2005 American Control Conference
June 8-10, 2005. Portland, OR, USA

Quadratic feedback linearization and normal forms for two-input discrete-time systems

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Abstract: Two quadratic normal forms and their invariants are introduced for 2-input discrete-time dynamics. The study is based on a differential/difference representation of those dynamics.

Keywords: Nonlinear discrete-time systems, quadratic approximations, normal forms.

I. INTRODUCTION

Controller normal forms have been introduced and further discussed in the continuous-time control literature (see [8], [7], [17] and the references therein). The idea, first developed by Poincaré in the context of dynamical systems [18], to classify intrinsic nonlinearities and so bifurcations brings, in the control context, to the design of coordinates change and feedback simplifying as much as possible the nonlinearities. The procedure is iterative and worked out step-by-step on homogeneous polynomial approximations of increasing order around the linear part. Assuming controllability of the linear approximation, if complete simplification is achievable, one speaks about the property of feeedback linearizability up to a certain degree of approximation, if not, the remaining terms are the resonance terms describing the normal form at a certain degree of approximation. Normal forms techniques have been more recently introduced in discrete-time. While this is not the case for dynamical systems, specific studies are in fact necessary when controlled dynamics are investigated mainly because of the nonlinearity in the control variable. In ([1] [5], [9]), quadratic and cubic normal forms are described for single-input dynamics in the usual formalism of nonlinear difference representations. Normal form of degree m are described in [6]. In [13], the authors introduced quadratic normal forms for nonlinear discrete-time dynamics represented as two coupled differential/difference equations (DDR). The structure of the equations which describe the normal forms in this context, though different from the one obtained on the bases of the usual representation, makes it possible, for the first time in discrete-time, to define two types of normal forms, their invariants and the links between invariants and resonance terms (see [13] for the quadratic case, [15] for a more general traitment). It is whorty to note that, when dealing with dynamics controllable in first approximation, such a DDR can be assumed the starting point without any loss of generality.

The object of the present paper is to generalize these results to the multi-input case. To simplify the statement, the study is developed for 2-input dynamic whit controllable linear part and equal controllability indices; quadratic approximations are investigated but their generalization to approximations of degree m or the investigation of the multi-input case does not bring to major difficulties except the notations. Up to the authors knowledge, normal forms for 2-input dynamics are investigated in the continuous-time case in [16] only, where one type of normal forms of degree m is characterized with their invariants.

The paper is organized as follows. Section 2 is devoted to define the context, recall the adopted differential/difference representation and set the problem. The concepts of quadratic feedback transformation, feedback and linear feedback equivalences are formulated in the proposed differential/difference set up. Section 3 contains the results. Quadratic feedback equivalence to a linear dynamics is characterized in terms of the solvability of the quadratic homological equations. Two types of normal forms are introduced depending if one privileges full cancellation of the quadratic terms either in the drift of in the control vector fields. A set of quadratic invariants which specify the resonance terms are also defined.

II. 2-INPUT DDR AND PROBLEM STATEMENT

We consider a 2-input discrete-time dynamics, $\zeta \rightarrow F(\zeta, v_1, v_2)$, controllable in first approximation around the equilibrium pair $(0,0)$. As in [16], we assume that the input channels have the same controllability index, $equi-controllability\ assumption$. Without loss of generality, as explained lateron, we set the study on the differential/difference representation (DDR) of discrete-time dynamics (see [12] and in the 2-input case [14]). Given an analytic map F and two analytic vector fields $({}_1G(.,v), {}_2G(.,v))$, on R^n, parameterized by $v=(v_1,v_2) \in \mathcal{V}$, a neighbourhood of 0 in R^2, and satisfying the *compatibility conditions* (1) below; i.e. for any $(i \neq j) \in (1,2)$

$$\left[{}_iG(.,v), {}_jG(.,v)\right]\Big|_x = \frac{\partial_i G(.,v)}{\partial v_j} - \frac{\partial_j G(.,v)}{\partial v_i} \quad (1)$$

the DDR of a discrete-time dynamics is described as three set of equations; a difference one modeling the free evolution and two partial-derivatives ones modeling the control action

$$\begin{aligned} \zeta^+ &= F(\zeta); \quad \zeta^+(0) = \zeta^+ & (2) \\ \frac{\partial \zeta^+(v)}{\partial v_i} &= {}_iG(\zeta^+(v), v); \quad i = (1,2). & (3) \end{aligned}$$

$\zeta^+(v)$ figurates a curve in R^n, parameterized by v.

• Under (1), the completeness of the ${}_iG(.,v)$'s ensures the existence of the flow associated with (3), then a nonlinear difference equation, $\zeta(k+1) = F(\zeta(k), v(k))$, can be recovered by integrating (3) successively between 0 and $v_2(k)$ for $v_1(k) = cst$ and 0 and $v_1(k)$ for $v_2(k) = 0$ with state initialization at $\zeta^+ = F(\zeta(k))$.

