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Neural Network Adaptive Control for Nonlinear Uncertain Dynamical Systems with Asymptotic Stability Guarantees

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Abstract

A neuro adaptive control framework for nonlinear uncertain dynamical systems with input-to-state stable internal dynamics is developed. The proposed framework is Lyapunov-based and unlike standard neural network controllers guaranteeing ultimate boundedness, the framework guarantees partial asymptotic stability of the closed-loop system, that is, asymptotic stability with respect to part of the closed-loop system states associated with the system plant states. The neuro adaptive controllers are constructed without requiring explicit knowledge of the system dynamics other than the assumption that the plant dynamics are continuously differentiable and that the approximation error of uncertain system nonlinearities lie in a small gain-type norm bounded conic sector. This allows us to merge robust control synthesis tools with neural network adaptive control tools to guarantee system stability. Finally, an illustrative numerical example is provided to demonstrate the efficacy of the proposed approach.

1. Introduction

One of the main motivation for developing neural network adaptive control algorithms is their capability to approximate a large class of continuous nonlinear maps from the collective action of very simple, autonomous processing units that are connected in simple ways. These processing units involve a weighted interconnection of fundamental elements called *neurons*, which are functions consisting of a summing junction and a nonlinear operation involving an activation function. In addition, neural networks have attracted attention due to their inherently parallel and highly redundant processing architecture that makes it possible to develop parallel weight update laws. This parallelism makes it possible to effectively update a neural network on line. Consequently, the use of the neural networks for system identification and control of complex highly uncertain dynamical systems has become an active area of research [1–9].

Unlike adaptive controllers which guarantee asymptotic stability of the closed-loop system states associated with the system plant states, neural network adaptive controllers guarantee *ultimate boundedness* of the closed-loop system states [10]. This fundamental difference between adaptive control and neuro adaptive control can be traced back to the modeling and treatment of the system uncertainties. In particular, adaptive control is based on *constant, linearly parameterized* system uncertainty models of a known structure but unknown variation [11–13], while neuro adaptive control is based on the universal function approximation property, wherein any continuous system uncertainty can be *approximated* arbitrarily closely on a compact set using a neural network with appropriate weights [14]. This system uncertainty parametrization makes it impossible to construct a system Lyapunov function whose time derivative along the closed-loop system trajectories is guaranteed to be negative definite. Instead, the Lyapunov derivative can only be shown to be negative on a sublevel set of the system Lyapunov function. This shows that, in this sublevel set, the Lyapunov function will decrease monotonically until the system trajectories enter a compact set containing the desired system equilibrium point, and thus, guaranteeing ultimate boundedness. This analysis is often conservative since standard Lyapunov-like theorems used to show ultimate boundedness of the closed-loop system states provide only sufficient conditions, while neural network controllers often achieve plant state convergence to a desired equilibrium point.

In this paper, we develop a neuro adaptive control framework for a class of nonlinear uncertain dynamical systems which guarantees asymptotic stability of the closed-loop system states associated with the system plant states, as well as boundedness of the neural network weighting gains. The proposed framework is Lyapunov-based and guarantees partial asymptotic stability of the closed-loop system, that is, Lyapunov stability of the overall closed-loop system states and convergence of the plant states [15]. The neuro adaptive controllers are constructed without requiring explicit knowledge of the system dynamics other than the assumption that the plant dynamics are continuously differentiable and that the approximation error of uncertain system nonlinearities lie in a small gain-type norm bounded conic sector. Furthermore, the proposed neuro control architecture is modular in the sense that if a nominal linear design model is available, then the neuro adaptive controller can be augmented to the nominal design to account for system nonlinearities and system uncertainty.

The notation used in this paper is fairly standard. Specifically, $(\cdot)^{\mathrm{T}}$ denotes transpose, $\mathrm{tr}(\cdot)$ denotes the trace operator, $\sigma_{\max}(\cdot)$ denotes the maximum singular value of a matrix, $\mathrm{vec}(\cdot)$ denotes the column stacking operator for a matrix, and $\|\cdot\|$ denotes the Euclidean vector norm.

2. Partial Stability

In this section we present partial stability theorems for nonlinear dynamical systems. Specifically, consider the nonlinear interconnected dynamical system

$$\dot{x}_1(t) = f_1(x_1(t), x_2(t)), \quad x_1(0) = x_{10}, \quad t \in \mathcal{I}_{x_0}, \quad (1)$$
$$\dot{x}_2(t) = f_2(x_1(t), x_2(t)), \quad x_2(0) = x_{20}, \quad (2)$$

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where $x_1 \in \mathcal{D}$, $\mathcal{D} \subseteq \mathbb{R}^{n_1}$ is an open set such that $0 \in \mathcal{D}$, $x_2 \in \mathbb{R}^{n_2}$, $f_1 : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$ is such that, for every $x_2 \in \mathbb{R}^{n_2}$, $f_1(0, x_2) = 0$ and $f_1(\cdot, x_2)$ is locally Lipschitz in x_1, $f_2 : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$ is such that, for every $x_1 \in \mathcal{D}$, $f_2(x_1, \cdot)$ is locally Lipschitz in x_2, and $\mathcal{I}_{x_0} \triangleq [0, \tau_{x_0})$, $0 < \tau_{x_0} \leq \infty$, is the maximal interval of existence for the solution $(x_1(t), x_2(t))$, $t \in \mathcal{I}_{x_0}$, to (1), (2). Note that under the above assumptions the solution $(x_1(t), x_2(t))$ to (1), (2) exists and is unique over $\mathcal{I}_{x_0}$. The following definition introduces several types of partial stability, that is, stability with respect to x_1, for the nonlinear dynamical system (1), (2). For the following definition we assume that $\mathcal{I}_{x_0} = [0, \infty)$.

