describing the value function is the unique positive definite solution of the second equation above. In particular, the value functions V_u, W_u are convex functions conjugate to each other.

III. Main Results

To shorten the notation, we will use a subscript to denote time dependence. That is, instead of $x(t)$ we write x_t, etc.

For all y, w, we have that

$$\frac{1}{2}y^T Q y + \frac{1}{2}w^T Q^{-1} w \geq y^T w,$$

and this holds as an equation if and only if $w = Qy$. Similarly, for all $u \in U$, q,

$$\frac{1}{2}u^T R u + \rho(q) \geq u^T q,$$

and this holds as an equation if and only if $u = \nabla\rho(q)$. The two inequalities and the dynamics (2) and (10) show that, for any primal feasible (x_t, u_t) and dual feasible (p_t, w_t)

$$\frac{1}{2}x_t^T C^T Q C x_t + \frac{1}{2}u_t^T R u_t + \rho(B^T p_t) + \frac{1}{2}w_t^T Q^{-1} w_t \geq \frac{d}{dt}(x_t^T p_t).$$

Moreover, an equality holds if and only if (x_t, u_t) and (p_t, w_t) satisfy (6), (7), and (8). As for feasible processes $x_t \to 0$, $p_t \to 0$, integrating the above inequality yields: for any primal feasible (x_t, u_t) and dual feasible (p_t, w_t)

$$\int_0^\infty \frac{1}{2}x_t^T C^T Q C x_t + \frac{1}{2}u_t^T R u_t \, dt \\ \int_0^\infty \rho(B^T p_t) + \frac{1}{2}w_t^T Q^{-1} w_t \, dt \geq -x_0^T p_0$$

and an equality holds if and only if (x_t, u_t) and (p_t, w_t) satisfy (6), (7), and (8). The arguments and inequalities above form the basis for most of the results that follow.

Theorem 3.1: (open loop optimality)

(a) *A feasible process (x_t, u_t) is optimal for $\mathcal{CLQR}$ if and only if $x_t \to 0$ and there exists an arc p_t such that* (6), (7) *hold and $p_t \to 0$.*

(b) *A feasible process (p_t, w_t) is optimal for the dual problem* (9), (10) *if and only if $p_t \to 0$ and there exists an arc x_t such that* (6), (8) *hold and $x_t \to 0$.*

The sufficient condition in (a) can be weakened to say only that $\lim_{t\to\infty} x(t)^T p(t) = 0$, similarly for (b).

Theorem 3.2: (value function conjugacy) *The following (equivalent to each other) formulas are true:*

$$W(p_0) = \sup_{x\in\boldsymbol{R}^n}\{-p_0^T x - V(x)\}, \\ V(x_0) = \sup_{p\in\boldsymbol{R}^n}\{-x_0^T p - W(p)\}. \tag{13}$$

The first supremum is attained for every p_0, while the second is attained for every x_0 such that $V(x_0) < +\infty$.

In other words, the convex functions V and W are conjugate to each other, up to a minus sign. That is, $W(p_0) = V^*(-p_0)$, equivalently, $V(x_0) = W^*(-x_0)$. For any pair of convex functions, differentiability of one of the functions is (essentially) equivalent to strict convexity of the other; see [23], Theorem 26.3. This suggests an easy way to show differentiability of both V and W: show that each is strictly convex.

Lemma 3.3: (strict convexity) *The function V is strictly convex on $\operatorname{dom} V$. The function W is strictly convex.*

We comment that for any differentiable convex function, the gradient is continuous; see [23], Theorem 25.5.

Corollary 3.4: (differentiability of value functions) *The value function V is continuously differentiable at every point of $\operatorname{dom} V$ and $\|\nabla V(x_i)\| \to +\infty$ for any sequence of points $x_i \in \operatorname{dom} V$ converging to a point not in $\operatorname{dom} V$. The value function W is continuously differentiable.*

For the algorithm in Section IV, the result below, in particular description of ∇V given by (c), is the key.

