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A Robust Controller for Uncertain Nonlinear Systems and Its Application to a Motor-Driven System

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***Abstract* — This paper introduces a robust controller for a class of nonlinear systems in the presence of unmatched uncertainties. The requirement of the proposed control is that the uncertainties are bounded by some known functions. The uncertainties are divided into the matched parts and the unmatched parts, and then are compensated respectively by the specially constructed robust controller. The design of the robust controller includes a class of continuous function *$\varphi(t)$*, such that the robust controller can guarantee the asymptotic stability of the closed-loop control system. Then, the robust controller design is applied to the motor-driven electromechanical system of a certain two-axes, yaw-pitch motion test bed. The simulation results are given.**

Keywords: uncertain nonlinear system, robust control, motor-driven system.

I. Introduction

Over the last several decades, considerable effort [1]–[13] has been directed in the robust stabilization of nonlinear control systems with the presence of uncertainties. The uncertainties involved in the nonlinear systems may be due to unknown dynamics, parameter variations, input disturbances or measurement errors, etc.

In general, uncertainties are difficult to model or measure accurately. However, it is usually possible to obtain some knowledge of their behaviors, e.g. the structure characteristics, the estimated dynamics and certain boundedness definitions, etc. In most of the existing literature, robust controllers are designed to deal with these uncertain systems. There are two main streams in the research of the uncertain nonlinear control systems [10], one is robust stabilization by state feedback [1]-[9], another is robust output tracking and robust output regulation by state feedback linearization or state feedback input/output decoupling[11]-[13].

This paper focuses on the robust stabilization method, which is to design the robust state feedback controller that can guarantee the stability in different conditions of the closed-loop systems. In this kind of research, the Lyapunov direct method is usually employed in the controller design and stability analysis of the nonlinear systems. [1] first considers the matched uncertainties in nonlinear systems, and proposed the min-max control, which can make the closed-loop system asymptotically stable with the discontinuous control law. [2] extends [1] to a continuous saturation-typed control guaranteeing the uniform ultimate boundedness of the uncertain nonlinear system. Instead of the conventional Lyapunov asymptotic stability, [3] and [4] further present the practical stability of the nonlinear systems without the matching assumption. Based on assumptions that the bounded functions exist for the uncertainties, [5] constructs a class of robust controllers which can stabilize the nonlinear systems with matched uncertainties asymptotically in the large. [6] investigates the global asymptotical stabilization of nonlinear systems with unmatched, but equivalently matched uncertainties. When there is insufficient knowledge on the bounding function, the uncertainties could be estimated either by an adaptive robust controller [8] which combines the robust and adaptive control in a Lyapunov-based direct adaptive control framework, or by a reduced-order or full-order nonlinear observer [9].

In this paper, a generalized robust controller is introduced. The controller design is based on the existing results in [6] which can be considered as a special case, and extends the result in [5] to the unmatched uncertainties case. The robust controller is designed constructively. It acts in a more generalized and continuous form. The controller consists of two parts with respect to the matched uncertainties and the unmatched uncertainties respectively, also it can guarantee the asymptotic stability for the closed-loop nonlinear control systems with the presence of both matched and unmatched uncertainties.

This paper is organized as follows: Section II gives the general problem description and the key assumptions that the nonlinear system with unmatched uncertainties must satisfy. Section III is devoted to the design of the robust controller and the proof of asymptotic stability of the closed-loop nonlinear control system. Section IV illustrates the robust controller design procedure with a motor-driven 2 degree-of-freedom electromechanical system. Section V offers some summary remarks.

II. Problem Formulation

In this paper, we consider the following uncertain nonlinear system described by (1),

$$\dot{x} = f(x,t) + \Delta f(x,t) + [B(x,t) + \Delta B(x,t)]u(t) \quad (1)$$
$$x_0 = x(t_0)$$

where $t \in R$ is time, $x(t) \in R^n$ is the state variable, $u(t) \in R^m$ is the control variable, $\Delta f(x, t)$ and $\Delta B(x, t)$ are the uncertainties in system matrix and input matrix, respectively. The corresponding system without uncertainties, called the nominal system, is described by

$$\dot{x} = f(x,t) + B(x,t)u(t) \quad x_0 = x(t_0) \quad (2)$$

Matching Conditions: For all (x,t), there exists continuous vector function $\Delta f_1(x, t): R^n \times R \to R^n$ and continuous matrix $\Delta B_1(x, t): R^n \times R \to R^{m \times m}$ such that :

a) $\Delta f(x, t) = B(x, t)\Delta f_1(x, t)$

b) $\Delta B(x, t) = B(x, t)\Delta B_1(x, t)$

The above a)-b) is referred to as Matching Conditions [1].