Definition 2.1. *i*) The nonlinear dynamical system (1), (2) is *Lyapunov stable with respect to* x_1 *uniformly in* x_{20} if, for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $\|x_{10}\| < \delta$ implies that $\|x_1(t)\| < \varepsilon$ for all $t \geq 0$ and for all $x_{20} \in \mathbb{R}^{n_2}$.

ii) The nonlinear dynamical system (1), (2) is *asymptotically stable with respect to* x_1 *uniformly in* x_{20} if it is Lyapunov stable with respect to x_1 uniformly in x_{20} and there exists $\delta > 0$ such that $\|x_{10}\| < \delta$ implies that $\lim_{t\to\infty} x_1(t) = 0$ for all $x_{20} \in \mathbb{R}^{n_2}$.

iii) The nonlinear dynamical system (1), (2) is *globally asymptotically stable with respect to* x_1 *uniformly in* x_{20} if it is Lyapunov stable with respect to x_1 uniformly in x_{20} and $\lim_{t\to\infty} x_1(t) = 0$ for all $x_{10} \in \mathbb{R}^{n_1}$ and $x_{20} \in \mathbb{R}^{n_2}$.

Next, we present sufficient conditions for partial stability of the nonlinear dynamical system (1), (2). For the following result define $\dot{V}(x_1, x_2) \triangleq V'(x_1, x_2) f(x_1, x_2)$, where $f(x_1, x_2) \triangleq [f_1^{\mathrm{T}}(x_1, x_2), f_2^{\mathrm{T}}(x_1, x_2)]^{\mathrm{T}}$, for a given continuously differentiable function $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$. Furthermore, we assume that the solution $(x_1(t), x_2(t))$ to (1), (2) exists and is unique for all $t \geq 0$. It is important to note that unlike standard theory the existence of a Lyapunov function $V(x_1, x_2)$ satisfying the conditions in Theorem 2.1 below is not sufficient to ensure that all solutions of (1), (2) starting in $\mathcal{D} \times \mathbb{R}^{n_2}$ can be extended to infinity since neither of the states of (1), (2) serve as an independent variable. We do note however that continuous differentiability of $f_1(\cdot, \cdot)$ and $f_2(\cdot, \cdot)$ provides a sufficient condition for the existence and uniqueness of solutions to (1), (2) for all $t \geq 0$.

Theorem 2.1 [15]. Consider the nonlinear dynamical system (1), (2). Then the following statements hold:

i) If there exists a continuously differentiable function $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$ and class $\mathcal{K}$ functions $\alpha(\cdot), \beta(\cdot)$ such that

$$\alpha(\|x_1\|) \leq V(x_1, x_2) \leq \beta(\|x_1\|), \quad (x_1, x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}, \quad (3)$$

$$\dot{V}(x_1, x_2) \leq 0, \quad (x_1, x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}, \quad (4)$$

then the nonlinear dynamical system given by (1), (2) is Lyapunov stable with respect to x_1 uniformly in x_{20}.

ii) If there exists a continuously differentiable function $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$ and class $\mathcal{K}$ functions $\alpha(\cdot), \beta(\cdot), \gamma(\cdot)$ satisfying (3) and

$$\dot{V}(x_1, x_2) \leq -\gamma(\|x_1\|), \quad (x_1, x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}, \quad (5)$$

then the nonlinear dynamical system given by (1), (2) is asymptotically stable with respect to x_1 uniformly in x_{20}.

iii) If $\mathcal{D} = \mathbb{R}^{n_1}$ and there exists a continuously differentiable function $V : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, a class $\mathcal{K}$ function $\gamma(\cdot)$, and class $\mathcal{K}_\infty$ functions $\alpha(\cdot), \beta(\cdot)$ satisfying (3) and (5), then the nonlinear dynamical system given by (1), (2) is globally asymptotically stable with respect to x_1 uniformly in x_{20}.

By setting $n_1 = n$ and $n_2 = 0$, Theorem 2.1 specializes to the case of nonlinear autonomous systems of the form $\dot{x}_1(t) = f_1(x_1(t))$. In this case, Lyapunov (respectively, asymptotic) stability with respect to x_1 and Lyapunov (respectively, asymptotic) stability with respect to x_1 uniformly in x_{20} are equivalent to the classical Lyapunov (respectively, asymptotic) stability of nonlinear autonomous systems. In particular, note that it follows from converse Lyapunov theory that there exists a continuously differentiable function $V : \mathcal{D} \to \mathbb{R}$ such that (3) and (5) hold if and only if $V(\cdot)$ is such that $V(0) = 0$, $V(x_1) > 0$, $x_1 \neq 0$, $V'(x_1) f_1(x_1) < 0$, $x_1 \neq 0$. In addition, if $\mathcal{D} = \mathbb{R}^{n_1}$ and there exist class $\mathcal{K}_\infty$ functions $\alpha(\cdot), \beta(\cdot)$ and a continuously differentiable function $V(\cdot)$ such that (3) and (5) hold if and only if $V(\cdot)$ is such that $V(0) = 0$, $V(x_1) > 0$, $x_1 \neq 0$, $V'(x_1) f_1(x_1) < 0$, $x_1 \neq 0$, and $V(x_1) \to \infty$ as $\| x_1 \| \to \infty$. Hence, in this case, Theorem 2.1 collapses to the classical Lyapunov stability theorem for autonomous systems.

