Stability and Optimality of Constrained Model Predictive Control with Future Input Buffering in Networked Control Systems

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Abstract— **This paper focuses on the stability and optimality of a novel control strategy for Networked Control Systems (NCS). The developed control strategy hones the potential of constrained Model Predictive Control (MPC) by buffering the predicted control sequence at the actuator in anticipation of the occurrence of typical data transmission errors associated with NCS. Global closed-loop stability in the sense of Lyapunov is guaranteed by bounding the projected receding horizon costs by lower- and upper-bounding terms using a predetermined terminal cost. The developed stability theorem, although suboptimal in real-time, is a sufficient measure to estimate the worst-case transmission delay that can be handled by the developed control buffering strategy.**

I. Introduction

Flexibility, reconfigurability, modularity, low installation cost, and ease of maintenance and diagnosis, are some of the advantages of networked control systems (NCS), which have brought forth increasing interest of this field among industries. However, the streams of data exchange between NCS components are prone to deterministic or non-deterministic delay, losses, and mis-synchronization, which degrade the performance and stability of the feedback control loop, if the controllers are designed using traditional approaches. The research of NCS may focus in two general directions. One direction may involve the development of robust communication protocols to ensure constant delay or minimum jitter in the data stream. Then, controllers can be designed without considering the dynamics of the network. The second direction in NCS research treats the particular communication protocol as fixed, and focuses on the development of feedback control methodologies. This latter direction is considered here.

This paper extends an NCS control strategy previously developed by the authors [1, 2], which is based on Generalized Predictive Control (GPC) with the buffering of future control sequence to overcome the transmission problems at the controller-to-actuator lines. The main objective of the present paper is to determine the conditions under which the stability of the developed control strategy is guaranteed, in a broader realm of constrained Model Predictive Control (MPC). The next section briefly introduces the developed strategy. Section 3 presents the main contribution of the paper, specifically the developed stability conditions with the utilization of buffered optimal input signals. Section 4 provides guidelines for the controller design that will maintain closed-loop stability, and evaluates the stability boundaries numerically.

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II. The NCS Strategy

Since the communication delay between network nodes (sensors, actuators and controllers) is non-deterministic, with random and frequent occurrence of data losses and vacant sampling, it is anticipated that the design of the networked feedback control solution requires no prior knowledge of the transmission dynamics. The only model used here is the linear, time-invariant discrete-time model of the system under control, given by

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}\boldsymbol{x}_k + \boldsymbol{B}\boldsymbol{u}_k \tag{1}$$

where $\boldsymbol{x}_k \in \mathbb{R}^n$ are the states of the system, $\boldsymbol{u}_k \in \mathbb{R}^m$ are the control inputs, and $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{B} \in \mathbb{R}^{n\times m}$. Fig. 1 shows the general architecture of the developed NCS strategy. The physical system consisting of sensor and actuator nodes are connected to the controller through a communication medium. All the nodes are set to a common sampling interval h with a synchronized clock. The sensor-to-controller and controller-to-actuator communication delays are represented by τ_{sc} and τ_{ca}, respectively. In order to fill the missing sensor data up to the current sample time k, an extended state observer is used. The key to the developed strategy is the use of an MPC policy, which is a form of receding horizon optimal control law, to *pre*-compute a string of optimal future control input sequence $\boldsymbol{\pi}_k^* = \left\{\boldsymbol{u}_{k|k}^*, ..., \boldsymbol{u}_{k+H|k}^*\right\} \in \mathbb{R}^{Hm}$, where H is the length of the prediction horizon. This is done at each time step k, by optimizing a finite quadratic cost function given by

$$V_H(\boldsymbol{x}_k, \boldsymbol{\pi}_k) = \min_{\boldsymbol{\pi}(\cdot|k)} \left\{ \left\| x_{k+H|k} \right\|_{\boldsymbol{P}_0}^2 + \sum_{i=0}^{H-1} W_k(\boldsymbol{x}_{k+i}, \boldsymbol{u}_{k+i}) \right\} \tag{2}$$

s.t. $\boldsymbol{x}_{k+i|k} \in \mathcal{X}_i$ and $\boldsymbol{u}_{k+i|k} \in \mathcal{U}$ for $\forall i = 1, ..., H$

where $W_k(\boldsymbol{x}_{k+i}, \boldsymbol{u}_{k+i}) = \left\|\boldsymbol{x}_{k+i|k}\right\|_{\boldsymbol{Q}}^2 + \left\|\boldsymbol{u}_{k+i|k}\right\|_{\boldsymbol{R}}^2$, $\boldsymbol{P}_0$ is the terminal weight matrix, and $\boldsymbol{Q}$ and $\boldsymbol{R}$ are the weighting