If the uncertainties in the system do not satisfy the Matching Conditions, generally, they can be written in the forms of the following descriptions

$$\Delta f(x,t) = B(x,t)\Delta f_1(x,t) + \Delta f'(x,t)$$
$$\Delta B(x,t) = B(x,t)\Delta B_1(x,t) + \Delta B'(x,t)$$

where the $\Delta f'(x, t)$ and $\Delta B'(x, t)$ are the unmatched parts of uncertainties.

By the above description, we obtain,

$$\dot{x} = f(x,t) + e'(x,t) + B(x,t)[u(t) + e(x,t)]$$
$$x(t_0) = x_0 \quad (3)$$

$$e(x,t) = \Delta f_1(x,t) + \Delta B_1(x,t)u(t)$$
$$e'(x,t) = \Delta f'(x,t) + \Delta B'(x,t)u(t) \quad (4)$$

where $e(x,t)$ consists of matched uncertainties, and $e'(x,t)$ consists of unmatched uncertainties.

Concerning the system (3), we introduce the following assumptions.

Assumption 1: There exists a scalar function $V(x, t): R^n \times R \to R$, $\forall (x, t) \in R^n \times R$, satisfying :

1) $V(x, t)$ is positive definite ;

2) $\|x\| \to \infty$, $V(x, t) \to \infty$;

3) $W_0(x, t) = (\partial (x, t)/\partial t) + \nabla_x^T V(x, t) f(x, t)$ is negative definite.

Assumption 2: The origin, $x=0$, is uniformly asymptotically stable for the nominal system (2), In particular, there is a Lyapunov function $V(\cdot,\cdot): R^n \times R \to R+$ and continuous, strictly increasing functions $\gamma_i()$: $R+ \to R+$, $i=1, 2, 3$, which satisfy

$$\gamma_i(0) = 0 \quad i=1, 2, 3$$

$$\gamma_1(\|x\|) \le V(x, t) \le \gamma_2(\|x\|)$$

$$\partial V(x,t)/\partial t + \nabla_x^T V(x, t) f(x, t) \le -\gamma_3(\|x\|)$$

Assumption 3: For system (3) with presence of both matched and unmatched uncertainties, it satisfies

1) The matched uncertainty $e(x, t)$ is bounded by a known function in Euclidean Norm, that is,

$$e(x, t) \in \mathrm{H} = \{e(x, t) \mid \|e(x, t)\| \le \rho(x,t)\}$$

$\forall (x, t)$. $\rho(x, t)$: $R^n \times R \to R+$ is a non-negative Caratheodory function, $\rho(0,t)=0$.

2) The unmatched uncertainty $e'(x, t)$ is bounded by a known function in Euclidean norm,

$$\frac{\left\|\nabla_x^T V(x,t) e'(x,t)\right\|}{\left\|\nabla_x^T V(x,t) B(x,t)\right\|} \le \rho'(x,t)$$

$\rho'(.)$: $R^n \times R \to R+$ is a non-negative Caratheodory function, $\rho'(0,t)=0$.

III. The Controller Design

Based on the assumptions in the stated in the previous section, we propose the following controller.

$$u(x, t) = u_1(x,t) + u_2(x,t) \quad (5)$$

$$u_1(x,t) = -\rho(x,t) \frac{\mu_1(x,t)}{\|\mu_1(x,t)\| + \varphi(t)\varepsilon}$$

$$u_2(x,t) = -\rho'(x,t) \frac{\mu_2(x,t)}{\|\mu_2(x,t)\| + \varphi(t)\varepsilon}$$

where

$$\mu_1(x,t) = B^T(x, t) \nabla_x V(x, t)\rho(x, t)$$

$$\mu_2(x,t) = B^T(x, t) \nabla_x V(x, t)\rho'(x, t)$$

And, we introduce a class of continuous functions $\varphi(t)$ that satisfy $\varphi(t) \in [0, 1]$, with their indefinite integral function $\omega(t) = \int \varphi(t)dt \le 0$.