In the case of time-invariant systems the Barbashin-Krasovskii-LaSalle invariance theorem shows that bounded system trajectories of a nonlinear dynamical system approach the largest invariant set $\mathcal{M}$ characterized by the set of all points in a compact set $\mathcal{D}$ of the state space where the Lyapunov derivative identically vanishes. In the case of partially stable systems, however, it is not generally clear on how to define the set $\mathcal{M}$ since $\dot{V}(x_1, x_2)$ is a function of both x_1 and x_2. However, if $\dot{V}(x_1, x_2) \leq -W(x_1) \leq 0$, where $W : \mathcal{D} \to \mathbb{R}$ is continuous and nonnegative definite, then a set $\mathcal{R} \supset \mathcal{M}$ can be defined as the set of points where $W(x_1)$ identically vanishes; that is, $\mathcal{R} = \{x_1 \in \mathcal{D} : W(x_1) = 0\}$. In this case, as shown in the next theorem, the partial system trajectories $x_1(t)$ approach $\mathcal{R}$ as t tends to infinity.

Theorem 2.2 [15]. Consider the nonlinear dynamical system (1), (2) and assume $\mathcal{D} \times \mathbb{R}^{n_2}$ is a positive invariant set with respect to (1), (2) and $f_1(\cdot, \cdot)$ is Lipschitz continuous in x_1 uniformly in x_2. Furthermore, assume there exist functions $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$, $W, W_1, W_2 : \mathcal{D} \to \mathbb{R}$ such that $V(\cdot, \cdot)$ is continuously differentiable, $W_1(\cdot)$ and $W_2(\cdot)$ are continuous and positive definite, $W(\cdot)$ is continuous and nonnegative definite, and, for all $(x_1, x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}$,

$$W_1(x_1) \leq V(x_1, x_2) \leq W_2(x_1), \quad (6)$$

$$\dot{V}(x_1, x_2) \leq -W(x_1). \quad (7)$$

Then there exists $\mathcal{D}_0 \subseteq \mathcal{D}$ such that for all $(x_{10}, x_{20}) \in D_0 \times \mathbb{R}^{n_2}$, $x_1(t) \to \mathcal{R} \triangleq \{x_1 \in \mathcal{D} : W(x_1) = 0\}$ as $t \to \infty$. If, in addition, $\mathcal{D} = \mathbb{R}^{n_1}$ and $W_1(\cdot)$ is radially unbounded, then for all $(x_{10}, x_{20}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, $x_1(t) \to \mathcal{R} \triangleq \{x_1 \in \mathbb{R}^{n_1} : W(x_1) = 0\}$ as $t \to \infty$.

Theorem 2.2 shows that the partial system trajectories $x_1(t)$ approach $\mathcal{R}$ as t tends to infinity. However, since the positive limit set of the partial trajectory $x_1(t)$ is a subset of $\mathcal{R}$, Theorem 2.2 is a weaker result than the standard invariance principle wherein one would conclude that the partial trajectory $x_1(t)$ approaches the *largest invariant set* $\mathcal{M}$

contained in $\mathcal{R}$. This is not true in general for partially stable systems since the positive limit set of a partial trajectory $x_1(t)$, $t \geq 0$, is not an invariant set. However, in the case where $f_1(\cdot, x_2)$ is periodic, almost-periodic, or asymptotically independent of x_2, then an invariance principle for partially stable systems can be derived.

3. Stable Neuro Adaptive Control for Nonlinear Uncertain Systems

In this section we consider the problem of characterizing neural adaptive feedback control laws for nonlinear uncertain dynamical systems. Specifically, consider the controlled nonlinear uncertain dynamical system $\mathcal{G}$ given by

$$\dot{x}(t) = f_x(x(t), z(t)) + G(x(t), z(t))u(t), \quad x(0) = x_0, \quad t \geq 0, \quad (8)$$

$$\dot{z}(t) = f_z(x(t), z(t)), \quad z(0) = z_0, \quad (9)$$

where $x(t) \in \mathbb{R}^{n_x}$, $t \geq 0$, and $z(t) \in \mathbb{R}^{n_z}$, $t \geq 0$, are the state vectors, $u(t) \in \mathbb{R}^m$, $t \geq 0$, is the control input, $f_x : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^{n_x}$ satisfies $f_x(0, z) = 0$, $z \in \mathbb{R}^{n_z}$, $f_x(\cdot, z)$ is *continuously differentiable* on $\mathbb{R}^{n_x}$ for each $z \in \mathbb{R}^{n_z}$, $f_x(x, \cdot)$ is continuous on $\mathbb{R}^{n_z}$ for each $x \in \mathbb{R}^{n_x}$, $f_z : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^{n_z}$ satisfies $f_z(x, 0) = 0$, $x \in \mathbb{R}^{n_x}$, and $G : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^{n_x \times m}$. The dynamics (9) typically describe the internal dynamics of the system $\mathcal{G}$. The control input $u(\cdot)$ in (8) is restricted to the class of *admissible controls* consisting of measurable functions such that $u(t) \in \mathbb{R}^m$, $t \geq 0$. Furthermore, for the nonlinear uncertain system $\mathcal{G}$ we assume that the required properties for the existence and uniqueness of solutions are satisfied, that is, $f_x(\cdot, \cdot)$, $f_z(\cdot, \cdot)$, $G(\cdot)$, and $u(\cdot)$ satisfy sufficient regularity conditions such that (8), (9) has a unique solution forward in time.