• Reversing the arguments and starting from a nonlinear discrete-time difference equation, $\zeta(k+1) = F(\zeta(k), v(k))$, the existence of ${}_iG(.,v)$ satisfying

$$\frac{\partial F(\cdot, v)}{\partial v_i} = {}_iG(F(\cdot, v), v) \quad (4)$$

is s ufficient to prove the existence of (2-3) with $F := F(.,0)$. The invertibility of F and thus that of $F(.,v)$ for $v \in \mathcal{V}$, ensure the existence of ${}_iG(.,v)$ satisfying (4) setting ${}_iG(\zeta, v) := \frac{\partial F(\zeta,v)}{\partial v_i}\Big|_{\zeta = F^{-1}(\zeta,v)}$. Dynamics controllable in first approximation admit under feedback such a represetna-tion.

• Computing the expansion of ${}_iG(.,v)$ in powers of v around 0, we get

$$ {}_iG(.,v) = {}_iG_1(.) + \sum_{p\geq 1} \sum_{i_1,\ldots,i_p=1}^{2} {}_{ii_1\cdots i_p}G_{p+1}(.) \frac{v_{i_1}\ldots v_{i_p}}{p!}$$

so defining the set of vector fields ${}_{ii_1\cdots i_p}G_{p+1}$. By construction, for any permutation σ of a generic multi-index of lenght p, say $\eta = (i_1 \ldots i_p)$, we have ${}_{ii_1 i_2 \ldots i_p}G_{p+1} = {}_{i\sigma(i_1)\sigma(i_2)\ldots\sigma(i_p)}G_{p+1}$; any element ${}_{i_1\cdots i_p}G_p$ is denoted by ${}_\eta G_p$ and is
said by convention of degree p.

Before to proceed in terms of homogeneous approximations, interpreating the existence of a normal form of a certain degree m as the capability of rendering it linear under feedback up to this degree, let us recall the necessary and sufficient conditions [3], ensuring full feedback linearization of a discrete-time dynamics represented as a DDR; see also ([4], [10], [11]) in the usual setting.

Theorem 2.1: A DDR (2-3) is locally equivalent through coordinates change and feedback to a linear dynamics if and only if for any η and $l = (0, \ldots, r-2)$

$$\begin{aligned} &(i) && span({}_\eta G_2, {}_\eta G_3, \ldots) \subset span(G_1) \\ &(ii) && \text{the distribution } \mathcal{L}(G_1, \ldots, F_*^l G_1) \\ & && \text{is involutive around } 0 \\ &(iii) && Rank(G_1(0), \ldots, F_*^{r-1}G_1(0)) = n. \end{aligned}$$

F_*G denotes the transport of G along F, defined as the vector field verifying $F_*G|_F = (J_xF)G$ ($J_x(.)$ indicates the jacobian of the function into the parentheses); analogously indicating by $F^p = F \circ \ldots \circ F$, the $p-$times composition of F, F_*^pG denotes the transport of G along F^p.

We note that $Rank({}_1G_1(0), {}_2G_1(0)) = 2$ in a neighbourhood of 0 and the linear controllability assumption imply (iii).

In the sequel, we thus consider without loss of generality a DDR (2-3), say $\Sigma^{[\infty]}$, with linear approximation around $(0,0)$ given by

$$\begin{aligned} \zeta^+ &= A\zeta = J_\zeta F\Big|_{\zeta=0}\zeta; \quad \zeta^+(0) = \zeta^+ & (5) \\ \frac{\partial \zeta^+(v)}{\partial v_i} &= B_i = {}_iG_1(0); & (6) \end{aligned}$$

with

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}; \quad A_i = \begin{pmatrix} 1 & 1 & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ & & & \ddots & 1 \\ 0 & \ldots & \ldots & 0 & 1 \end{pmatrix} \quad (7)$$

$$B_1 = (0\ldots010\ldots0)^T; \quad B_2 = (0\ldots00\ldots01)^T \quad (8)$$

The following assumptions are set:

• $rank(B_i, AB_i, \ldots, A^{r_i-1}B_i) = r_i$ with $r_1 + r_2 = n$

• $r_1 = r_2 = r : equi - controllability\ assumption$

• the distribution $\mathcal{G}^{[2]} : span\Big(B_1 +_1 G_1^{[1]}, B_2 +_2 G_1^{[1]}\Big)$ is involutive, modulo terms in $R^{\geq 1}(\zeta)$

• The compatibility conditions (1) are satisfied modulo terms in $R^{\geq 1}(\zeta)$.

Throughout the paper the upperscrip $(.)^{[m]}$ stands for the homogeneous term of order m of the Taylor series expansion of the function or vector field into the parentheses. Analogously, $R^{[m]}(.)$ (resp. $R^{\geq m}(.)$) stands for the space of vector fields or functions whose components are polyno mial of degree m (resp. formal power series of degree $\geq m$) in the variables into the parentheses.
Let $\Sigma^{[2]}$ be the quadratic approximation around (A, B_1, B_2) of $\Sigma^{[\infty]}$

$$\begin{aligned} \zeta^+ &= A\zeta + F^{[2]}(\zeta); \quad \zeta^+(0) = \zeta^+ \\ \frac{\partial \zeta^+(v)}{\partial v_i} &= B_i + {}_iG_1^{[1]}(\zeta^+(v)) + \sum_{j=1}^{2} {}_{ij}G_2^{[0]} v_j \end{aligned}$$

with (A, B_1, B_2) as in (7-8). $(F^{[2]}, {}_iG_1^{[1]}, {}_{ij}G_2^{[0]})$ denote respectively the quadratic, linear and constant parts of $(F, {}_iG_1, {}_{ij}G_2)$.
In terms of quadratic approximations, the compatibility conditions (1) reduce, modulo terms in $R^{\geq 1}(\zeta)$, to

$$J_x({}_2G_1^{[1]})B_1 - J_x({}_1G_1^{[1]})B_2 = {}_{12}G_2^{[0]} - {}_{21}G_2^{[0]} \quad (9)$$

which implies, due to the involutivity of $\mathcal{G}^{[2]}$, $({}_{12}G_2^{[0]} - {}_{21}G_2^{[0]}) \in span(B_1, B_2)$ and thus ${}_{12}G_2^{[0]}$=${}_{21}G_2^{[0]}$ except the $r-th$ and $n-th$ components;

