2005 American Control Conference
June 8-10, 2005. Portland, OR, USA

Towards a Packet-based Control Theory - Part I: Stabilization Over a Packet-based Network

Ling Shi and Richard M. Murray

Abstract— **In this paper, we study the classical problem of stabilizing a Linear Time Invariant (LTI) system in a packet-based network setting. We assume that the LTI system is unstable but both controllable and observable. The state information is transmitted to the controller over a packet-based network. We also assume that there is a perfect link from the controller to the plant. We give a set of sufficient conditions under which the system can be stabilized for a given data rate C. In particular, these conditions can yield an upper bound on the minimum C for which the system can be stabilized. A recursive encoding-decoding scheme and an associated control law are proposed to achieve stability for rate exceeding this bound. An optimal bit allocation problem is investigated in which we ask about how to allocate the bits in a single packet for a subsystem of a general LTI system such that a minimum upper bound on the data rate is achieved. We then formulate the optimal bit allocation problem as a Linear Matrix Inequality (LMI) optimization problem which can be solved efficiently using standard Semi-definite Programming (SDP) solvers. Examples and simulations are given to demonstrate the results.**

I. INTRODUCTION

Classical information theory has become a hot area of current cutting-edge research ever since Shannon published his seminal paper [12] in 1948. At the same time, control technology has become so important and popular that we can see its applications everywhere in our daily lives, such as military applications, aircraft control, chemical process control, manufacturing control and so on. While classical control problems have been widely studied by the control community (see [6], [13], [2] for some typical applications), until recently, little attention has been paid to the "information" aspect of them. This is largely because classical information theory has several fundamental assumptions incompatible with control problems, such as in classical information theory, proofs often assume infinite delays which is inappropriate for the realtime setting where the delayed information might in fact be useless. Classical information theory also does not often admit a sense of the "quality" or the "importance" of information.

Consequently, information theoretical issues are traditionally decoupled from decision and control problems. An implicit assumption made by the control community is that information processing and data processing are done with zero time delay and infinite precision and as a result, infinite communication bandwidth is usually assumed.

Control and Dynamical Systems, California Institute of Technology; Pasadena, CA 91106.

Email:{shiling,murray}@cds.caltech.edu; Tel: (626) 395-2313, Fax: (626) 796-8914.

Work supported in part by NSF ITR grant CCR-0326554.

However, as new applications keep emerging, the standard assumptions may need to be revisited. As an example, in the context of networked control, the sensed data is sent to the controller through a data network which has finite bandwidth or finite data rate and as a consequence, zero time delay in the information processing is not guaranteed. These new application domains can be clustered together as control under finite communication constraints or control with partial information.

Many researchers have already seen the limitations of traditional information theory and progress towards a more real-time and interactive version of information theory is underway. In recent years there has been much attention and interest in the problem of control under communication or information constraints. Delchamps [3] investigated the issue of stabilizing a discrete time linear system with quantized state feedback. In his work, he proposed feedback control strategies that can bring the closed loop trajectories arbitrarily close to origin with arbitrarily long time. The problem of state estimation and stabilization of a LIT system with a finite bandwidth digital communication channel capacity was introduced by Wong and Brockett [15], [16] and further pursued by [9], [4], [14], [5]. Mitter in [8] described the need for a unified approach to control, communication, and computation. His former PhD students, Tatikonda [14] and Sahai [10] have also presented some interesting resutls in the area of control under communication constraints. The issue of determining the minimum bit-rate to achieve stability has been studied in [4], [14] but in a different setting than what we study here. They consider a discrete communication channel in the feedback control loop while we consider a packet-based network in the feedback control loop. Nair and Evans [4] also considered stability problem of noiseless discrete-time linear systems with communication constraints but setting the initial condition as a scalar random variable. Xiao [7] et al considered the problem of jointly optimizing the communication rate and linear systems with bit rate constraints.

As an extension of these and other similar works, we take initial steps towards developing a packet-based control theory. In the current work, we concentrate on the stability problem of an unstable but both controllable and observable LTI system. We assume that between the observer and the controller, there is a encoder-decoder pair and a packet-based network which has a finite fixed data rate C bits/s (see Figure 1 for a system block diagram). A simple example of this kind would be to remotely control an inverted pendulum through the Internet. If $C = 0$, then the system is in the open loop, hence by our assumption, it is unstable. On the other extreme, if $C = \infty$ and assume that there are no other delays caused by the network, such as propagational delay or queueing delay, we return to the case of classical control problem and it is stabilizable by assumptions. The natural question to ask here is that what is the minimum C ($C_{\inf}$) required to stabilize the unstable system? Given such a $C > C_{\inf}$, what is the corresponding control law that stabilizes the system ? By study these problems, we wish to understand more about the tradeoff between the information rate and stability of a system. and understand better how to design a stable system by using the minimum resources.