In this paper, we assume that $f_x(\cdot, \cdot)$ and $f_z(\cdot, \cdot)$ are unknown functions, and $f_x(\cdot, \cdot)$ and $G(\cdot, \cdot)$ are given by

$$f_x(x, z) = Ax + \Delta f(x, z), \quad (10)$$

$$G(x, z) = BG_{\rm n}(x, z), \quad (11)$$

where $A \in \mathbb{R}^{n_x \times n_x}$ and $B \in \mathbb{R}^{n_x \times m}$ are known matrices, $G_{\rm n} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^{m \times m}$ is a known matrix function such that $\det G_{\rm n}(x, z) \neq 0$, $(x, z) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z}$, and $\Delta f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^{n_x}$ is an uncertain function belonging to the uncertainty set $\mathcal{F}$ given by

$$\mathcal{F} = \{\Delta f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^{n_x} : \Delta f(0, \cdot) = 0, \Delta f(x, z) = B\delta(x, z), (x, z) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z}\}, \quad (12)$$

where $\delta : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^m$ is an uncertain function such that $\delta(\cdot, z)$ is continuously differentiable on $\mathbb{R}^{n_x}$ for each $z \in \mathbb{R}^{n_z}$, $\delta(x, \cdot)$ is continuous on $\mathbb{R}^{n_z}$ for each $x \in \mathbb{R}^{n_x}$, and $\delta(0, \cdot) = 0$. Furthermore, we assume that (9) is input-to-state stable with $x(t)$ viewed as the input. It is important to note that since $\delta(x, z)$ is continuously differentiable in x and $\delta(0, z) = 0$, $z \in \mathbb{R}^{n_z}$, it follows that there exists a continuous matrix function $\Delta : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^{m \times n_x}$ such that $\delta(x, z) = \Delta(x, z)x$, $(x, z) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z}$. We assume that the continuous matrix function $\Delta(\cdot, \cdot)$ can be approximated over a compact set $\mathcal{D}_{{\rm c}x} \times \mathcal{D}_{{\rm c}z} \subset \mathbb{R}^{n_x} \times \mathbb{R}^{n_z}$ by a linear in the parameters neural network up to a desired accuracy so that

$$\mathrm{col}_i(\Delta(x, z)) = W_i^{\rm T}\sigma(x, z) + \varepsilon_i(x, z), \quad (x, z) \in \mathcal{D}_{{\rm c}x} \times \mathcal{D}_{{\rm c}z}, \quad i = 1, \cdots, n_x, \quad (13)$$

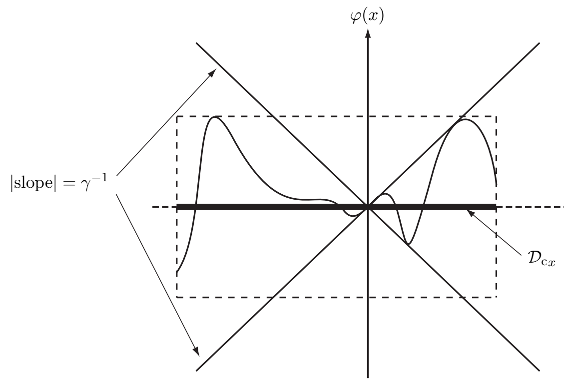


Figure 3.1: Visualization of function $\varphi(\cdot)$

where $\mathrm{col}_i(\Delta(\cdot, \cdot))$ denotes the ith column of $\Delta(\cdot, \cdot)$, $W_i^{\rm T} \in \mathbb{R}^{m \times s}$, $i = 1, \cdots, n_x$, are optimal *unknown* (constant) weights that minimize the approximation error over $\mathcal{D}_{{\rm c}x} \times \mathcal{D}_{{\rm c}z}$, $\varepsilon_i : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^m$, $i = 1, \cdots, n_x$, are modeling errors such that $\sigma_{\max}(\Upsilon(x, z)) \leq \gamma^{-1}$, $(x, z) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z}$, where $\Upsilon(x, z) \triangleq [\varepsilon_1(x, z), \cdots, \varepsilon_{n_x}(x, z)]$ and $\gamma > 0$, and $\sigma : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^s$ is a given basis function such that $0 \leq \sigma(x, z) \leq 1$, $(x, z) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z}$.

Next, defining

$$\varphi(x, z) \triangleq \delta(x, z) - W^{\rm T}[x \otimes \sigma(x, z)], \quad (14)$$

where $W^{\rm T} \triangleq [W_1^{\rm T}, \cdots, W_{n_x}^{\rm T}] \in \mathbb{R}^{m \times n_x s}$ and $\otimes$ denotes Kronecker product, it follows from (13) and Cauchy-Schwartz inequality that

$$\begin{aligned} \varphi^{\rm T}(x, z)\varphi(x, z) &= \|\Delta(x, z)x - W^{\rm T}(x \otimes \sigma(x, z))\|^2 \\ &= \|\Delta(x, z)x - \Sigma(x, z)x\|^2 \\ &= \|\Upsilon(x, z)x\|^2 \\ &\leq \gamma^{-2}x^{\rm T}x, \quad (x, z) \in \mathcal{D}_{{\rm c}x} \times \mathcal{D}_{{\rm c}z}, \quad (15) \end{aligned}$$

where $\Sigma(x, z) \triangleq [W_1^{\rm T}\sigma(x, z), \cdots, W_{n_x}^{\rm T}\sigma(x, z)]$. In the case where $\mathcal{G}$ does not possess internal dynamics (i.e., $n_z = 0$), (13) and (15) specialize to

$$\mathrm{col}_i(\Delta(x)) = W_i^{\rm T}\sigma(x) + \varepsilon_i(x), \quad x \in \mathcal{D}_{{\rm c}x}, \quad i = 1, \cdots, n_x, \quad (16)$$

and

$$\varphi^{\rm T}(x)\varphi(x) \leq \gamma^{-2}x^{\rm T}x, \quad x \in \mathcal{D}_{{\rm c}x}, \quad (17)$$

respectively. This corresponds to a nonlinear small gain-type norm bounded uncertainty characterization for $\varphi(\cdot)$ (see Figure 3.1).