Corollary 3.5: (Hamiltonian system) *The following are equivalent:*

(a) $p_0 = -\nabla V(x_0)$,

(b) $x_0 = -\nabla W(p_0)$,

(c) *There exist arcs x_t, p_t on $[0, +\infty)$, originating at x_0, p_0, such that* (6) *holds and $(x_t, p_t) \to (0, 0)$.*

Suppose x_t, p_t satisfy (6). Then $\frac{d}{dt}H(x_t, p_t) = 0$, where H is the Hamiltonian (4). (The equation can be verified directly.) As $H(0, 0) = 0$, if each arc converges to 0, then $H(x_t, p_t) = 0$ for all t. In light of Corollary 3.5, we obtain:

Corollary 3.6: (Hamilton-Jacobi equations) *For all x, $H(x, -\nabla V(x)) = 0$. For all p, $H(-\nabla W(p), p) = 0$.*

This can be used to find the value function for one-dimensional problems. We do this to illustrate that V need not be piecewise quadratic.

Example 3.7: (lack of piecewise quadratic structure) Consider minimizing $\frac{1}{2}\int_0^\infty x^2(t) + u^2(t)\, dt$ subject to $\dot{x}(t) = u(t)$ and $u(t) \in [-1, 1]$. The Hamiltonian is $H(x, p) = -\frac{1}{2}x^2 + \bar{\rho}(p)$, where $\bar{\rho}(q) = -q - \frac{1}{2}$ if $q < -1$, $\frac{1}{2}q^2$ if $-1 \leq q \leq 1$, and $q - \frac{1}{2}$ if $1 < q$. Since by convexity of V, ∇V is a nondecreasing function, we obtain

$$\nabla V(x) = \begin{cases} -\frac{1}{2}(x^2+1) & \text{if} \quad x < -1, \\ x & \text{if} \quad -1 \leq x \leq 1, \\ \frac{1}{2}(x^2+1) & \text{if} \quad x > 1. \end{cases}$$

Thus, ∇V is piecewise quadratic (it is not clear if this holds in higher dimensions), and then V has the piecewise structure, but the pieces are not quadratic.

Theorem 3.8: (feedback optimality)

(a) *A feasible process $(\bar{x}_t, \bar{u}_t)$ is optimal for $\mathcal{CLQR}$ if and only if $\bar{u}_t$ maximizes*

$$-u^T B^T \nabla V(\bar{x}_t) - \frac{1}{2}u^T R u$$

over $u \in U$.

(b) *A feasible process $(\bar{p}_t, \bar{w}_t)$ is optimal for the dual problem (9), (10) if and only if $\bar{w}_t$ maximizes*

$$-w^T C \nabla W(\bar{p}_t) - \frac{1}{2} w^T Q^{-1} w.$$

The maximum conditions can be alternatively written as $u_t = \nabla\rho(-B^T \nabla V(x_t))$ and $w_t = -QC\nabla W(p_t)$.

Most of the results stated so far can be extended to more general convex optimal control problems, see [12], [13], but there they require a less direct approach. Here, although only in Theorem 3.1 and Lemma 3.3, we did take advantage of the fact that V is quadratic near 0. Further use of this fact, and a result of [14], lead to stronger regularity of V.

Theorem 3.9: (local Lipschitz continuity) *The mappings ∇V, respectively ∇W, are locally Lipschitz continuous on $\operatorname{dom} V$, respectively, on $\mathbb{R}^n$.*

IV. Algorithm

The theory of Section III suggests a numerical procedure for computing the optimal feedback for $\mathcal{CLQR}$ (and thus a stabilizing feedback for the linear system subject to input constraints). From Theorem 3.8 and Corollary 3.5 one sees that given the state x of the system, the optimal control is $\nabla\rho(B^T p)$, where p is such that there exist a Hamiltonian trajectory (i.e. a pair of arcs x_t, p_t such that (6) holds) on $[0, +\infty)$, originating at (x, p) and converging to $(0, 0)$. As in a neighborhood of $(0, 0)$ we have $p = -Px$, integrating the Hamiltonian system (6) backwards from points $(x, -Px)$ should lead to values of the adjoint arc p corresponding to any state in $\operatorname{dom} V$. More precisely, the idea of the algorithm is as follows:

(1) Find the matrix P by solving the Riccati equation (12), and the corresponding optimal feedback matrix for the unconstrained problem $F_u = -R^{-1}B^T P$.
(2) Find a neighborhood N of 0 so that for all $x_0 \in N$ one has $F_u x_0 \in U$ and such that N is invariant under $\dot{x} = (A + BF_u)x$.
(3) For each point x on the boundary of N, find the solution of the backward Hamiltonian system

$$\begin{aligned} \dot{x}(t) &= -Ax(t) - B\nabla\rho(B^T p(t)), \\ \dot{p}(t) &= A^T p(t) - C^T QCx(t), \end{aligned} \tag{14}$$

on $[0, +\infty)$, originating from $(x, -Px)$.