Theorem: Consider the system (3) satisfying *Assumptions* 1, 2 and 3, under the controller in the form of (5) $u(x, t) = u_1(x,t) + u_2(x,t)$, the state $x(t)$, $x(0)=0$, of the closed-loop system is asymptotically stable.

Proof. Using the same Lyapunov function $V(x,t)$ in *Assumption 1*, we have

$$\dot{V}(x,t) = \frac{\partial V(x,t)}{\partial t} + \nabla_x^T V(x,t)\{ f(x,t) + e'(x,t)$$
$$+ B(x,t)[e(x,t) + u(x,t)]\}$$

$$\leq -\gamma_3(\|x\|) + \nabla_x^T V(x,t)e'(x,t) + \nabla_x^T V(x,t)B(x,t)u_2(x,t)$$
$$+ \nabla_x^T V(x,t)B(x,t)e(x,t) + \nabla_x^T V(x,t)B(x,t)u_1(x,t)$$
$$\leq -\gamma_3(\|x\|) + \|\nabla_x^T V(x,t)B(x,t)\|\rho'(x,t)$$
$$+ \nabla_x^T V(x,t)B(x,t)u_2(x,t)$$
$$+ \|\mu_1\| + \nabla_x^T V(x,t)B(x,t)u_1(x,t)$$

Using the inequality

$$0 \leq \frac{ab}{a+b} \leq a \qquad \forall a,b \geq 0$$

We see that

$$\|\mu_1(x,t)\| + \nabla_x^T V(x,t)B(x,t)u_1(x,t)$$
$$= \|\mu_1(x,t)\| - \frac{\nabla_x^T V(x,t)B(x,t)\rho(x,t)\mu_1(x,t)}{\|\mu_1(x,t)\| + \varphi(t)\varepsilon}$$
$$= \|\mu_1(x,t)\| - \frac{\|\mu_1(x,t)\|^2}{\|\mu_1(x,t)\| + \varphi(t)\varepsilon}$$
$$= \frac{\|\mu_1(x,t)\|\varphi(t)\varepsilon}{\|\mu_1(x,t)\| + \varphi(t)\varepsilon}$$
$$\leq \varphi(t)\varepsilon$$

Similarly,

$$\|\nabla_x^T V(x,t)B(x,t)\|\rho'(x,t) + \nabla_x^T V(x,t)B(x,t)u_2(x,t)$$
$$= \|\mu_2(x,t)\| - \frac{\nabla_x^T V(x,t)B(x,t)\rho'(x,t)\mu_2(x,t)}{\|\mu_2(x,t)\| + \varphi(t)\varepsilon}$$
$$= \|\mu_2(x,t)\| - \frac{\|\mu_2(x,t)\|^2}{\|\mu_2(x,t)\| + \varphi(t)\varepsilon}$$
$$\leq \varphi(t)\varepsilon$$

Hence

$$\dot{V}(x,t) \leq -\gamma_3(\|x\|) + 2\varphi(t)\varepsilon$$

From *Assumption 2,* we have

$$\gamma_1(\|x\|) \leq V(x,t) \leq \gamma_2(\|x\|)$$

$$0 \leq \gamma_1(\|x\|) \leq V(x,t)$$
$$\leq V(x_0,t_0) + \int_{t_0}^{\tau} \dot{V}(x,\tau)d\tau$$
$$\leq \gamma_2(\|x_0\|) + \int_{t_0}^{\tau}[-\gamma_3(\|\tau\|) + 2\varphi(\tau)\varepsilon]d\tau$$
$$= \gamma_2(\|x_0\|) + \int_{t_0}^{\tau}[-\gamma_3(\|\tau\|)d\tau + 2\varepsilon[\omega(t) - \omega(t_0)]$$
$$\leq \gamma_2(\|x_0\|) + \int_{t_0}^{\tau}[-\gamma_3(\|\tau\|)d\tau + 2\varepsilon|\omega(t\ \ )|$$

Therefore

$$\lim_{t\to\infty}\int_0^t \gamma_3(\|\tau\|)d\tau \leq \gamma_2(\|x_0\|) + 2\varepsilon|\omega_0(t)| \langle \infty \qquad (6)$$

Because $\gamma_3(.)$ is continuous and positive definite, we obtain

$$\lim_{t\to\infty}\gamma_3(\|x\|) = 0 \Rightarrow \lim_{t\to\infty}\|x\| = 0 \qquad (7)$$

From these results, it can be seen that: as *t* goes to infinity, the state *x(t)* approaches zero asymptotically.