Theorem 3.1. Consider the nonlinear uncertain dynamical system $\mathcal{G}$ given by (8) and (9) where $f_x(\cdot, \cdot)$ and $G(\cdot, \cdot)$ are given by (10) and (11), respectively, and $\Delta f(\cdot, \cdot)$ belongs to $\mathcal{F}$. Assume there exists a matrix $K \in \mathbb{R}^{m \times n_x}$ such that $A_{\rm s} \triangleq A + BK$ is asymptotically stable. Furthermore, for a given $\gamma > 0$, assume there exist positive-definite matrices $P \in \mathbb{R}^{n_x \times n_x}$ and $R \in \mathbb{R}^{n_x \times n_x}$ such that

$$0 = A_{\rm s}^{\rm T}P + PA_{\rm s} + \gamma^{-2}PBB^{\rm T}P + I_{n_x} + R. \quad (18)$$

In addition, assume that (9) is input-to-state stable with $x(t)$ viewed as the input. Finally, let $Q \in \mathbb{R}^{m \times m}$ and $Y \in$

$\mathbb{R}^{n_x s \times n_x s}$ be positive definite. Then the neural adaptive feedback control law

$$u(t) = G_{\mathrm{n}}^{-1}(x(t),z(t))\Big[Kx(t) - \hat{W}^{\mathrm{T}}(t)[x(t)\otimes\sigma(x(t),z(t))]\Big], \quad (19)$$

where $\hat{W}^{\mathrm{T}}(t) \in \mathbb{R}^{m\times n_x s}$, $t \geq 0$, and $\sigma : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^s$ is a given basis function, with update law

$$\dot{\hat{W}}^{\mathrm{T}}(t) = QB^{\mathrm{T}}Px(t)[x(t)\otimes\sigma(x(t),z(t))]^{\mathrm{T}}Y, \quad \hat{W}^{\mathrm{T}}(0) = \hat{W}_0^{\mathrm{T}}, \quad (20)$$

guarantees that there exists a positively invariant set $\mathcal{D}_\alpha \subset \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{m\times n_x s}$ such that $(0,0,W^{\mathrm{T}}) \in \mathcal{D}_\alpha$, where $W^{\mathrm{T}} \in \mathbb{R}^{m\times n_x s}$, and the solution $(x(t),z(t),\hat{W}^{\mathrm{T}}(t)) \equiv (0,0,W^{\mathrm{T}})$ of the closed-loop system given by (8), (9), (19), (20) is Lyapunov stable and $(x(t),z(t)) \to (0,0)$ as $t\to\infty$ for all $\Delta f(\cdot,\cdot) \in \mathcal{F}$ and $(x_0,z_0,\hat{W}_0^{\mathrm{T}}) \in \mathcal{D}_\alpha$.

Proof. First, note that with $u(t)$, $t \geq 0$, given by (19) it follows from (8), (10), and (11) that

$$\dot{x}(t) = Ax(t) + \Delta f(x(t),z(t)) + BKx(t) - B\hat{W}^{\mathrm{T}}(t)[x(t)\otimes\sigma(x(t),z(t))], \quad x(0)=x_0, \quad t\geq 0, \quad (21)$$

or, equivalently, using (14),

$$\dot{x}(t) = A_{\mathrm{s}}x(t) + B\Big[\varphi(x(t),z(t)) - \tilde{W}^{\mathrm{T}}(t) \cdot[x(t)\otimes\sigma(x(t),z(t))]\Big], \quad x(0)=x_0,\ t\geq 0, \quad (22)$$

where $\tilde{W}^{\mathrm{T}}(t) \triangleq \hat{W}^{\mathrm{T}}(t) - W^{\mathrm{T}}$. To show Lyapunov stability of the closed-loop system (9), (20), and (22) consider the Lyapunov function candidate

$$V(x,z,\tilde{W}^{\mathrm{T}}) = x^{\mathrm{T}}Px + \mathrm{tr}\,Q^{-1}\tilde{W}^{\mathrm{T}}Y^{-1}\tilde{W}. \quad (23)$$

Note that (23) satisfies (3) and (6) with $x_1 = [x^{\mathrm{T}}, (\mathrm{vec}\,\tilde{W})^{\mathrm{T}}]^{\mathrm{T}}$, $x_2 = z$, $\alpha(\|x_1\|) = \beta(\|x_1\|) = W_1(x_1) = W_2(x_1) = x^{\mathrm{T}}Px + \mathrm{tr}\,\tilde{W}Q^{-1}\tilde{W}^{\mathrm{T}}$. Now, letting $x(t)$, $t\geq 0$, denote the solution to (22) and using (15), (18), and (20), it follows that the Lyapunov derivative along the closed-loop system trajectories is given by

$$\begin{aligned}
&\dot{V}(x(t),z(t),\tilde{W}^{\mathrm{T}}(t)) \\
&= 2x^{\mathrm{T}}(t)P\Big[A_{\mathrm{s}}x(t) + B[\varphi(x(t),z(t)) \\
&\quad -\tilde{W}^{\mathrm{T}}(t)[x(t)\otimes\sigma(x(t),z(t))]]\Big] \\
&\quad +2\mathrm{tr}\,Q^{-1}\tilde{W}^{\mathrm{T}}(t)Y^{-1}\dot{\hat{W}}(t) \\
&= -x^{\mathrm{T}}(t)(R+\gamma^{-2}PBB^{\mathrm{T}}P+I_{n_x})x(t) \\
&\quad +2x^{\mathrm{T}}(t)PB\Big[\varphi(x(t),z(t)) \\
&\quad -\tilde{W}^{\mathrm{T}}(t)[x(t)\otimes\sigma(x(t),z(t))]\Big] \\
&\quad +2\mathrm{tr}\,\tilde{W}^{\mathrm{T}}(t)\left(B^{\mathrm{T}}Px(t)[x(t)\otimes\sigma(x(t),z(t))]^{\mathrm{T}}\right)^{\mathrm{T}} \\
&= -x^{\mathrm{T}}(t)Rx(t) - x^{\mathrm{T}}(t)(\gamma^{-2}PBB^{\mathrm{T}}P + I_{n_x})x(t) \\
&\quad +2x^{\mathrm{T}}(t)PB\varphi(x(t),z(t)) \\
&\leq -x^{\mathrm{T}}(t)Rx(t) \\
&\quad -x^{\mathrm{T}}(t)[\gamma^{-1}B^{\mathrm{T}}P+\gamma I_{n_x}]^{\mathrm{T}}[\gamma^{-1}B^{\mathrm{T}}P+\gamma I_{n_x}]x(t) \\
&\leq -x^{\mathrm{T}}(t)Rx(t) \\
&\leq 0, \quad t\geq 0. \qquad (24)
\end{aligned}$$