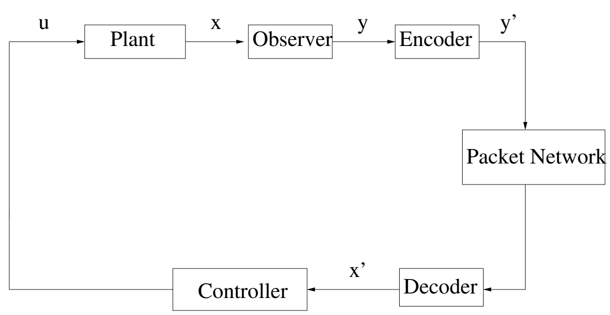

Fig. 1. System Block Diagram

The paper is organized as follows. In section 2, we present the mathematical model of the closed loop system and state our assumptions. In section 3, we consider the special case of a linear scalar system and give a set of sufficient conditions on the $C_{\inf}$ described above. We then consider the general LTI system in section 4. We propose some particular control laws that lead to upper bounds on $C_{\inf}$. In particular, we formulate the optimal control algorithm as an LMI optimization problem. We conclude the current work and propose some future work directions in section 5.

II. Modelling and Problem Setting

For simplicity, the systems that we are interested are assumed to be controllable and observable but are nominally unstable (see [2] for definitions of controllability and observability of LTI systems). The system dynamics is represented by

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t), & (1)\\ \mathbf{y}(t) &= H\mathbf{x}(t). & (2) \end{aligned}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state of the system, and $\mathbf{u} \in \mathbb{R}^m$ is the control input. The initial condition $\mathbf{x(0)} \in \mathbb{R}^n$ is assumed to be bounded. The matrix A is assumed to be unstable, that is, A has at least one eigenvalue λ such that $\mathrm{Real}(\lambda) > 0$. We ignore the cases where $\lambda = 0$ can also cause the system to be unstable. We also assume that the pair (A, B) is controllable and (A, H) is observable.

We assume that the packet network has a finite data rate C bits/s and for now we ignore other possible effects of the network such as packet loss or reordering. For simplicity, we assume that each packet that the encoder transmits is l bits long. Thus the transmission delay is $\delta = l/C$. Assume the total delay induced by the network besides the transmission delay is D at every time step.

We first discuss the case of a linear scalar systems and then consider general LTI systems. In each case, a set of sufficient conditions is given on $C_{\inf}$ such that if $C > C_{\inf}$, there is a encoding-decoding scheme and an associated control law such that the system (1) can be made exponentially stable through the packet-based network.

III. Linear Scalar Systems

In this section we consider a simple linear scalar system

$$\begin{aligned} \dot{x}(t) &= ax(t) + u(t), x, u \in \mathbb{R}, a > 0, & (3)\\ y(t) &= x(t). & (4) \end{aligned}$$

Since $a > 0$, the system is unstable but controllable and observable. Following the notation in section II, let $\delta + D$ be the sampling time for the continuous process (*i.e.*, the controller is updating whenever it receives a packet hence the system is equivalent to a sampled system with zero order hold), thus the system is represented in discrete time form as

$$\begin{aligned} x(k+1) &= \tau x(k) + u(k), & (5)\\ y(k) &= x(k), & (6) \end{aligned}$$

where $\tau_k = e^{a(\delta+D)}$. Without loss of generality, assume that $x(0) > 0$ and

$$x(0) = \sum_{i=-\infty}^{M} \alpha_i 2^i \qquad (7)$$

be the binary expansion of $x(0)$, where $\alpha_i = 0$ or 1. Clearly $x(0) < 2^{M+1}$. We assume that the controller knows the system parameter a, then we have the following result.

Lemma 1: If a is known to the controller, then a sufficient condition for the overall system to be exponentially stabilizable is that

$$C > C_{\inf} = \frac{al\log e}{l - 1 - aD\log e},$$

where C is the network data rate, l is the packet length, D is the total delay induced by the network and $\log$ is assumed to the base 2.