There are many kinds of choices for the function *φ(t).* The function $e^{-\beta t}$ in [6] can be thought as a special case of *φ(t)*, [5] gives some other examples for the function *φ(t).*

From the *Theorem* and the *Proof* we can know, the robust controller is constructed with respect to the uncertainties deterministically. The design procedure is not complex and the controller is continuous. All these are the benefit advantages in practical implementations.

IV. EXAMPLE

To illustrate the robust controller design, we apply the above method to design the robust controller of a two-axes yaw-pitch electromechanical test bed which is driven by motor systems [14]. Figure 1 indicates the structure of the test bed. The test bed consists of two frames, the inner frame carries out the pitch motion and the outer frame carries out the yaw motion. The inner frame is attached at the center of the outer frame. Each frame can rotate around its own axis, the outer frame rotates along the vertical axis while the inner frame along the horizontal axis. The rotations are driven by separate motor systems. Three reference frames can be introduced: (X, Y, Z) is ground-fixed, (Xo, Yo, Zo) is fixed to the outer frame, (Xi, Yi, Zi) is fixed to the inner frame. The structure of the test bed results that the motion dynamics of the inner frame will be coupled with the motion dynamics of the outer frame.

As an illustrative example, this paper will design the control system of the inner frame, which indeed is a loaded motor-driven system. The motion dynamics can be simplified by the moment equation of pitch motion:

$$J\ddot{\theta} = M + M_f + M_c$$

where θ is the motor rotating angle, J is the moment of inertia of rotor, M is the control torque, M_f is the fiction torque, M_c is the disturbance torque with fluctuant dynamics $M_c(x,t) = \Delta M_c(x,t)\sin(N\theta)$, N is the number of polar pairs of the stator of motor.

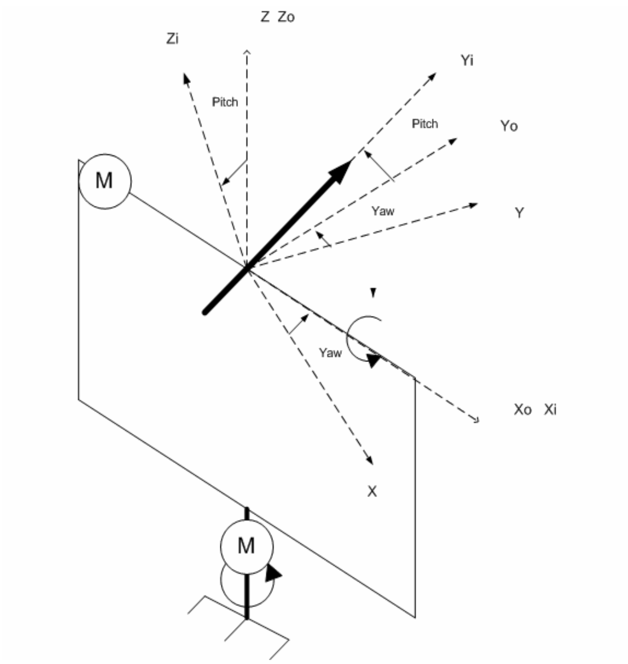

Figure 1.Two-axes testbed system

Both M_f and M_c are time varying unknown dynamics, and are hard to be modeled or measured, so they are considered as the matched uncertainties in the system. Also, when the test bed rotates around its two axes simultaneously, the motion of the inner axis is affected by the motion of the outer axis because of the motion coupling dynamics. In this case, this kind of coupling dynamics can be considered as the unmatched uncertainties.