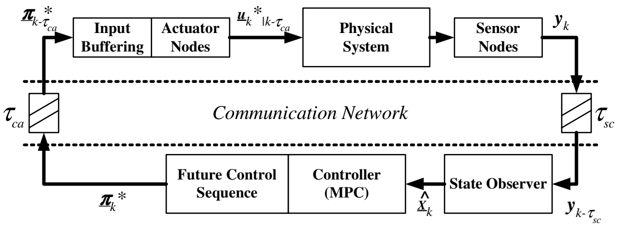

Fig. 1. Overview of the developed NCS strategy.

sequence matrices ($\boldsymbol{P}_0 \succ 0$, $\boldsymbol{Q} \succ 0$ and $\boldsymbol{R} \succ 0$). This yields the optimal predicted future control sequence as $\boldsymbol{\pi}_k^* = \arg^{\min}_{\boldsymbol{\pi}} V_H(\boldsymbol{x}_k, \boldsymbol{\pi}_k)$. In a typical application of MPC, the first control input $\boldsymbol{u}_{k|k}^*$ is used. In the developed NCS strategy, however, the subsequent control sequence is employed depending on the level of transmission delay. This is accomplished by retrofitting the actuator with a data buffering mechanism to schedule an appropriate control input $\boldsymbol{u}_{k+i|k}^*$ from $\boldsymbol{\pi}_k^*$ based on the time-stamps of data packets.

The developed NCS strategy has been benchmarked and shown in [1,2] to perform well with satisfactory results. Besides being able to cope with transmission delays, it is also able to compensate for various levels of packet losses and vacant sampling. The main concern then is whether the pre-computed and buffered string of optimal control input sequence is still feasible and optimal at the future sampling instant of the actuator. The conditions whereby the stability of the closed-loop NCS is guaranteed will be established.

III. Stability of Future Input Buffering

The approach adopted here to analyze the stability of the developed future control input buffering originates from the work by Primbs and Nevistić [3] where the terminal states $\boldsymbol{x}_{k+H}$ are related to the current states $\boldsymbol{x}_k$ by establishing a bound on the terminal cost. The various levels of projected cost, depending on the controller-to-actuator delay τ_{ca} , are essentially bounded by predetermined upper and lower bound terms. Stability is achieved without end constraints. Here, the state measurements presented to the constrained MPC controller are assumed to be perfect.

A. *Preliminaries*

In order to establish conditions under which the closed-loop NCS is guaranteed to be asymptotically stable in the sense of Lyapunov, first a valid Lyapunov function is identified.

Definition 1 [4]: A continuous function $V(\cdot): \mathbb{R}^n \to \mathbb{R}_+$ is a valid Lyapunov function for system (1) if there are some $\mathcal{K}_\infty$-functions $\gamma_1(\cdot)$, $\gamma_2(\cdot)$ and $\gamma_3(\cdot)$ such that $\gamma_1(\|\boldsymbol{x}\|) \le V(\boldsymbol{x}) \le \gamma_2(\|\boldsymbol{x}\|)$ and $V(f(\boldsymbol{x},\boldsymbol{u})) - V(\boldsymbol{x}) \le -\gamma_3(\|\boldsymbol{x}\|)$.

Theorem 1 [5]: A system under control is asymptotically stable in $\mathcal{X} \subseteq \mathbb{R}^n$ if:

1. a function $V(\cdot): \mathbb{R}^n \times \mathbb{R}^{m\times H} \to \mathbb{R}$ with $V(0,0) = 0$ exists such that $V(x,\pi) \ge \alpha_1(\|\boldsymbol{x}\|)$ where $\alpha_1(\cdot)$ is a $\mathcal{K}_\infty$-function.
2. $\mathcal{X}$ exists and contains an open neighborhood of the origin, such that every realization $\{\boldsymbol{x}_k, \boldsymbol{\pi}_k\}$ of the controlled system with $\boldsymbol{x}_0 \in \mathcal{X}$ satisfies $\boldsymbol{x}_k \in \mathcal{X}$ for $\forall k \ge 0$ and $V(\boldsymbol{x}_{k+1}, \boldsymbol{\pi}_{k+1}) - V(\boldsymbol{x}_k, \boldsymbol{\pi}_k) \le -\alpha_2(\|(\boldsymbol{x}_k, \boldsymbol{u}_k)\|)$ where $\alpha_2(\cdot)$ is a $\mathcal{K}_\infty$-function.
3. a constant $r > 0$ exists for $\boldsymbol{x}_k \in \mathcal{B}_r^n := \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\| \le r\}$, such that every realization $\{\boldsymbol{x}_k, \boldsymbol{\pi}_k\}$ of the controlled system with $\boldsymbol{x}_k \in \mathcal{B}_r^n$ satisfies $\|\boldsymbol{\pi}_k\| \le \alpha_3(\|\boldsymbol{x}_k\|)$ where $\alpha_3(\cdot)$ is a $\mathcal{K}$-function.

Hence, in the nominal case, if initial optimal sequence of control input $\boldsymbol{\pi}_0^*$ is feasible, the elements $\boldsymbol{u}_{k+i|k}$ of subsequent sequence of control input $\boldsymbol{\pi}_k^*$ computed using the identical MPC policy of (2) are sufficiently stabilizing in the future as long as it is within *H*.

B. *Maintaining Feasibility with Suboptimal MPC*

In less than nominal conditions, Theorem 1 alone is insufficient to ensure the feasibility of control inputs $\boldsymbol{u}_{k+i|k}$ from $\boldsymbol{\pi}_k^*$ in subsequent time steps $k+i$ for $i \le \tau_{ca} \le H$ as applied in the developed NCS buffering strategy. This can be overcome by enforcing an additional condition to the MPC policy. Consider the states and control inputs at k and $k+\tau_{ca}$. At sample time k , the optimizing $\boldsymbol{\pi}_k^*$ is found. Denote the corresponding cost as $V_H(\boldsymbol{x}_k, \boldsymbol{\pi}_k^*)$.

At the sample time $k+\tau_{ca}$, the corresponding control input of the previously computed $\boldsymbol{\pi}_k$ stored at the buffer is used. Denote $\boldsymbol{\pi}_{pre} = \{\boldsymbol{u}_{k+\tau_{ca}|k}, ..., \boldsymbol{u}_{k+\tau_{ca}+H-1|k}, \boldsymbol{0}, ..., \boldsymbol{0}\} \in \mathbb{R}^{Hm}$ and the corresponding cost as $V_H(\boldsymbol{x}_k, \boldsymbol{\pi}_{pre})$. The following condition guarantees that the closed-loop NCS is asymptotically stabilizing within τ_{ca} by satisfying the second condition of Theorem 1:

$$V_H(\boldsymbol{x}_{k+\tau_{ca}}, \boldsymbol{\pi}_{pre}) - V_H(\boldsymbol{x}_k, \boldsymbol{\pi}_k^*) \le -\mu \sum_{i=0}^{\tau_{ca}-1} W_k(\boldsymbol{x}_{k+i}, \boldsymbol{u}_{k+i}) \quad (3)$$

where $\mu \in (0,1]$;

$$V_H(\boldsymbol{x}_k, \boldsymbol{\pi}_k^*) = \|\boldsymbol{x}_{k+H|k}\|_{\boldsymbol{P}_0}^2 + \sum_{i=0}^{H-1} \|\boldsymbol{x}_{k+i|k}\|_{\boldsymbol{Q}}^2 + \sum_{i=0}^{H-1} \|\boldsymbol{u}_{k+i|k}\|_{\boldsymbol{R}}^2 \text{ ; and}$$