Proof: Using the notation in equation (7), let the first packet contain l bits consisting of α_{M-l+1} through α_M. Let

$$\bar{x}(0) = \sum_{i=M-l+1}^{M} \alpha_i 2^i$$

be the first l bits truncated version of $x(0)$, and let $\epsilon(0) = x(0) - \bar{x}(0)$ be the remainder. Then one can easily obtain

$$\epsilon(0) = \sum_{i=-\infty}^{M-l} \alpha_i 2^i < 2^{M-l+1}. \qquad (8)$$

After the first round of communication,

$$x(1) = e^{a(\delta+D)}(\bar{x}(0) + \epsilon(0)) + u(0).$$

As the controller knows $\bar{x}(0)$ and a, $u(0)$ can be set as $-e^{a(\delta+D)}\bar{x}(0)$, hence, by equation (8), we obtain

$$x(1) = e^{a(\delta+D)}\epsilon(0) < e^{a(\delta+D)}2^{M-l+1}.$$

Let $e^{a(\delta+D)}2^{M-l+1} < 2^M$, or $e^{a(\delta+D)} < 2^{l-1}$, which can be rewritten as

$$C > \frac{al\log e}{l - 1 - aD\log e}. \tag{9}$$

Now, repeat the same process, by making $x(1)$ the new initial condition. Note that this time, $x(1) < 2^M$, *i.e.*, the upper bound of $x(1)$ is half the size of the upper bound of $x(0)$.

After the second round of communication,

$$x(2) = e^{a(\delta+D)}(\bar{x}(1) + \epsilon(1)) + u(1),$$

where $\epsilon(1) < 2^{M-l}$. By setting $u(1) = -e^{a(\delta+D)}(\bar{x}(1))$ and if inequality (9) is satisfied, we have $x(2) < 2^{M-1}$, and continuing this way, we get $x(k) < 2^{M+1-k}$, hence the closed loop system is exponentially stabilizable.

QED

IV. General LTI System

In this section we consider the same problem for a general LTI system described in equation (1), *i.e.*,

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t), \\ \mathbf{y}(t) &= H\mathbf{x}(t). \end{aligned}$$

For simplicity, we assume that from now on, the controller knows A. Also assume $H = I$, so $\mathbf{y}(t) = \mathbf{x}(t)$. Again let $\delta + D$ be the sampling time for the continuous process, then the system in discrete time looks like:

$$\mathbf{x}(k+1) = \tau\mathbf{x}(k) + B\mathbf{u}(k), \tag{10}$$

$$\mathbf{y}(k) = \mathbf{x}(k), \tag{11}$$

where $\tau = e^{A(\delta+D)}$.

We first give a general result on the network data rate C, packet length l and the system parameter A such that the closed loop system is exponentially stable. The equal bit allocation scheme (see the proof of lemma 2 for an explanation) is used to derive the result. We then discuss different types of bit allocation schemes which provide lower $C_{\inf}$.

A. Equal Bit Allocation

We allocate the packet in such a way that l/n bits are used for the i^{th} component of $\mathbf{y}(k)\ \forall\ k$.

Lemma 2: With the assumptions above, a sufficient condition for the closed loop to be exponentially stabilizable is that:

$$C > \frac{l\log(||e^A||)}{\frac{l}{n} - 1 - D\log(||e^A||)}$$

where C, l, D, the log function are the same as in Lemma 1, and $||e^A||$ is the induced L_2 norm or the largest singular value of the matrix exponential e^A.

Proof: The proof is similar as in Lemma 1.

Remark 3: The result above includes the linear scalar case, *i.e.*, when $n = 1$, we have the same result, as in this case $||e^A|| = e^a$.

In the above control scheme, we equally divide the packet length into n parts. Is this the optimal scheme to do? Are there any other bit allocation schemes so that we can get a lower value of $C_{\inf}$? We explore this problem in the next two subsections. Assume from now on that $A =$ **diag** $\{\lambda_1, \cdots, \lambda_n\}$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. This is the easiest A to begin with and we will consider general A in the future work. Further assume that $x_i(0) > 0\ \forall\ i$ and $B = I$, $\mathbf{x}$ and $\mathbf{u}$ are of the same dimension, *i.e.*, the controller needs to control n independent subsystems together. Then the lemma 2 shows that

$$C_{\inf} = \frac{l\lambda_1\log e}{\frac{l}{n} - 1 - D\lambda_1\log e},$$

as in this case $||e^A|| = e^{\lambda_1}$.