Then, given any point $x_0 \in \operatorname{dom} V$, the optimal feedback is found as follows:

- Among all the stored pairs (x_t, p_t) corresponding to all initial points and all times, find the one with x_t equal to x_0 and let the control equal $\nabla\rho(B^T p)$.

Of course, in practice one needs to pick a grid of points on the boundary of N in step (2), use a bounded time interval $[0, T]$ in step (3). With a reasonable choice of the grid and a sufficiently large T, the x-coordinates of the backward Hamiltonian trajectories "fill out" any bounded subset of $\operatorname{dom} V$. Then, for any x_0 in this subset, to find the optimal control, one picks x_t that is closest to x_0.

Lemma 4.1: (invariant set) *For $U = [-1, 1]$ and $R = 1$, the set*

$$N = \left\{x \in \mathbb{R}^n \mid x^T Px \le (b^T Pb)^{-1}\right\}$$

meets the conditions required in step (2) of the algorithm. In fact, the N above is the largest ellipse given by P that meets the condition that $F_u x \in U$ for all $x \in N$.

The Hamiltonian system (6), and the backwards one (14), involve the gradient of the function ρ. As ρ is given by (5), its gradient can be found without computing ρ itself, as the optimal value in a simple optimization problem. That is,

$$\nabla\rho(q) = \arg\max\left\{q^T u - \frac{1}{2}u^T Ru \mid u \in U\right\}, \tag{15}$$

see Example 11.18 in [27]. This formula simplifies greatly in many cases. In particular, for single input systems ($u \in \mathbb{R}$) and the standard saturation (that is, $U = [-1, 1]$), and when (without loss of generality) $R = 1$, the gradient $\nabla\rho = \sigma$, where σ stands for the standard saturation function:

$$\nabla\rho(q) = \sigma(q) = \begin{cases} -1 & \text{for} \quad q < -1, \\ q & \text{for} \quad -1 \le q \le 1, \\ 1 & \text{for} \quad 1 < q. \end{cases}$$

More generally, similar formula holds whenever U is a closed interval, and, for multiple input cases, when R is diagonal and U is a product of intervals.

V. Numerical examples

A. Double Integrator

Consider the double integrator

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t),$$

and the weights $Q = 1$, $R = 0.1$, studied in [20]. We calculate the feedback using the algorithm outlined in the previous section. Solving the Riccati equation (12) yields

$$P = \begin{bmatrix} 0.795 & 0.316 \\ 0.316 & 0.256 \end{bmatrix}.$$

Initial points $x_i(0)$, $i = 1, 2, \ldots, 72$ are chosen on the boundary of the invariant ellipse

$$N = \left\{x \in \mathbb{R}^2 \mid x^T Px \le 0.0398\right\}.$$

Solutions to (14), from $(x_i(0), -Px_i(0))$ are computed on $[0, T]$ with $T = 10$ sec. First, we set $\Delta T = 0.005$ sec. and store the points $(x_i(j\Delta T), p_i(j\Delta T))$ for $i = 1, 2, \ldots, 72$, $j = 0, 1, \ldots, 2000$. The corresponding trajectories $x_i(t)$ are shown in Figure 1, first on the square $[-3, 3] \times [-3, 3]$ (the darker shade indicates the "strip" in the plane where the control is not saturated), then on the whole region they fill out. Figure 1 also shows the trajectory starting at $x(0) = (1, -2.5)$ for the closed-loop system with noise.