$V_H(\boldsymbol{x}_{k+\tau_{ca}},\boldsymbol{\pi}_{pre}) = \left\| \boldsymbol{x}_{k+\tau_{ca}+H|k+\tau_{ca}} \right\|_{\boldsymbol{P}_0}^2 + \sum_{i=0}^{H-1} \left\| \boldsymbol{x}_{k+\tau_{ca}+i|k+\tau_{ca}} \right\|_{\boldsymbol{Q}}^2$ $+ \sum_{i=0}^{H-\tau_{ca}-1} \left\| \boldsymbol{u}_{k+\tau_{ca}+i|k} \right\|_{\boldsymbol{R}}^2$. Note that since $W_k(\boldsymbol{x}_{k+i},\boldsymbol{u}_{k+i}) > 0$, by forcing the τ_{ca}-step cost difference $V_H(\boldsymbol{x}_{k+\tau_{ca}},\boldsymbol{\pi}_{pre}) - V_H(\boldsymbol{x}_k,\boldsymbol{\pi}_k^*) < 0$, (3) ensures that $V_H(\boldsymbol{x}_k,\boldsymbol{\pi}_k)$ will decrease along the trajectories of the closed-loop system. The design parameter μ is introduced to reduce the effect of model uncertainty and disturbances. It is desirable to make μ sufficiently large in order to provide a less conservative system. The controller-to-actuator delay τ_{ca} is another design parameter, which specifies the worst-case delay the NCS strategy has to handle. Although the introduction of (3) into the MPC policy causes the controller to be suboptimal, feasibility is still maintained, which implies that asymptotic stability is still maintained [5].

C. Establishing the Bounds

Before proceeding to establish the stability condition for the developed future control input buffering strategy for NCS, the terminal cost $\left\| \boldsymbol{x}_{k+H|k} \right\|_{\boldsymbol{P}_0}^2$ has to be bounded in order to formulate the cost reduction terms of (3). This follows directly from [3]. Assume that the finite horizon quadratic cost $V_H(\boldsymbol{x}_k,\boldsymbol{\pi}_k)$ is bounded such that $\boldsymbol{x}_k^T \boldsymbol{U}_H^{\boldsymbol{P}_0} \boldsymbol{x}_k \le V_H(\boldsymbol{x}_k,\boldsymbol{\pi}_k) \le \boldsymbol{x}_k^T \boldsymbol{L}_H^{\boldsymbol{P}_0} \boldsymbol{x}_k$. This leads to an upper bound on the cost at any $j < H$, given by

$$V_{H-(j+1)}(\boldsymbol{x}_{k+(j+1)|k},\boldsymbol{\pi}_{k+(j+1)}) \le \boldsymbol{x}_k^T \left(\boldsymbol{U}_H^{\boldsymbol{P}_0} - \boldsymbol{L}_j^{\boldsymbol{Q}} \right) \boldsymbol{x}_k \qquad (4)$$

Consequently, the terminal cost is bounded in terms of the current state $\boldsymbol{x}_k$ as

$$\left\| \boldsymbol{x}_{k+H|k} \right\|_{\boldsymbol{P}_0}^2 \le \min_{0\le j\le H-1} \kappa_{H-(j+1)}(N-(j+1)) \left\| \boldsymbol{x}_k \right\|_{\boldsymbol{U}_H^{\boldsymbol{P}_0} - \boldsymbol{L}_j^{\boldsymbol{Q}}}^2 \qquad (5)$$

where $\kappa_p(j)$ is defined as

$$\kappa_p(j) = \begin{cases} 1 & \text{for } j=0 \\ \min_{0\le i\le j-1} \kappa_p(i) \left(1 - 1/\bar{\lambda}\left(\boldsymbol{U}_{p-i}^{\boldsymbol{P}_0} \left(\boldsymbol{L}_{(j-i)-1}^{\boldsymbol{Q}} \right)^{-1} \right) \right) & \text{for } j>0 \end{cases}$$

with $\bar{\lambda}(\boldsymbol{C})$ as the maximum eigenvalue of $\boldsymbol{C}$.

The lower bounds $\boldsymbol{L}_j$ are computed by considering the unconstrained MPC policy, which is then solved iteratively by the following Riccati difference equation:

$$\boldsymbol{L}_{j+1} = \boldsymbol{A}^T \left[\boldsymbol{L}_j - \boldsymbol{L}_j \boldsymbol{B}(\boldsymbol{B}^T \boldsymbol{L}_j \boldsymbol{B} + \boldsymbol{R})^{-1} \boldsymbol{B}^T \boldsymbol{L}_j \right] \boldsymbol{A} + \boldsymbol{Q} \qquad (6)$$