We study another bit allocation scheme in the second subsection and the optimal bit allocation scheme in the last subsection.

B. Proportional Bit Allocation

A naturally proposed different bit allocation scheme is that $\frac{\lambda_i}{\lambda}l$ bits are used for the i^{th} subsystem, where $\lambda = \sum_{i=1}^{n}\lambda_i$. So the bit allocation is proportional to the size of the eigenvalues. Proceeding the same proof as in the above, the sufficient conditions are

$$e^{\lambda_i(\delta+D)} < 2^{\frac{\lambda_i}{\lambda}l - 1}\ \forall\ i,$$

which turn out to be

$$C > \max_i \frac{l\lambda\lambda_i\log e}{\lambda_i l - \lambda - \lambda\lambda_i D\log e}.$$

Note that in this case,

$$\begin{aligned} C_{\inf} &= \max_i \frac{l\lambda\lambda_i\log e}{\lambda_i l - \lambda - \lambda\lambda_i D\log e} \\ &= \max_i \frac{l\lambda\log e}{l - \lambda D\log e - \frac{\lambda}{\lambda_i}} \\ &= \frac{l\lambda\log e}{l - \lambda D\log e - \frac{\lambda}{\lambda_n}}. \end{aligned}$$

Hence if the following condition

$$\frac{l\lambda_1\log e}{\frac{l}{n} - 1 - \lambda_1 D\log e} > \frac{l\lambda\log e}{l - \lambda D\log e - \frac{\lambda}{\lambda_n}},$$

is satisfied, which can be equivalently written as

$$l > \frac{(\frac{\lambda_1}{\lambda_n} - 1)\lambda}{\lambda_1 - \frac{\lambda}{n}}, \tag{12}$$

then the proportional allocation scheme can give a lower value of C_{inf} that the equal allocation scheme produces (see example 4 for a comparison).

In the proportional bit allocation scheme,

$$C_{\text{inf}} = \frac{l\lambda \log e}{l - \lambda D \log e - \frac{\lambda}{\lambda_n}},$$

i.e., the smallest eigenvalue determines the largest C needed? which is counter-intuitive. Ideally we wish that the largest eigenvalue determine the largest C. Then a natural question to ask is that is the proportional bit allocation scheme the optimal scheme ? If not, how can we find the optimal one? We consider these problems in the following subsection.

C. Optimal Bit Allocation

In this subsection, we consider the optimal bit allocation scheme . We start with variable portions of the bits need to be allocated to each individual subsystems, and then perform an optimization on those variables. Suppose that we use $\beta_i l$ bits for the i^{th} subsystem, where $\sum_{i=1}^{n} \beta_i = 1$, $\beta_i > 0$. Similar to proportional allocation scheme, we have

$$e^{\lambda_i(\delta + D)} < 2^{\beta_i l - 1} \; \forall \, i,$$

which turns out to be

$$C_{\text{inf}} = \max_{\lambda_i} \{ \frac{\lambda_i l \log e}{\beta_i l - 1 - \lambda_i D \log e} \}.$$

Denote $\beta = [\beta_1 \cdots \beta_n]$, we form an optimization problem as follows:

$$\min_{\beta} C_{\text{inf}}$$

subjected to

$$\begin{aligned} \sum_{i=1}^{n} \beta_i &= 1, \beta_i > 0 \; \forall \, i, \\ \beta_i l &> 1 + \lambda_i D \log e \; \forall \, i. \end{aligned}$$

where

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$$

are given.

By introducing a dummy variable t, we write the above problem as follows.