Let $x_1 = \theta$ and $x_2 = \dot{\theta}$, then

$$\dot{x}_1 = x_2 + \Delta f_1'(x,t)$$
$$\dot{x}_2 = \Delta f'_2(x,t) + \Delta f_1(x,t) + \frac{K_M}{J}u(t)$$

where the $\Delta f_1(x,t)$ is the matched uncertainty and $\Delta f_1'(x,t)$ and $\Delta f_2'(x,t)$ are the unmatched uncertainties due to the coupling dynamics between the two axes.

$$\Delta f_1(x,t) = \frac{M_f(x,t)}{J} + \frac{\Delta M_c(x,t)\sin(Nx_1)}{J}$$

$$\Delta f'(x,t) = \begin{bmatrix} \Delta f'_1(x,t) \\ \Delta f'_2(x,t) \end{bmatrix}$$

Let the controller be of the form

$$u(t)=u_0(x,t)+u_1(x,t)+u_2(x,t)$$

where $u_0(x,t)$ is the controller to stabilize the nominal system, $u_1(x,t)$ and $u_2(x,t)$ is the part of robust control to compensate the matched and unmatched uncertainties.

From the measurements, we get

$J = 55.16\ kg.m^2$
$K_M = 96.272\ N.m/V$
$|M_f(x,t)| \le 39.2\ N.m$
$|M_c(x,t)| \le 20\ N.m$

It is easy to design the nominal control $u_0(t)$ that can guarantee the global asymptotic stability of the nominal system.

From the Ricatti equation

$$PA+A^TP-PBR^{-1}B^TP+Q=0$$

We can obtain the positive matrix P,

$$P = \begin{bmatrix} 1.230 & 0.573 \\ 0.573 & 0.705 \end{bmatrix}$$

Then

$$u_0(t) = -R^{-1}B^TPx = -x_1-1.23x_2$$

To design the robust controller $u_1(x,t)$ and $u_2(x,t)$, the following Lyapunov function is chosen,

$$V(x) = \frac{1}{2}x^TPx$$
$$\nabla_x^TV(x) = \begin{bmatrix}1.23x_1 + 0.573x_2 & 0573x_1 + 0.705x_2\end{bmatrix}$$

There should exist $\rho(x,t)$ and $\rho'(x,t)$, that

$$|\Delta f_1(x)| \le \left|\frac{M_f(t)}{J}\right| + \left|\frac{\Delta M_c(t)}{J}\right| \le \rho(x,t) \le 1.073$$

$$\frac{\left|\nabla_x^TV(x)\cdot\begin{bmatrix}\Delta f'_1(x,t) & \Delta f'_2(x,t)\end{bmatrix}^T\right|}{\left|\nabla_x^TV(x)\begin{bmatrix}0, K_M/J\end{bmatrix}^T\right|} \le \rho'(x,t)$$

Let

$$\mu_1(x,t) = B^T(x,t)\nabla_x^T V(x,t)\rho(x,t)$$
$$= \left[0, {K_M}/{J}\right]^T \cdot \nabla_x V(x) \cdot \rho(x,t)$$
$$= (x_1 + 1.23x_2) \cdot \rho(x,t)$$
$$\mu_2(x,t) = B^T(x,t)\nabla_x^T V(x,t)\rho'(x,t)$$
$$= \left[0, {K_M}/{J}\right]^T \cdot \nabla_x V(x) \cdot \rho'(x,t)$$
$$= (x_1 + 1.23x_2) \cdot \rho'(x,t)$$

Finally, the controller can be

$$u(x,t) = -x_1 - 1.23x_2 + u_1(x,t) + u_2(x,t)$$

$$u_1(x,t) = -\rho(x,t)\frac{(x_1 + 1.23x_2)\cdot\rho(x,t)}{\|(x_1 + 1.23x_2)\cdot\rho(x,t)\| + \varphi(t)\varepsilon}$$

$$u_2(x,t) = -\rho'(x,t)\frac{(x_1 + 1.23x_2)\cdot\rho'(x,t)}{\|(x_1 + 1.23x_2)\cdot\rho'(x,t)\| + \varphi(t)\varepsilon}$$

Simulation is performed using MatLab, with the following choice:

$\varphi(t)=e^{-\beta t}$, where $\beta=1$, $\varepsilon=0.1$;

In this simple case, $\rho(x,t)$ and $\rho'(x,t)$ can be chosen as constants, that is, $\rho(x,t) = 1.73$ and $\rho'(x,t) = 0.1$.