As we do not want to modify the linear part, a quadratic feedback transformation $\Gamma^{[2]}$ is defined by a quadratic coordinates change, $x = \zeta + \phi^{[2]}(\zeta)$ and a quadratic regular static-state feedback $v_i = u_i + {}_i\gamma^{[2]}(\zeta, u); i = (1, 2)$, with

$$\begin{array}{rcl} {}_i\gamma^{[2]}(\zeta, u) & = & {}_i\gamma_0^{[2]}(\zeta) + u_1 {}_i\gamma_{10}^{[1]}(\zeta) + u_2 {}_i\gamma_{01}^{[1]}(\zeta) \\ & + & \dfrac{u_1^2}{2!} {}_i\gamma_{20}^{[0]} + \dfrac{u_2^2}{2!} {}_i\gamma_{02}^{[0]} + u_1 u_2 {}_i\gamma_{11}^{[0]}. \end{array}$$

$\phi^{[2]}$ and the ${}_i\gamma_{kl}^{[2-k-l]}$'s, are respectively R^n and R-valued mappings with $\frac{\partial \gamma^{[2]}(\zeta,u)}{\partial u}|_{u=0}$ full rank. In the sequel, when no ambiguity is possible, we note

$$\gamma_{kl}^{[2-k-l]}B = {}_1\gamma_{kl}^{[2-k-l]}B_1 + {}_2\gamma_{kl}^{[2-k-l]}B_2.$$

It is now possible to formulate the general question asked in this paper: up to what extend is it possible to simplify the nonlinearities of $\Sigma^{[2]}$ and thus to achieve quadratic transformation $\Gamma^{[2]}$?

III. Quadratic equivalence and normal forms

Given $\Sigma^{[\infty]}$, quadratic linear feedback equivalence corresponds to complete cancellation under $\Gamma^{[2]}$ of the terms of degree 2 in (2) and 1 in (3); when this is not achievable, the remaining terms define the so-called *resonance terms* and *quadratic normal forms*. Thanks to the introduced formalism, quadratic linear feedback equivalence can be reported either to the solvability of a family of equations, the *homological equations* or to satisfy Theorem 2.1 modulo quadratic approximations. Failure of their solvability enables us to define either the *quadratic normal forms* or the *quadratic invariants*. To do so, let $\tilde{\Sigma}^{[2]}$ be another quadratic dynamics.

Definition 3.1: $\Sigma^{[2]}$ is quadratically feedback equivalent to $\tilde{\Sigma}^{[2]}$ if there exists an homogeneous feedback transformation $\Gamma^{[2]}$ which brings $\Sigma^{[2]}$ into $\tilde{\Sigma}^{[2]}$ modulo terms in $R^{\geq 3}(\zeta, v)$.

Definition 3.2: $\Sigma^{[2]}$ is quadratically linear feedback equivalent if there exists a quadratic feedback transformation $\Gamma^{[2]}$ which brings $\Sigma^{[2]}$ into (A, B_1, B_2) modulo terms in $R^{\geq 3}(\zeta, v)$.

In the sequel, referring to Definitions 3.1 and 3.2 respectively, we will speak about *quadratic feedback equivalence* (resp. *quadratic linear feedback equivalence*).

A. The action of quadratic feedback transformation

Let us work out the action of $\Gamma^{[2]}$ over $\Sigma^{[2]}$. The transformation $\Gamma^{[2]}$ being composed w ith two parts, $\phi^{[2]}$ acts as a usual coordinates change so that $(F^{[2]}, {}_iG_1^{[1]}, {}_{ij}G_2^{[0]})$ are simply transformed into $(\bar{F}^{[2]}, {}_i\bar{G}_1^{[1]}, {}_{ij}\bar{G}_2^{[0]})$ below

$$\begin{array}{rcll} \bar{F}^{[2]}(.) & = & F^{[2]}(.) + \phi^{[2]}(A.) - A\phi^{[2]}(.) & (10) \\ {}_i\bar{G}_1^{[1]}(.) & = & {}_iG_1^{[1]}(.) + \dfrac{d\phi^{[2]}}{d\zeta}B_i; \quad i = (1, 2) & (11) \\ {}_\eta\bar{G}_2{}^{[0]} & = & {}_\eta G_2^{[0]} & (12) \end{array}$$