$$\min_{\beta} t$$

subjected to

$$\frac{\lambda_i l \log e}{\beta_i l - 1 - \lambda_i D \log e} \leq t \; \forall \, i.$$

$$\sum_{i=1}^{n} \beta_i = 1, \beta_i > 0 \; \forall \, i.$$

$$\beta_i l > 1 + \lambda_i D \log e \; \forall \, i.$$

Equivalently, it can be written as:

$$\min_{\beta} t$$

subjected to

$$\begin{bmatrix} t & \sqrt{\lambda_i \log e} \\ \sqrt{\lambda_i \log e} & \beta_i - \frac{1}{l} - \frac{\lambda_i D \log e}{l} \end{bmatrix} \geq 0 \; \forall \, i.$$

$$\sum_{i=1}^{n} \beta_i = 1, \beta_i > 0 \; \forall \, i.$$

$$\beta_i l > 1 + \lambda_i D \log e \; \forall \, i.$$

This is a standard LMI with variables t and β (see [1] for an introduction to convex optimization and LMI problems) which can be solved efficiently using standard SDP solvers. For example, using the SDP solver SeDuMi [11] and the Matlab interface Yalmip [17], the above problem can be solved very efficiently.
We now consider an example to illustrate the differences between the three bit allocation schemes we have discussed so far.

Example 4: Consider a simple LTI system with dimension 10 and $\lambda_i = 3i$ for $i = 1, 2, \cdots, 10$. Figure 2 and Figure 3 show the minimum channel data rate as a function of the channel maximum delay for different allocation schemes. In Figure 2, $l = 200$ and inequality (12) is satisfied, hence the proportional scheme is better than the equal scheme. In Figure 3, $l = 100$ and inequality (12) is not satisfied, therefore the equal scheme turns out to be better than the proportional scheme. Figure 4 shows the different β values corresponding to the three different bit allocation schemes where we set $D = 0$ and $l = 200$. In any cases, the optimal scheme is always the best. Figure 5 gives a plot of the minimum data rate produced by the three schemes as a function of the packet length, where we assume $D = 0$. It is clear from the result that if l is small, the proportional scheme is the worst of all. However, as l gets big enough, the proportional scheme in fact converges to the optimal scheme. The intersection point of the proportional scheme and the equal scheme is determined by inequality (12) and in this case, the value is $l = 110$. We give detailed analysis in below.

Remark 5: The optimal scheme tries to give more weight to the most unstable eigenvalue and less weight to the least unstable eigenvalue which intuitively makes sense. Also notice that if $\lambda_i' s$ are all equal, the three bit allocation schemes then give the same result and $\beta_i = 1/n$ in this case.

If $l \gg n$ is true, then simulation result (Figure 6) shows that

$$\frac{C_{\inf equal}}{C_{\inf optimal}} \approx \frac{n\lambda_1}{\sum_{i=1}^{n} \lambda_i}.$$