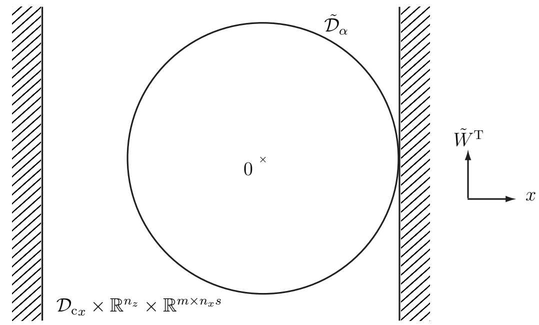


Figure 3.2: Visualization of sets used in the proof of Theorem 3.1

Next, let

$$\tilde{\mathcal{D}}_\alpha \triangleq \Big\{(x,z,\tilde{W}^{\mathrm{T}}) \in \mathbb{R}^{n_x}\times\mathbb{R}^{n_z}\times\mathbb{R}^{m\times n_x s} : V(x,z,\tilde{W}^{\mathrm{T}}) \leq \alpha\Big\}, \quad (25)$$

where α is the maximum value such that $\tilde{\mathcal{D}}_\alpha \subseteq \mathcal{D}_{\mathrm{c}x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{m\times n_x s}$ (see Figure 3.2). Now, since $\dot{V}(x(t),z(t),W^{\mathrm{T}}(t)) \leq 0$ for all $(x(t),z(t),W^{\mathrm{T}}(t)) \in \tilde{\mathcal{D}}_\alpha$ and $t\geq 0$, it follows that $\tilde{\mathcal{D}}_\alpha$ is positively invariant. Furthermore, it follows from Theorem 2.1 that the solution $(x(t),z(t),\hat{W}^{\mathrm{T}}(t)) \equiv (0,0,W^{\mathrm{T}})$ to (9), (20), and (22) is Lyapunov stable with respect to x and $\hat{W}^{\mathrm{T}}$ (uniformly in z_0) for all $\Delta f(\cdot,\cdot) \in \mathcal{F}$ and $(x_0,z_0,\hat{W}_0) \in \tilde{\mathcal{D}}_\alpha$. In addition, since $R>0$, it follows from Theorem 2.2 that $x(t)\to 0$ as $t\to\infty$.

Next, since (9) is input-to-state stable with $x(t)$ viewed as the input, it follows from Theorem 1 of [16] that there exist a continuously differentiable, radially unbounded positive-definite function $V_z : \mathbb{R}^{n_z} \to \mathbb{R}$ and class $\mathcal{K}$ functions $\gamma_1(\cdot), \gamma_2(\cdot)$ such that

$$V_z'(z)f_z(x,z) \leq -\gamma_1(\|z\|), \quad \|z\| \geq \gamma_2(\|x\|). \quad (26)$$

Since $\|x(t)\|$ is bounded for all $t\geq 0$, it follows that the set given by

$$\mathcal{D}_z \triangleq \Big\{z\in\mathbb{R}^{n_z} : V_z(z) \leq \max_{\|z\|=\gamma_2(\sup_{t\geq 0}\|x(t)\|)} V_z(z)\Big\} \quad (27)$$

is also positively invariant as long as[1] $\mathcal{D}_z \subset \mathcal{D}_{\mathrm{c}z}$. Now, since $\tilde{\mathcal{D}}_\alpha$ and $\mathcal{D}_z$ are positively invariant, it follows that

$$\mathcal{D}_\alpha \triangleq \Big\{(x,z,\hat{W}^{\mathrm{T}}) \in \mathbb{R}^{n_x}\times\mathbb{R}^{n_z}\times\mathbb{R}^{m\times n_x s} : (x,z,\hat{W}^{\mathrm{T}} - W^{\mathrm{T}}) \in \tilde{\mathcal{D}}_\alpha,\ z\in\mathcal{D}_z\Big\} \quad (28)$$

[1] See Remark 3.2.

is also positively invariant. Furthermore, since (9) is input-to-state stable with $x(t)$ viewed as the input, it follows from Lemma 4.7 of [10] that the solution $(x(t), z(t), \hat{W}^{\rm T}(t)) \equiv (0, 0, W^{\rm T})$ to (9), (20), and (22) is Lyapunov stable and $x(t) \to 0$ and $z(t) \to 0$ as $t \to \infty$ for all $\Delta f(\cdot,\cdot) \in \mathcal{F}$ and $(x_0, z_0, \hat{W}_0) \in \mathcal{D}_\alpha$. □

Remark 3.1. Note that the conditions in Theorem 3.1 imply partial asymptotic stability, that is, the solution $(x(t), z(t), \hat{W}^{\rm T}(t)) \equiv (0, 0, W^{\rm T})$ of the overall closed-loop system is Lyapunov stable and $(x(t), z(t)) \to (0, 0)$ as $t \to \infty$. Hence, it follows from (20) that $\dot{\hat{W}}^{\rm T}(t) \to 0$ as $t \to \infty$.