Given the stored grid and a state of the system x, the control $u(x)$ is found as $\sigma(R^{-1}B^T p_i(j\Delta T))$, where $(x_i(j\Delta T), p_i(j\Delta T))$ is the grid point with $x_i(j\Delta T)$ closest to x. The response of the system to this feedback, from

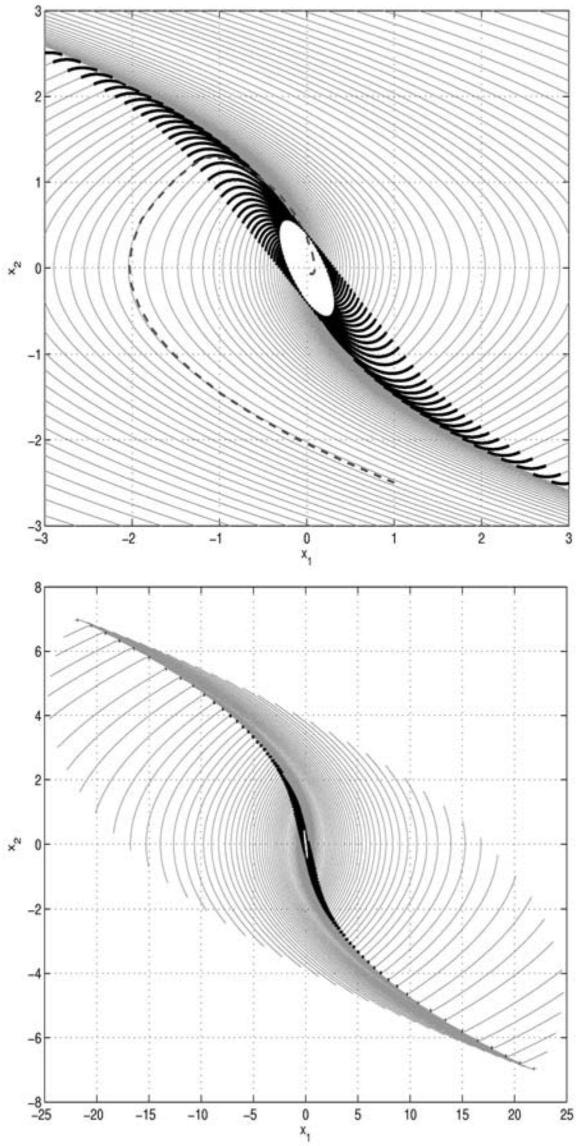

Fig. 1. Double Integrator: optimal trajectories obtained via backward integration of the Hamiltonian system.

$x(0) = (1, -2.5)$, and the corresponding sequence of used controls (sample time is $\tau = 0.25$ sec.) is in Figure 2. There are two sets of control signals and states on the figure. One set corresponds to the system without noise, the other to $\dot{x}(t) = Ax(t) + Bu(x(t) + m(t)) + n(t)$ with zero mean Gaussian random noises $m(\cdot)$ and $n(\cdot)$ with variances 0.01. The response is essentially the same as in [20].

To reduce the size of the lookup table, we also solved (14) on $[0, 10]$ sec. with $\Delta T = 0.5$ sec. and $j = 0, 1, \ldots, 20$. The resulting grid of $x_i(j\Delta T)$'s, restricted to $[-3, 3] \times [-3, 3]$, is on Figure 3. Also in this figure is the trajectory from $x(0) = (1, -2.5)$ for a feedback with the new grid. The response of the system is essentially the same as that shown in Figure 2, thought the size of the feedback table is much smaller.

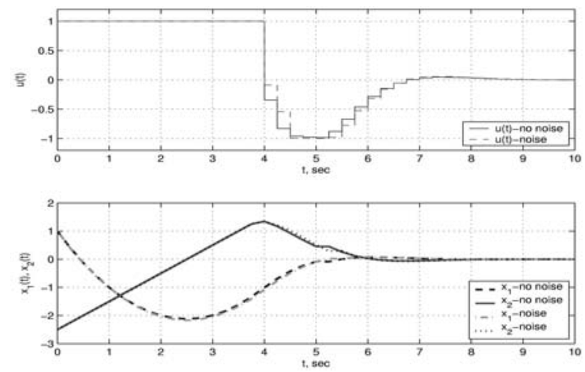

Fig. 2. Double integrator: control inputs and the responses of the system with and without noise.