In the case $l \gg n$, we obtain

$$C_{\inf equal} \approx n\lambda_1 \log e,$$

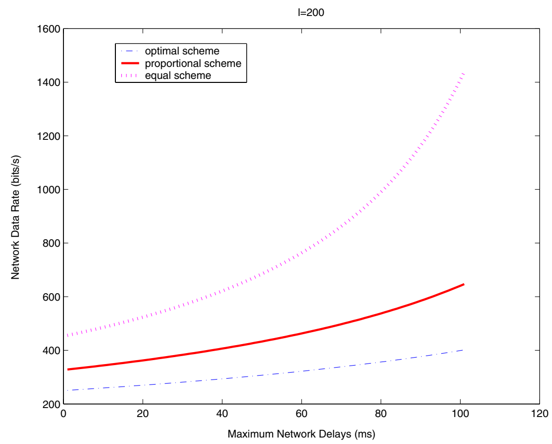

Fig. 2. Minimum Channel Data Rate versus Maximum Channel Delay for $l = 200$

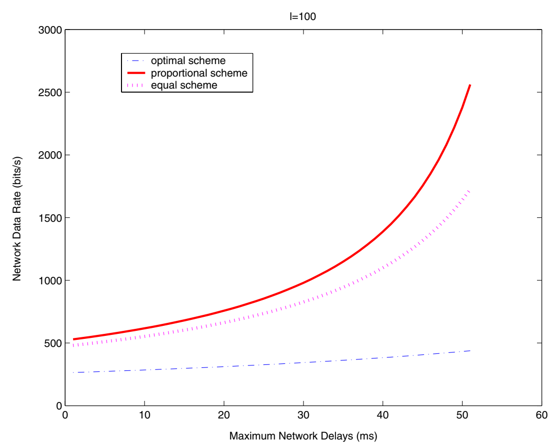

Fig. 3. Minimum Channel Data Rate versus Maximum Channel Delay for $l = 100$

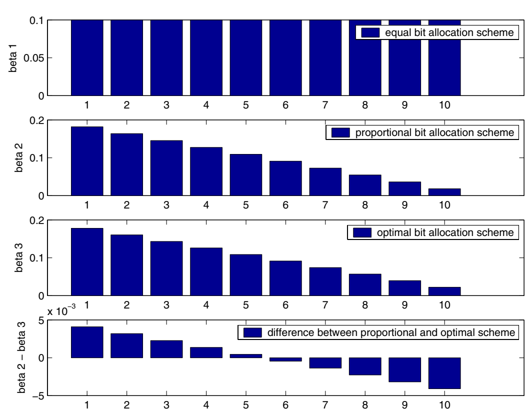

Fig. 4. Different bit allocation schemes

hence

$$C_{\inf\, optimal} \approx \sum_{i=1}^{n} \lambda_i \log e.$$

We can even show this result analytically for some special cases. For example, by assuming $D = 0$, and letting l go

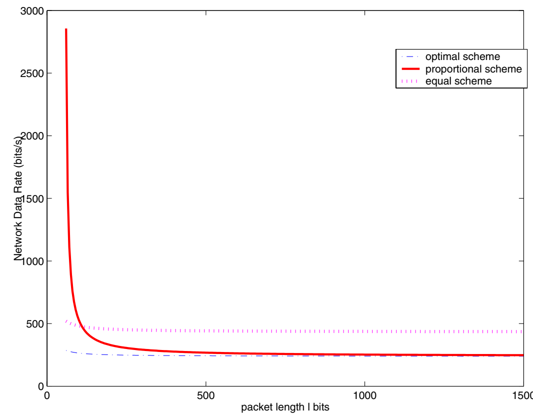

Fig. 5. Minimum Channel Data Rate versus Packet Length l

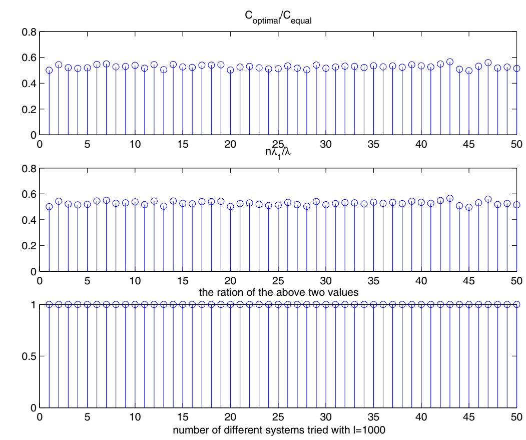

Fig. 6. Equal versus optimal schemes

to ∞, we obtain the following optimization problem

$$\min_{\beta} t$$

subjected to

$$\sum_{i=1}^{n} \beta_i = 1, \beta_i > 0 \; \forall \; i.$$

$$t \geq \frac{\lambda_i \log e}{\beta_i} \; \forall \; i.$$

For this problem, we can actually derive a closed form solution for t which is

$$t = \sum_{i=1}^{n} \lambda_i \log e.$$

In this case, the optimal bit allocation scheme turns out to coincide with the proportional bit allocation scheme which also explains the result from Figure 5. Notice that by the proportional scheme, *i.e.*, by letting

$$\beta_i = \frac{\lambda_i}{\sum_{k=1}^{n} \lambda_k},$$

we have

$$\frac{\lambda_i \log e}{\beta_i} = \sum_{k=1}^{n} \lambda_k \log e \; \forall \; i.$$

Hence

$$t \geq \sum_{k=1}^{n} \lambda_k \log e.$$

We argue that for any other β, there exists at least one i such that the following is true

$$\frac{\lambda_i \log e}{\beta_i} \geq \sum_{k=1}^{n} \lambda_k \log e.$$

Otherwise, assume that

$$\frac{\lambda_i \log e}{\beta_i} < \sum_{k=1}^{n} \lambda_k \log e \; \forall \; i.$$

Or equivalently,

$$\lambda_i \log e < \beta_i \sum_{k=1}^{n} \lambda_k \log e \; \forall \; i.$$

Add the n inequalities together, we obtain

$$\sum_{i=1}^{n} \lambda_i < \sum_{k=1}^{n} \lambda_k$$

which is clearly a contradiction. Hence

$$t_{\inf} = \sum_{i=1}^{n} \lambda_i \log e.$$

In the case where $l < \infty$ and $D \neq 0$, the proportional scheme will then not return the optimal value, but we can solve the corresponding LMI optimization problem.