The simulation results of the system step response are shown in Figure 2. First, the simulation results of the nominal system with nominal controller $u_0(x,t)$ and without uncertainties are given in a), then the bad performance of the uncertain system without robust controllers can be seen in b). The last one, c) shows the improved results of uncertain system with the robust controllers $u_1(x,t)$ and $u_2(x,t)$. From Figure 2, it can be seen that the proposed controller achieves much more robust stability with the presence of uncertainties, and ensures the more satisfactory performance of the closed-loop control systems.

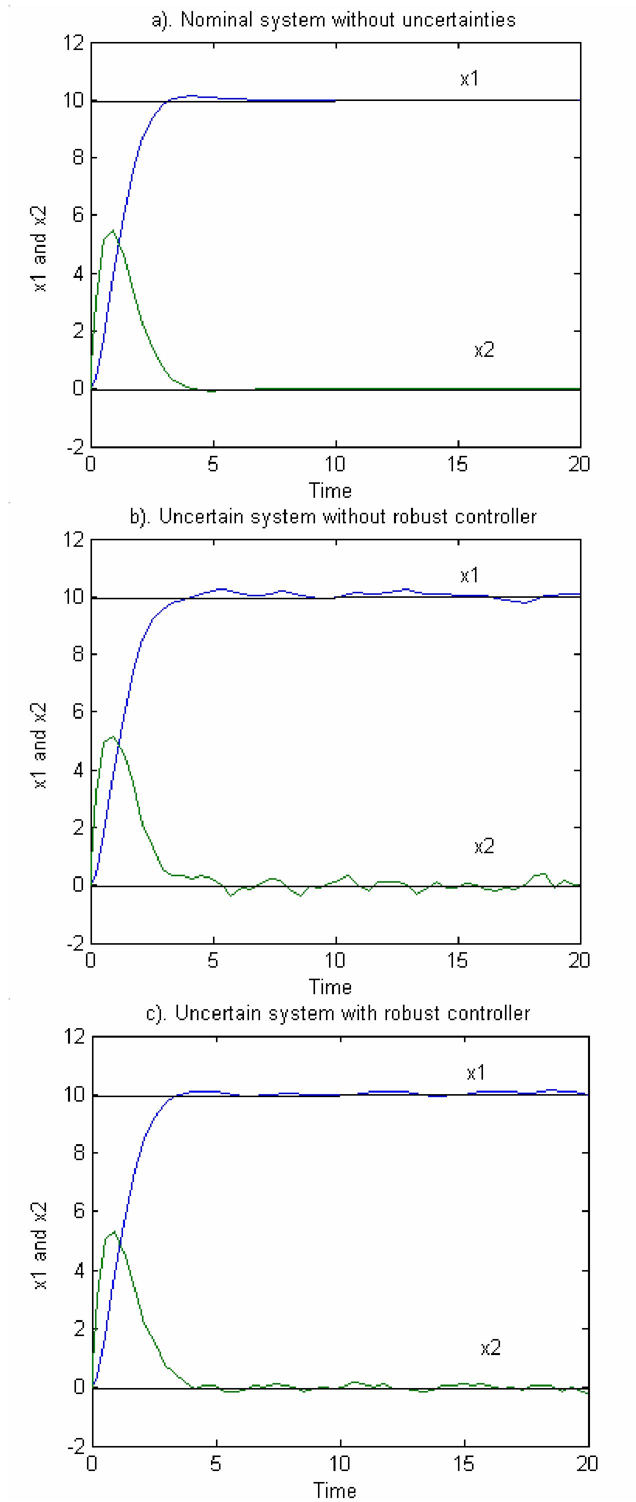

Figure 2. Simulation results of the motor-driven system

V. SUMMARY

This paper proposes a method for the design of a robust controller for a class of nonlinear systems in the presence of unmatched uncertainties, and then the method is applied to a motor-driven electromechanical system. Here, uncertainties in a dynamic system are not assumed to match the conventional matching condition. Instead, as long as the unknown dynamics of the uncertainties can be bounded by some known functions, a robust stabilizing controller can be constructed with the introduction of a class of functions. The controller is continuous and easy to implement and guarantees the asymptotic stability of the closed-loop nonlinear control systems.

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