Remark 3.2. Since $\alpha(\|x_1\|)$ used in the proof of Theorem 3.1 is a class $\mathcal{K}_\infty$ function, the assumption $\mathcal{D}_z \subset \mathcal{D}_{{\rm c}z}$ invoked in the proof of Theorem 3.1 is automatically satisfied in the case where the neural network approximation holds in $\mathbb{R}^{n_x} \times \mathbb{R}^{n_z}$. Furthermore, in this case the control law (19) ensures global asymptotic stability with respect to x and z. However, the existence of a global neural network approximator for an uncertain nonlinear map cannot in general be established. Hence, as is common in the neural network literature, for a given arbitrarily large compact set $\mathcal{D}_{{\rm c}x} \times \mathcal{D}_{{\rm c}z} \subset \mathbb{R}^{n_x} \times \mathbb{R}^{n_z}$, we assume that there exists an approximator for the unknown nonlinear map up to a desired accuracy in the sense of (15). This assumption ensures that there exists a nontrivial Lyapunov level set such that $\mathcal{D}_z \subset \mathcal{D}_{{\rm c}z}$. In the case where $\Delta(\cdot,\cdot)$ is continuous on $\mathbb{R}^{n_x} \times \mathbb{R}^{n_z}$, it follows from the Stone-Weierstrass theorem that $\Delta(\cdot,\cdot)$ can be approximated over an arbitrarily large compact set $\mathcal{D}_{{\rm c}x} \times \mathcal{D}_{{\rm c}z}$. In this case, our neuro adaptive controller guarantees semiglobal partial asymptotic stability.

Remark 3.3. Note that the neuro adaptive controller (19) and (20) can be constructed to guarantee partial asymptotic stability using standard *linear* H_∞ theory. Specifically, it follows from standard H_∞ theory [17] that $\|G(s)\|_\infty < \gamma$, where $G(s) = E(sI_n - A_{\rm s})^{-1}B$ and E is such that $E^{\rm T}E = I_{n_x} + R$, if and only if there exists a positive-definite matrix P satisfying the bounded real Riccati equation (18). It is well known that (18) has a positive-definite solution if and only if the Hamiltonian matrix

$$\mathcal{H} = \begin{bmatrix} A_{\rm s} & \gamma^{-2}BB^{\rm T} \\ -E^{\rm T}E & -A_{\rm s}^{\rm T} \end{bmatrix}, \qquad (29)$$

has no purely imaginary eigenvalues.

It is important to note that the adaptive control law (19) and (20) does not require the explicit knowledge of the optimal weighting matrix W. Furthermore, no specific structure on the nonlinear dynamics $f_x(x,z)$ and $f_z(x,z)$ is required to apply Theorem 3.1. However, if (8) is in normal form [18] with (9) being input-to-state stable with x viewed as the input, then we can always construct a neuro adaptive control law *without* requiring knowledge of the system dynamics $f_x(x,z)$ and $f_z(x,z)$. To see this, assume that the nonlinear uncertain system $\mathcal{G}$ is generated by

$$q_i^{(r_i)}(t) = f_{{\rm u}i}(q(t), z(t)) + \sum_{j=1}^{m} G_{{\rm s}(i,j)}(q(t), z(t))u_j(t), \quad t \ge 0, \quad i = 1, \cdots, m, \qquad (30)$$

$$\dot{z}(t) = f_z(q(t), z(t)), \quad z(0) = z_0, \qquad (31)$$

where $q = [q_1, \cdots, q_1^{(r_1-1)}, \cdots, q_m, \cdots, q_m^{(r_m-1)}]^{\rm T}$, $q(0) = q_0$, $q_i^{(r_i)}$ denotes the r_ith derivative of q_i, and r_i denotes the relative degree with respect to the output q_i. Here we assume that the square matrix function $G_{\rm s}(q,z)$ composed of the entries $G_{{\rm s}(i,j)}(q,z)$, $i,j = 1, \cdots, m$, is such that $\det G_{\rm s}(q,z) \neq 0$, $(q,z) \in \mathbb{R}^{\hat{r}} \times \mathbb{R}^{n_z}$, where $\hat{r} = r_1 + \cdots + r_m$ is the (vector) relative degree of (30) and $\hat{r} = n_x$. Furthermore, we assume that $f_{{\rm u}i}(\cdot, z)$ is continuously differentiable on $\mathbb{R}^{n_x}$ for each $z \in \mathbb{R}^{n_z}$, $f_{{\rm u}i}(x,\cdot)$ is continuous on $\mathbb{R}^{n_z}$ for each $x \in \mathbb{R}^{n_x}$, and $f_{{\rm u}i}(0,\cdot) = 0$. In addition, we assume that the dynamics given by (31) is input-to-stable with q viewed as the input.

Next, define $x_i \triangleq \left[q_i, \cdots, q_i^{(r_i-2)}\right]^{\rm T}$, $i = 1, \cdots, m$, $x_{m+1} \triangleq \left[q_1^{(r_1-1)}, \cdots, q_m^{(r_m-1)}\right]^{\rm T}$, and $x \triangleq \left[x_1^{\rm T}, \cdots, x_{m+1}^{\rm T}\right]^{\rm T}$, so that (30) can be described by (8) with

$$A = \begin{bmatrix} A_0 \\ 0_{m\times n_x} \end{bmatrix}, \quad \Delta f(x,z) = \begin{bmatrix} 0_{(n_x-m)\times 1} \\ f_{\rm u}(x,z) \end{bmatrix},$$

$$G(x,z) = \begin{bmatrix} 0_{(n_x-m)\times m} \\ G_{\rm s}(x,z) \end{bmatrix}, \qquad (32)$$

where $A_0 \in \mathbb{R}^{(n_x-m)\times n_x}$ is a known matrix of zeros and ones capturing the multivariable controllable canonical form representation [19], $f_{\rm u} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^{n_x}$ is an unknown function and satisfies $f_{\rm u}(0,\cdot) = 0$, and $G_{\rm s} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^{m\times m}$. Note that $\Delta f(\cdot,\cdot) \in \mathcal{F}$ with $B = [0_{m\times(n_x-m)}, I_m]^{\rm T}$ and $\delta(x,z) = f_{\rm u}(x,z)$. In this case, $G_{\rm n}(x,z) \equiv G_{\rm s}(x,z)$. Furthermore, since A is in multivariable controllable canonical form, we can always construct K such that $A + BK$ is asymptotically stable.