The feedback action further transforms (10) to (12) into

$$\begin{array}{rcl} \tilde{F}^{[2]}(.) & = & \bar{F}^{[2]}(.) + \gamma_0^{[2]}(.)B \\ {}_1\tilde{G}_1{}^{[1]}(.) & = & {}_1\bar{G}_1{}^{[1]}(.) + \gamma_{10}^{[1]}(A^{-1}.)B \\ {}_2\tilde{G}_1{}^{[1]}(.) & = & {}_2\bar{G}_1{}^{[1]}(.) + \gamma_{01}^{[1]}(A^{-1}.)B \\ {}_{11}\tilde{G}_2{}^{[0]} & = & {}_{11}\bar{G}_2{}^{[0]} + \gamma_{20}^{[0]}B - \gamma_{10}^{[1]}(A^{-1}B_1)B \\ {}_{12}\tilde{G}_2{}^{[0]} & = & {}_{12}\bar{G}_2{}^{[0]} + \gamma_{11}^{[0]}B - \gamma_{10}^{[1]}(A^{-1}B_2)B \\ {}_{21}\tilde{G}_2{}^{[0]} & = & {}_{21}\bar{G}_2{}^{[0]} + \gamma_{11}^{[0]}B - \gamma_{01}^{[1]}(A^{-1}B_1)B \\ {}_{22}\tilde{G}_2{}^{[0]} & = & {}_{22}\bar{G}_2{}^{[0]} + \gamma_{02}^{[0]}B - \gamma_{01}^{[1]}(A^{-1}B_2)B \end{array}$$

because, up to an error in $R^{\geq 2}(\zeta)$

$$\begin{array}{rcl} \zeta & = & A^{-1}\zeta^+(v) - vA^{-1}B \\ dv_i & = & du_i + {}_i\gamma_{10}^{[1]}(A^{-1}x^+(u) - uA^{-1}B)du_1 \\ & + & {}_i\gamma_{01}^{[1]}(A^{-1}x^+(u) - uA^{-1}B)du_2 + {}_i\gamma_{11}^{[0]}u_2du_1 \\ & + & {}_i\gamma_{11}^{[0]}u_1du_2 + {}_i\gamma_{20}^{[0]}u_1du_1 + {}_i\gamma_{02}^{[0]}u_2du_2 \end{array}$$

and up to an error in $R^{\geq 3}(\zeta, v)$

$$\begin{array}{rcl} \gamma^{[2]}(., u)|_{u=0} & = & \gamma_0^{[2]}; \quad \zeta^+(v)|_{v=\gamma_0^{[2]}} = \zeta^+(0) + \gamma_0^{[2]}B \\ x^+(v) & = & \zeta^+(v) + \phi^{[2]}(\zeta^+(v)). \end{array}$$

In conclusion, $\Gamma^{[2]}$ transforms $\Sigma^{[2]}$ into $\tilde{\Sigma}^{[2]}$ described by

$$\begin{array}{rcll} x^+ & = & Ax + F^{[2]}(x) & (13) \\ & + & \phi^{[2]}(Ax) - A\phi^{[2]}(x) + \gamma_0^{[2]}(x)B & (14) \\ \dfrac{\partial x^+(u)}{\partial u_1} & = & B_1 + {}_1G_1^{[1]}(x^+(u)) + \left.\dfrac{d\phi^{[2]}}{dx}\right|_{x^+(u)} B_1 & \\ & + & \gamma_{10}^{[1]}(A^{-1}x^+(u))B & \\ & + & u_1({}_{11}G_2^{[0]} - \gamma_{10}^{[1]}(A^{-1}B_1)B + \gamma_{20}^{[0]}B) & \\ & + & u_2({}_{12}G_2^{[0]} - \gamma_{10}^{[1]}(A^{-1}B_2)B - \gamma_{11}^{[0]}B) & (15) \\ \dfrac{\partial x^+(u)}{\partial u_2} & = & B_2 + {}_2G_1^{[1]}(x^+(u)) + \left.\dfrac{d\phi^{[2]}}{dx}\right|_{x^+(u)} B_2 & \\ & + & \gamma_{01}^{[1]}(A^{-1}x^+(u))B & \\ & + & u_1({}_{21}G_2^{[0]} - \gamma_{01}^{[1]}(A^{-1}B_1)B + \gamma_{11}^{[0]}B) & \\ & + & u_2({}_{22}G_2^{[0]} - \gamma_{01}^{[1]}(A^{-1}B_2)B + \gamma_{02}^{[0]}B). & (16) \end{array}$$

It is a matter of computations to verify that the involutivity of $\tilde{\mathcal{G}}^{[2]}$ as well as the compatibility conditions are maintained on the transformed system $\tilde{\Sigma}^{[2]}$. The following results are immediately deduced from equalities (14) to (16).

Proposition 3.1: Given $\Sigma^{[2]}$ and $\tilde{\Sigma}^{[2]}$, the quadratic feedback equivalence problem is solvable if and only if there exist