V. Conclusions and Future Work

As an initial step towards a complete theory of packet-based control, we have considered the classical stability issue of LTI system over a packet-based network. A set of sufficient conditions are given on the minimum network data rate ($C_{\inf}$) such that the overall closed loop system is exponentially stable. An encoding-decoding pair and an associated control law are proposed to make the system stable once the network data rate C is greater than $C_{\inf}$.

The lower bound we obtained on $C_{\inf}$ is quite conservative, therefore it does not mean that if $C \leq C_{\inf}$, the whole system is not stabilizable. $C_{\inf}$ can be further pushed down by other proper choices of control laws.

For this ground work of research, we have dealt with noiseless LTI systems. In real implementation, we always have to face all kinds of noises. Therefore a natural extension of this work will be to analyze the noise in the control loop. For example, we can consider the case where the observer can not obtain the perfect information of the state information, *i.e.*, $\mathbf{y}(t) = H\mathbf{x}(t) + \mathbf{w}(t)$ where $\mathbf{w}(t)$ is an independent sensor noise process. Intuitively, we might need to change our definition of stability to be just within certain neighborhood of the origin as we cannot drive the state back to the origin precisely. If we adopt the new definition, there is a hope that the same analysis can be generalized in this case.

As we pointed out earlier in section 2 that for simplicity, we have not considered the case where packet loss or reorder cannot be ignored. We can extend this work by incorporating those issues into account and ask similar questions like as in the above. We can also ask the question that what is the maximum packet loss rate that the closed loop system can handle given $C > C_{\inf}$?

In section 4, part B and C, we studied the cases where A is diagonal and it will be interesting to study the general case for arbitrary unstable A.

Those problems are of immediate interest and will be pursued in the near future.

References

[1] Stephen Boyd and Lieven Vandenberghe. *Convex Optimization*. Cambridge University Press, March 2004.

[2] F. Callier and C. Desoer. *Linear System Theory*. Springer Verlag, 1991.

[3] D.F.Delchamps. Stabilizing a linear system with quantized state feedback. *IEEE Trans. Automat. Contr.*, 35:916–924, 1990.

[4] G.N.Nair and R.J.Evans. Communication-limited stabilization of linear systems. In *Proceedings of the 39th Conf. on Decision and Contr.*, volume 1, pages 1005–1010, Dec 2000.

[5] I.R.Petersen and A.V.Savkin. Multi-rate stabilization of multivariable discrete-time linear systems via a limited capacity communication channel. In *Proceedings of the 40th Conf. on Decision and Contr.*, volume 1, pages 304–309, Dec 2001.

[6] Hassan K. Khalil. *Nonlinear Systems*. Prentice Hall, second edition edition, 1995.

[7] H.Hindi S.Boyd L.Xiao, M.Johansson and A. Goldsmith. Joint optimization of communication rates and linear systems. *IEEE Trans. Automat. Contr.*, 48, Jan 2003.

[8] S. K. Mitter. Control with limited information: the role of systems theory and information theory. In *ISIT 2000 Plenary Talk, IEEE Information Theory Society Newsletter*, page 50, 2000.

[9] R.W.Brockett and D.Liberzon. Quantized feedback stabilization of linear systems. *IEEE Trans. Automat. Contr.*, 45, July 2000.

[10] A. Sahai. *Anytime Information Theory*. PhD thesis, Massachusetts Institute of Technology, 2001.

[11] Sedumi. http://fewcal.kub.nl/sturm/software/sedumi.html.

[12] C. E. Shannon. A mathematical theory of communication. *Bell System Tech. J.*, 27:379–423; 623–656, 1948.

[13] S.S.Sastry. *Nonlinear Systems; Analysis, Stability, and Control*. Springer Verlag, 1999.

[14] S. C. Tatikonda. *Control Under Communication Constraints*. PhD thesis, Massachusetts Institute of Technology, 2000.

[15] W.S.Wong and R.W.Brochett. Systems with finite communication bandwidth-part i: State estimation problems. *IEEE Trans. Automat. Contr.*, 42, Sept 1997.

[16] W.S.Wong and R.W.Brochett. Systems with finite communication bandwidth-part ii: Stabilization with limited information feedback. *IEEE Trans. Automat. Contr.*, 44, May 1999.

[17] Yalmip. http://control.ee.ethz.ch/ joloef/yalmip.msql.