Assuming $\boldsymbol{u}_{k+i|k} = \boldsymbol{K}_i \boldsymbol{x}_{k+i|k}$, the upper bounds $\boldsymbol{U}_j$ are computed by minimizing the following unconstrained generalized eigenvalue problem in Linear Matrix Inequality (LMI) form [3,6]:

$$\min_{\beta,\boldsymbol{S}_i,\boldsymbol{Z}_i} \beta$$
s.t.
$$\begin{bmatrix} \beta \boldsymbol{L}_H^{\boldsymbol{P}_0} & \boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{S}_0 \end{bmatrix} \ge \boldsymbol{0}$$
$$\begin{bmatrix} \boldsymbol{S}_i & (\boldsymbol{A}\boldsymbol{S}_i + \boldsymbol{B}\boldsymbol{Z}_i)^T & \boldsymbol{S}_i^T \boldsymbol{Q}^{\frac{1}{2}} & \boldsymbol{Z}_i^T \boldsymbol{R}^{\frac{1}{2}} \\ \boldsymbol{A}\boldsymbol{S}_i + \boldsymbol{B}\boldsymbol{Z}_i & \boldsymbol{S}_{i+1} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{Q}^{\frac{1}{2}} \boldsymbol{S}_i & \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{R}^{\frac{1}{2}} \boldsymbol{Z}_i & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \ge \boldsymbol{0} \qquad (7)$$

for $0 \le i \le H-1$ where $\boldsymbol{S}_i = \boldsymbol{U}_{H-i}^{-1}$, $\boldsymbol{Z}_i = \boldsymbol{K}_i \boldsymbol{S}_i$, and $\boldsymbol{S}_H^{-1} = \boldsymbol{U}_0 = \boldsymbol{P}_0$. The flexibility of LMI conveniently allows further incorporation of various constraints into the problem (see Section IV).

D. Main Stability Results

Using the Principle of Optimality [7], the finite horizon quadratic cost is equivalent to

$$V_H(\boldsymbol{x}_k,\boldsymbol{\pi}_k) = V_{H-\tau_{ca}}(\boldsymbol{x}_{k+\tau_{ca}},\boldsymbol{\upsilon}_{k+\tau_{ca}}) + \sum_{i=0}^{\tau_{ca}-1} W_k(\boldsymbol{x}_{k+i},\boldsymbol{u}_{k+i}) \qquad (8)$$

where $\boldsymbol{\upsilon}_{k+\tau_{ca}} = \left\{ \boldsymbol{u}_{k+\tau_{ca}|k},\ldots,\boldsymbol{u}_{k+H|k} \right\} \in \mathbb{R}^{(H-\tau_{ca})m}$. Consequently, the τ_{ca}-step cost difference terms in (3) can be effectively recast so that all costs are considered from the time step k, yielding

$$\begin{aligned} & V_H(\boldsymbol{x}_k,\boldsymbol{\pi}_k) - V_H(\boldsymbol{x}_{k+\tau_{ca}},\boldsymbol{\pi}_{pre}) \\ &= \left[V_{H-\tau_{ca}}(\boldsymbol{x}_{k+\tau_{ca}},\boldsymbol{\upsilon}_{k+\tau_{ca}}) - V_H(\boldsymbol{x}_{k+\tau_{ca}},\boldsymbol{\pi}_{pre}) \right] \\ &\quad + \sum_{i=0}^{\tau_{ca}-1} W_k(\boldsymbol{x}_{k+i},\boldsymbol{u}_{k+i}) \\ &= \left[V_H(\boldsymbol{x}_{k+\tau_{ca}},\boldsymbol{\upsilon}_{k+\tau_{ca}}) - V_{H+\tau_{ca}}(\boldsymbol{x}_{k+\tau_{ca}},\tilde{\boldsymbol{\pi}}_k) \right] \\ &\quad + \sum_{i=0}^{\tau_{ca}-1} W_k(\boldsymbol{x}_{k+i},\boldsymbol{u}_{k+i}) \\ &= \left[V_H(\boldsymbol{x}_k,\boldsymbol{\pi}_k) - V_{H+\tau_{ca}}(\boldsymbol{x}_k,\tilde{\boldsymbol{\pi}}_k) \right] \\ &\quad + \sum_{i=0}^{\tau_{ca}-1} W_k(\boldsymbol{x}_{k+i},\boldsymbol{u}_{k+i}) \end{aligned} \qquad (9)$$

where $\tilde{\boldsymbol{\pi}}_k = \left\{ \boldsymbol{u}_{k|k},\ldots,\boldsymbol{u}_{k+H|k},\boldsymbol{0},\ldots,\boldsymbol{0} \right\} \in \mathbb{R}^{(H+\tau_{ca})m}$.