$(\phi^{[2]}, {}_i\gamma_{kl}^{[2-k-l]})$ satisfying

$$\begin{aligned}
\tilde F^{[2]}(.) - F^{[2]}(.) &= \phi^{[2]}(A.) - A\phi^{[2]}(.) + \gamma_0^{[2]}(.)B \\
{}_1\tilde G_1^{[1]}(.) - {}_1G_1^{[1]}(.) &= \frac{d\phi^{[2]}(.)}{d\zeta}B_1 + \gamma_{10}^{[1]}(A^{-1}.)B \\
{}_2\tilde G_1^{[1]}(.) - {}_2G_1^{[1]}(.) &= \frac{d\phi^{[2]}(.)}{d\zeta}B_2 + \gamma_{01}^{[1]}(A^{-1}.)B \\
{}_{11}\tilde G_2^{[0]} - {}_{11}G_2^{[0]} &= -\gamma_{10}^{[1]}(A^{-1}B_1)B + \gamma_{20}^{[0]}B \\
{}_{12}\tilde G_2^{[0]} - {}_{12}G_2^{[0]} &= -\gamma_{10}^{[1]}(A^{-1}B_2)B + \gamma_{11}^{[0]}B \\
{}_{21}\tilde G_2^{[0]} - {}_{21}G_2^{[0]} &= -\gamma_{01}^{[1]}(A^{-1}B_1)B + \gamma_{11}^{[0]}B \\
{}_{22}\tilde G_2^{[0]} - {}_{22}G_2^{[0]} &= -\gamma_{01}^{[1]}(A^{-1}B_2)B + \gamma_{02}^{[0]}B.
\end{aligned}$$

Proposition 3.2: Given $\Sigma^{[2]}$ with linear approximation (A, B_1, B_2), the quadratic linear feedback equivalence problem is solvable if and only if there exist $(\phi^{[2]}, {}_i\gamma_{kl}^{[2-k-l]})$ satisfying the *quadratic homological equations*

$$-F^{[2]}(.) = \phi^{[2]}(A.) - A\phi^{[2]}(.) + \gamma_0^{[2]}(.)B \quad (17)$$

$$-{}_1G_1^{[1]}(.) = \frac{d\phi^{[2]}(.)}{d\zeta}B_1 + \gamma_{10}^{[1]}(A^{-1}.)B \quad (18)$$

$$-{}_2G_1^{[1]}(.) = \frac{d\phi^{[2]}(.)}{d\zeta}B_2 + \gamma_{01}^{[1]}(A^{-1}.)B$$

$$\begin{aligned}
-{}_{11}G_2^{[0]} &= -\gamma_{10}^{[1]}(A^{-1}B_1)B + \gamma_{20}^{[0]}B \\
-{}_{22}G_2^{[0]} &= -\gamma_{01}^{[1]}(A^{-1}B_2)B + \gamma_{02}^{[0]}B \quad (19)\\
-{}_{12}G_2^{[0]} &= -\gamma_{10}^{[1]}(A^{-1}B_2)B + \gamma_{11}^{[0]}B \\
-{}_{21}G_2^{[0]} &= -\gamma_{01}^{[1]}(A^{-1}B_1)B + \gamma_{11}^{[0]}B.
\end{aligned}$$

B. Quadratic normal forms

Quadratic normal forms are exactly those dynamics which contain the resonance terms which cannot be cancelled according to (17-19) under an appropriate choice of $(\phi^{[2]}, \gamma_{kl}^{[2-k-l]})$.

Theorem 3.1: The 2-input discrete-time dynamics (2-3) with controllable linear approximation, equal controllability indices and involutive control distribution $\mathcal{G}^{[2]}$, is locally quadratically feedback equivalent to one of the two quadratic normal forms, $\Sigma_{NFA}^{[2]}$ or $\Sigma_{NFB}^{[2]}$.

First type of normal form - linearity of the drift - $\Sigma_{NFA}^{[2]}$

$$\begin{aligned}
x^+ &= Ax \quad \text{and for} \quad i = (1,2) \\
\frac{\partial x^+(u)}{\partial u_i} &= B_i + {}_iG_1^{[1]}(x^+(u)) + {}_{i1}G_2^{[0]}u_1 + {}_{i2}G_2^{[0]}u_2
\end{aligned}$$

with

$${}_iG_1^{[1]} = (0, {}_iG_{1;2}^{[1]}, \ldots, {}_iG_{1;r-1}^{[1]}, 0, 0, {}_iG_{1;r+2}^{[1]}, \ldots, {}_iG_{1;n-1}^{[1]}, 0)^T$$

and for $p = (2, ..., r-1)$, $q = (0, r)$

$${}_1G_{1;p+q}^{[1]}(x) = \sum_{l=r-p+2}^{r} ({}_1g_{p+q,l}x_l + {}_1g_{p+q,l+r}x_{l+r})$$

$${}_2G_{1;p+q}^{[1]} = \sum_{l=r-p+2}^{r} ({}_2g_{p+q,l}x_l + {}_2g_{p+q,l+r}x_{l+r} + {}_2k_{p+q,l-1}x_{l-1})$$

with ${}_1g_{p+q,2r} = {}_2g_{p+q,r}$, and for $(i,j) \in (1,2)$

$${}_{ij}G_2^{[0]} = ({}_{ij}g_1, \ldots, {}_{ij}g_{r-1}, 0, {}_{ij}g_r, \ldots, {}_{ij}g_{n-1}, 0)^T. \quad (20)$$

with ${}_{12}G_2^{[0]} = {}_{21}G_2^{[0]}$. The $({}_ig_{p,l}, {}_ik_{p,l}, {}_{ij}g_l)$'s are real coefficients.