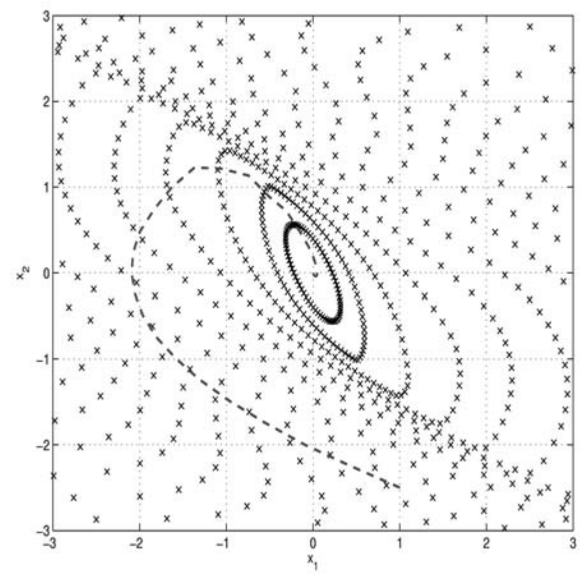

Fig. 3. Double integrator: grid and response, small lookup table.

B. Unstable system

Consider the system

$$\dot{x}(t) = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \quad y(t) = x(t), \quad (16)$$

and the weights $Q = I$, $R = 1$. As A is not semi-stable, $domV$ is not the whole plane. Thus, no matter how large T is, the x-trajectories of (14) will not fill out arbitrarily chosen compact sets. In Figure 4 we show the trajectories obtained with $T = 8$ sec., $\Delta T = 0.005$ sec., $i = 1, 2, \ldots, 72$ and $j = 0, 1, \ldots, 1600$. Figure 4 also shows the closed-loop system trajectory starting at $x(0) = (0.5, 1)$ with the presence of noise. We also calculated the approximate values of the value function V at the grid points. This is possible via the formula for the time derivative of V along optimal trajectories:

$$\frac{d}{dt} V(x_t) = -\frac{1}{2} x_t C^T Q C x_t - \frac{1}{2} u_t^T R u_t$$

This differential equation is included, together with the backward Hamiltonian system (14), in the system to be solved, while the initial points are taken to be $(x_i(0), -Px_i(0), \frac{1}{2}x_i(0)^T Px_i(0))$.

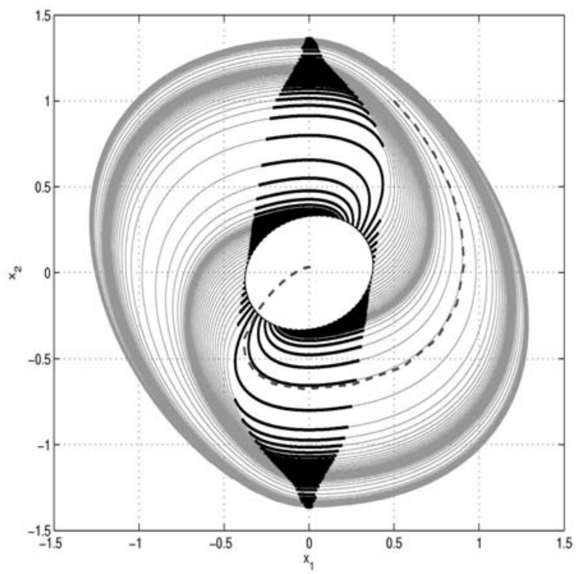

Fig. 4. Unstable system: optimal trajectories obtained via backward integration of the Hamiltonian system.

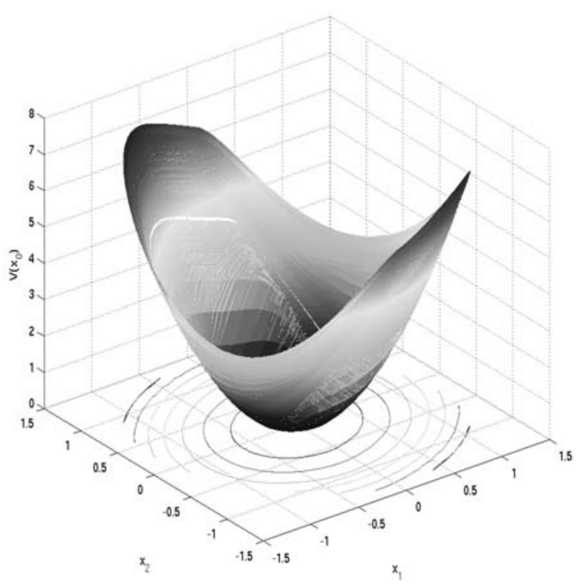

Fig. 5. The value function for (1) with dynamics (16) and $u(t) \in [-1, 1]$.

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