4. Illustrative Numerical Example

In this section we present a numerical example to demonstrate the utility of the proposed neuro adaptive control framework for adaptive stabilization. Specifically, consider the uncertain controlled Liénard system given by

$$\ddot{q}(t) + c(q(t))\dot{q}(t) + k(q(t)) = bu(t), \quad q(0) = q_0, \quad \dot{q}(0) = \dot{q}_0, \quad t \ge 0, \qquad (33)$$

where $c : \mathbb{R} \to \mathbb{R}$ and $k : \mathbb{R} \to \mathbb{R}$ are unknown, continuously differential functions. Note that with $x_1 = q$ and $x_2 = \dot{q}$, (33) can be written in state space form (8), (9), and (15) with $x = [x_1, x_2]^{\rm T}$, $z = \emptyset$, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\Delta f(x) = [0, -c(x_1)x_2 - k(x_1)]^{\rm T}$, $B = [0, b]^{\rm T}$, and $G_{\rm n}(x) = 1$. Here, we assume that the unknown function $\Delta f(x)$ can be written as $\Delta f(x) = B\delta(x)$, where $\delta(x) = \frac{1}{b}[-c(x_1)x_2 - k(x_1)]$ is an unknown, continuously differentiable function. Next, let $K = \frac{1}{b}[k_1, k_2]$, where k_1, k_2 are arbitrary scalars, so that $A_{\rm s} = A + BK = \begin{bmatrix} 0 & 1 \\ k_1 & k_2 \end{bmatrix}$. Now, with the proper choice of k_1 and k_2, it follows from Theorem 3.1 that if there exists $P > 0$ satisfying (18), then the neuro adaptive feedback controller (19) guarantees that $x(t) \to 0$ as $t \to \infty$. Specifically, here we choose $k_1 = -1$, $k_2 = -1$, $\gamma = 3$, and $R = I_2$, so that P satisfying (18) is given by

$$P = \begin{bmatrix} 3.1586 & 1.0627 \\ 1.0627 & 2.3765 \end{bmatrix}. \qquad (34)$$

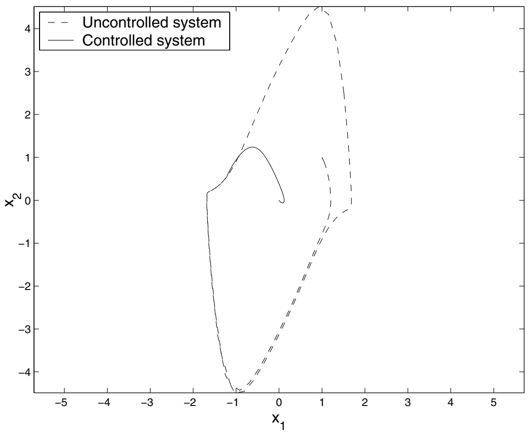
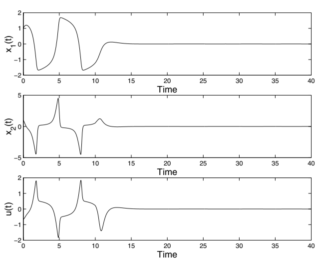


Figure 4.3: Phase portrait of controlled and uncontrolled Liénard system



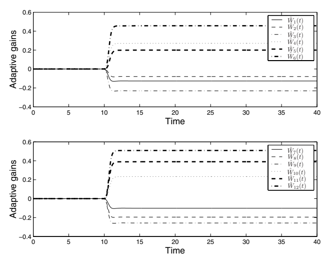


Figure 4.4: State trajectories and control signal versus time

Figure 4.5: Neural network weighting functions versus time

With $c(x_1) = 2(x_1^4 - 1)$, $k(x_1) = x_1 + \tanh(x_1)$, $b = 3$, $Q = 1$, $Y = 0.1 I_{12}$, $\sigma(x) = \left[\frac{1}{1+e^{-a_1 x_1}}, \cdots, \frac{1}{1+e^{-3a_1 x_1}}, \frac{1}{1+e^{-a_2 x_2}}, \cdots, \frac{1}{1+e^{-3a_2 x_2}}\right]$, where $a_1 = a_2 = 0.5$, and initial conditions $x(0) = [1, 1]^{\mathrm{T}}$ and $\hat{W}(0) = 0_{12\times 1}$, Figure 4.3 shows the phase portrait of the controlled and uncontrolled system. Note that the neuro adaptive controller is switched on at $t = 10$ sec. Figure 4.4 shows the state trajectories versus time and the control signal versus time. Finally, Figure 4.5 shows the neural network weighting functions versus time.

5. Conclusion

A neuro adaptive control framework for adaptive stabilization of nonlinear uncertain dynamical systems was developed. In particular, using Lyapunov methods along with the robust control techniques and partial stability notions, the proposed framework was shown to guarantee partial asymptotic stability of the closed-loop system, that is, asymptotic stability with respect to part of the closed-loop system states associated with the plant. Furthermore, in the case where the nonlinear system is represented in normal form with input-to-state stable internal dynamics of unknown order, the neuro adaptive controllers were constructed without requiring knowledge of the system dynamics other than the fact that the plant dynamics are continuously differentiable. Finally, an illustrative numerical example was presented to show the utility of the proposed neuro adaptive stabilization scheme.

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