Second type of normal form - linearity of G_1 - $\Sigma_{NFB}^{[2]}$

$$\begin{aligned}
x^+ &= Ax + F^{[2]}(x) \\
\frac{\partial x^+(u)}{\partial u_i} &= B_i + {}_{ii}G_2^{[0]}u_i; \quad i = (1,2)
\end{aligned}$$

with $F^{[2]} = (F_1^{[2]}, \ldots, F_{r-2}^{[2]}, 0, 0, F_{r+1}^{[2]}, ldots, F_{n-2}^{[2]}, 0, 0)^T$ and for $p = (1, ..., r-2)$, $q = (0, r)$

$$\begin{aligned}
F_{p+q}^{[2]}(x) &= \sum_{l=p+2}^{r} (P_{p+q,l}x_l^2 + P_{p+q,l+r}x_{l+r}^2 \\
&+ Q_{p+q,l}x_lx_{l+r} + R_{p+q,l}x_lx_{l+r-1}).
\end{aligned}$$

${}_{12}G_2^{[0]} = {}_{21}G_2^{[0]} = 0$ and for $i = (1,2)$, ${}_{ii}G_2^{[0]}$ is given by (20). The $(P_{p,l}, Q_{p,l}, R_{p,l})$'s are real coefficients.

Proof: From (17), we immediately deduce that $\gamma_0^{[2]}$ can be used to cancel all the terms into the components r and $2r = n$ of $F^{[2]}$. Analogously, from (18), $\gamma_{10}^{[1]}$ (resp. $\gamma_{01}^{[1]}$) can be used to cancel the components r and $2r$ of ${}_1G_1^{[1]}$ (resp. ${}_2G_1^{[1]}$) while the other components would satisfy for $p \neq r, p \neq n$ the equalities

$${}_1G_{1;p}^{[1]}(x) = \frac{\partial \phi_p^{[2]}(x)}{\partial x_r}; {}_2G_{1;p}^{[1]}(x) = \frac{\partial \phi_p^{[2]}(x)}{\partial x_n} \quad (21)$$

where $\phi_p^{[2]}$ (resp. ${}_iG_{1;p}^{[1]}$), indicates the p-th component of $\phi^{[2]}$ (resp. ${}_iG_1^{[1]}$). (21) can be simultaneously solved by a suitable $\phi_p^{[2]}$ due to the involutivity of $\mathcal{G}^{[2]}$ which ensures that $\frac{\partial_1 G_{1;p}^{[1]}(x)}{\partial x_n} = \frac{\partial_2 G_{1;p}^{[1]}(x)}{\partial x_r}$ except for $p = r$ and $p = n$. From (19), $\gamma_{20}^{[0]}$ (resp. $\gamma_{02}^{[0]}$) can be used to cancel the components r and n of ${}_{ii}G_2^{[0]}$ so getting (20) and $\gamma_{11}^{[0]}$ to achieve ${}_{12}G_2^{[0]} = 0$ and thus ${}_{21}G_2^{[0]} = 0$ because ${}_{12}G_2^{[0]} - {}_{21}G_2^{[0]} = 0$ due to the compatibility conditions and involutivity of $\mathcal{G}^{[2]}$. From (17) on one hand and (21) on the other hand, it can now be deduced that two classes of normal forms are in fact possible depending if one privileges complete cancellation of the terms in the drift term by solving (17) or in the forced terms by solving (21). Depending on our choice, resonance terms are present either in components 2 to $r-1$ and $r+2$ to $n-1$ of ${}_iG_1^{[1]}$ or in components 1 to $r-2$ and $r+1$ to $n-2$ of the drift $F^{[2]}$.

First type of normal form - linearity of the drift - $\Sigma_{NFA}^{[2]}$

Let the Taylor expansion of any $\phi_p^{[2]}$ be $\phi_p^{[2]}(x) := \sum_{1\le i\le j\le n} \phi_{p;ij}x_ix_j$. Writing down (17) for $i = (1, ..., n-1); i \neq r$)

$$-F_i^{[2]}(x) = \phi_i^{[2]}(Ax) - \phi_1^{[2]}(x) - \phi_{i+1}^{[2]}(x). \quad (22)$$

we verify that $\phi_2^{[2]}$ (resp. $\phi_{r+2}^{[2]}$) can be used to cancel all the terms of order 2 in $F_1^{[2]}$ (resp. $F_{r+1}^{[2]}$) up to $\phi_r^{[2]}$ (resp. $\phi_n^{[2]}$) which can be used to cancel all the terms in $F_{r-1}^{[2]}$ (resp. $F_{n-1}^{[2]}$). The drift term has been thus fully linearized. Doing so, we kept as freedom degrees all the coefficients in $\phi_1^{[2]}$

and $\phi_{r+1}^{[2]}$ which can be used to solve at most as possible (21). First, (21) can be satisfied for $p=(1,r+1)$ with an adequate choice of $(\phi_{1;ir},\phi_{1;in},\phi_{r+1;ir},\phi_{r+1;in})$. Then, noticing that the equalities (22) from $i=1$ to $n-1$ induce a relation between the coefficients $\phi_{p;ij}$, it results that for $p\geq 2$ and $j\geq 2$, the coefficients $\phi_{p-1;ij-1}$ can be used in place of the $\phi_{p;ij}$ to solve (21). An adequate choice of $(\phi_{1;ir-1},\phi_{r+1;ir-1}; i\neq (r,n))$ (resp. $\phi_{1;in-1},\phi_{r+1;in-1}; i\neq (r-1,r,n)$) enables us to solve (21) for $p=(2,r+2)$; the non removable terms being in x_r and x_n (resp. (x_r,x_{r-1},x_n)) in rows 2 and $r+2$ of ${}_1G_1{}^{[1]}$ (resp. ${}_2G_1{}^{[1]}$). Iterating the reasonning, (21) can be solved for $p=(r-1,n-1)$ with an adequate choice of $(\phi_{1;i2},\phi_{r+1;i2}; i\neq (3,...,r,r+3,...,n))$ and $(\phi_{1;ir+2},\phi_{r+1;ir+2}; i\neq (2,3,...,r,r+3,...,n))$; non removable terms are thus in $(x_3,...,x_r,x_{r+3},...,x_n)$ (resp. $(x_2,...,x_r,x_{r+3},...,x_n)$) in rows $r-1$ and $n-1$ of ${}_1G_1{}^{[1]}$ (resp. ${}_2G_1{}^{[1]}$).

Second type of normal form - linearity of G_1 - $\Sigma_{NFB}^{[2]}$

In this case, being (21) completely satisfied, the coefficients kept free are thus the $(\phi_{p;ij}; p\neq (r,n); i,j\neq (r,n))$ which can be used to partially solve (22). More precisely, the $(\phi_{2r-2;in-1}; i\neq (r,n))$ and $(\phi_{2r-2;ir-1}; i\neq (r,n-1,n))$ are used in place of $(\phi_{2r-1;in}; i\neq (r,n))$ and $(\phi_{2r-1;ir}; i\neq (r,n))$ to cancel terms in $F_{2r-2}^{[2]}$; the non removable terms are thus in $(x_r^2,x_n^2,x_rx_n,x_rx_{n-1})$. Then, the free coefficients $(\phi_{2r-3;in-2}; i\neq (r-1,r,n-1,n))$ and $(\phi_{2r-3;ir-2}; i\neq (r-1,r,n-2,n-1,n)$ are use d in place of $(\phi_{2r-2;in-1}; i\neq (r-1,r,n))$ and $(\phi_{2r-2;ir-1}; i\neq (r-1,r,n-1,n)$, to cancel terms in $F_{2r-3}^{[2]}$; the non removable terms are thus in

$$(x_{r-1}^2,x_r^2,x_{n-1}^2,x_n^2,x_{r-1}x_{n-1},x_{r-1}x_n,x_rx_n,x_rx_{n-1}).$$

Iterating the reasonning up to cancel the terms in $F_{r+1}^{[2]}$ and then in $F_{r-2}^{[2]}$ up to $F_1^{[2]}$, one finds $\Sigma_{NFB}^{[2]}$. As ${}_iG_1=B_i$, the compability conditions ensure ${}_{ij}G_2=0; i\neq j$. ■

C. The quadratic invariants

Finally, formulating Theorem 2.1 in terms of quadratic approximations.

Corollary 3.1: Given $\Sigma^{[2]}$, the quadratic linear feedback equivalence problem is solvable if and only if

(i) *span*$({}_{ij}G_2^{[0]})\subset$ *span*(B),

(ii) the distribution $(G_1,\ldots,F_*^{r-2}G_1)$ is involutive around 0 modulo terms in $R^{\geq 1}(\zeta,v)$

(iii) $rank(G_1(0),\ldots,F_*^{r-1}G_1(0))=n$, modulo terms in $R^{\geq 2}(\zeta,v)$.

quadratic linear feedback equivalence can be characterized by a set of integers which are invariant under quadratic feedback transformation. Let

$$C=\begin{pmatrix} 1 & ... & 0 & 0 & ... & 0\\ 0 & ... & 0 & 1 & ... & 0\end{pmatrix}.$$

Definition 3.3: The invariants of degree 2 of $\Sigma^{[2]}$ are the homogeneous parts of degree 0 of the polynomials defined below for $1\leq l\leq r-2$ and $0\leq p\leq r-l-2$

$$ {}_{ii}a_1^{lp+2}=C(A-I)^{l-1}[(F_*-I)^p{}_iG_1,(F_*-I)^{p+1}{}_iG_1] \quad (23)$$

$$ {}_{21}a_1^{lp+2}=C(A-I)^{l-1}[(F_*-I)^{p+1}{}_2G_1,(F_*-I)^{p+1}{}_1G_1] \quad (24)$$

$$ {}_{12}a_1^{lp+2}=C(A-I)^{l-1}[(F_*-I)^{p}{}_1G_1,(F_*-I)^{p+1}{}_2G_1] \quad (25)$$

$$ {}_{ij}a_2^{l}=C(A-I)^{l-1}{}_{ij}G_2;\quad i\neq j,\quad 1\leq l\leq r-1. \quad (26)$$

Theorem 3.2:
- The invariants fully characterize quadratic feedback equivalence; $\Sigma^{[2]}$ and $\tilde{\Sigma}^{[2]}$ have the same quadratic invariants.
- The quadratic linear feedback equivalence problem is solvable if and only if the quadratic invariants are equal to zero.
- The invariants of degree 2 verify w.r.t $\Sigma_{NFB}^{[2]}$ for $1\leq l\leq r-2$ and $0\leq p\leq r-l-2$

$$\begin{aligned}
{}_{11}a_1^{lp+2} &= C(A-I)^{l-1}\frac{\partial^2F^{[2]}(x)}{\partial x_{r-p}^2}\\
{}_{22}a_1^{lp+2} &= C(A-I)^{l-1}\frac{\partial^2F^{[2]}(x)}{\partial x_{2r-p}^2}\\
{}_{21}a_1^{lp+2} &= C(A-I)^{l-1}\frac{\partial^2F^{[2]}(x)}{\partial x_{r-p}\partial x_{2r-p-1}}\\
{}_{12}a_1^{lp+2} &= C(A-I)^{l-1}\frac{\partial^2F^{[2]}(x)}{\partial x_{r-p}\partial x_{2r-p}}
\end{aligned}$$

and for $1\leq l\leq r-1$, ${}_{ij}a_2^l=C(A-I)^{l-1}{}_{ij}G_2^{[0]}$.

Proof: • Computing $F_{*i}^lG_1=A^lB_i+(F_{*i}^lG_1)^{[1]}$ with

$$(F_{*i}^lG_1)^{[1]}=A(F_*^{l-1}{}_iG_1)^{[1]}|_{A^{-1}x}+\frac{\partial F^{[2]}(x)}{\partial x}\Big|_{A^{-1}x}A^{l-1}B_i \quad (27)$$

we show that, up to an error in $R^{\geq 1}(\zeta)$

$$(\tilde{F}_{*i}^l\tilde{G}_1)^{[1]}=(F_{*i}^lG_1)^{[1]}+\frac{d\phi^{[2]}}{dx}A^lB_i+span(A^{l-1}B,A^lB) \quad (28)$$

so that, after easy computations, it can be verified that the bracket of vector fields $[(F_*-I)^p{}_1G_1,(F_*-I)^{p+1}{}_1G_1]$ and $[(\tilde{F}_*-I)^p{}_1\tilde{G}_1,(\tilde{F}_*-I)^{p+1}{}_1\tilde{G}_1]$ differ from their components r up to $r-p-1$ so that the invariants (23) associated with $\Sigma^{[2]}$ and $\tilde{\Sigma}^{[2]}$ coincide. The same reasoning holds true for the invariants (24) and (25) due to the involutivity of $\mathcal{G}^{[2]}$.
• Being (iii) obviously verified due to the controllability of the linear canonical form, it is easy to verify that conditions $(i)-(ii)$ in Corollary 3.1 are in fact equivalent to require the nullity of *the quadratic invariants*. More precisely, due to the form of C, the parallelism condition (i) in Corollary 3.1 is equivalent to the nullity of the invariants (26) while the involutivity condition (ii) is equivalent to the nullity of the invariants (23-24-25). ■

D. Quadratic normal forms on R^6

As an example, let us write down the two types of normal forms on R^6.

First type of normal form (linearity of F): $\Sigma^{[2]}_{NFA}$

$$\begin{aligned}
x_1^+ &= x_1 + x_2; \quad x_2^+ = x_2 + x_3; \quad x_3^+ = x_3 \\
x_4^+ &= x_4 + x_5; \quad x_5^+ = x_5 + x_6; \quad x_6^+ = x_6 \\
\frac{\partial x^+(u)}{\partial u_1} &= \Big(u_1 f + u_2 k, a x_3 + b x_6 + u_1 g + u_2 l, 1, \\
&\quad u_1 \bar{f} + u_2 \bar{k}, \bar{a} x_3 + \bar{b} x_6 + u_1 \bar{g} + u_2 \bar{l}, 0\Big)^T \\
\frac{\partial x^+(u)}{\partial u_2} &= \Big(u_2 i + u_1 k, b x_3 + d x_6 + e x_2 + u_2 j + u_1 l, 0, \\
&\quad u_2 \bar{i} + u_1 \bar{k}, \bar{b} x_3 + \bar{d} x_6 + \bar{e} x_2 + u_2 \bar{j} + u_1 \bar{l}, 1\Big)^T
\end{aligned}$$

Second type of normal form (linearity of G_1) : $\Sigma^{[2]}_{NFB}$

$$\begin{aligned}
x_1^+ &= x_1 + x_2 + a x_3^2 + b x_6^2 + c x_3 x_6 + d x_3 x_5 \\
x_2^+ &= x_2 + x_3; \quad x_3^+ = x_3 \\
x_4^+ &= x_4 + x_5 + \bar{a} x_3^2 + \bar{b} x_6^2 + \bar{c} x_3 x_6 + \bar{d} x_3 x_5 \\
x_5^+ &= x_5 + x_6; \quad x_6^+ = x_6 \\
\frac{\partial x^+(u)}{\partial u_1} &= \Big(u_1 f, u_1, 1, u_1 \bar{f}, u_1 \bar{g}, 0\Big)^T \\
\frac{\partial x^+(u)}{\partial u_2} &= \Big(u_2 i, u_2 j, 0, u_2 \bar{i}, u_2 \bar{j}, 1\Big)^T.
\end{aligned}$$

IV. Conclusion

Two types of quadratic normal forms have been introduced for two-input nonlinear discrete-time dynamics with controllable linear part. The quadratic invariants, a set of numbers which are invariant under quadratic transformations, are introduced and it is shown haw those numbers characterize the terms which cannot be cancelled (resonance terms).

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