Theorem 2: The NCS-MPC policy of horizon H utilizing the optimal control sequence $\boldsymbol{\pi}_{pre}$ of previous time step up to a controller-to-actuator delay of τ_{ca} is asymptotically stabilizing within the region of attraction $\mathcal{R}$ if there exists some j such that

$$(1-\mu)\boldsymbol{\Lambda}_H^{\tau_{ca}} - \boldsymbol{\Psi}_{H,\tau_{ca}}^j \succ 0 \qquad (10)$$

where $\Lambda_H^{\tau_{ca}} = L_H^{P_0} - U_H^{P_0} + L_{\tau_{ca}}^{Q}$, and

$$\Psi_H^j = \max\left[0, \bar{\lambda}\left(U_{\tau_{ca}}^{P_0} P_0^{-1}\right) - 1\right] \kappa_{H-(j+1)}(H-(j+1))\left[U_H^{P_0} - L_j^{Q}\right]$$

Proof:

From the stability condition (3) combining with (9), the following is obtained:

$$(1-\mu)\sum_{i=0}^{\tau_{ca}-1} W_k(x_{k+i}, u_{k+i}) - \left[V_{H+\tau_{ca}}(x_k, \tilde{\pi}_k) - V_H(x_k, \pi_k)\right] > 0 \quad (11)$$

Since the first left-hand-side term may be expressed as $\sum_{i=0}^{\tau_{ca}-1} W_k(x_{k+i}, u_{k+i}) = V_H(x_k, \pi_k) - V_{H-\tau_{ca}}(x_{k+\tau_{ca}}, \pi_{k+\tau_{ca}})$, it can be bound from below by utilizing (4) to bound the costs as $V_H(x_k) \ge \|x_k\|^2_{L_H^{P_0}}$ and $V_{H-\tau_{ca}}(x_{k+\tau_{ca}}, \pi_{k+\tau_{ca}}) \le \|x_k\|^2_{U_H^{P_0} - L_{\tau_{ca}-1}^{Q}}$ resulting in

$$\begin{aligned}\sum_{i=0}^{\tau_{ca}-1} W_k(x_{k+i}, u_{k+i}) &> \|x_k\|^2_{L_H^{P_0}} - \|x_k\|^2_{U_H^{P_0} - L_{\tau_{ca}-1}^{Q}} \\ &= \|x_k\|^2_{L_H^{P_0} - U_H^{P_0} + L_{\tau_{ca}-1}^{Q}}\end{aligned} \quad (12)$$

Next, the second left-hand-side term of (11) is required to be bounded from above. The term $\left[V_H(x_k, \pi_k) - V_{H+\tau_{ca}}(x_k, \tilde{\pi}_k)\right]$ in the right-hand-side of (9) is equivalent to $\left[\|x_{k+H|k}\|^2_{P_0} - V_{\tau_{ca}}(x_{k+H|k}, \pi = 0)\right]$, which provides the means to bound it from below. Since, intuitively, $V_{\tau_{ca}}(x_{k+H|k}, \pi = 0) \le V_{\tau_{ca}}(x_{k+H|k}, \pi_{k+H})$, the following bound can be enforced:

$$\begin{aligned}&V_{H+\tau_{ca}}(x_k, \tilde{\pi}_k) - V_H(x_k, \pi_k) \\ &\le V_{\tau_{ca}}(x_{k+H|k}, \pi_{k+H}) - \|x_{k+H|k}\|^2_{P_0}\end{aligned} \quad (13)$$

On the other hand, it can be evidently obtained that $V_{\tau_{ca}}(x_{k+H|k}, \pi_{k+H}) \le \|x_{k+H|k}\|^2_{U_{\tau_{ca}}^{P_0}}$ since $V_H(x_k, \pi_k) \le \|x_k\|^2_{U_H^{P_0}}$. Therefore, using (5) the upper bound for $\left[V_{H+\tau_{ca}}(x_k, \tilde{\pi}_k) - V_H(x_k, \pi_k)\right]$ is resolved as

$$\begin{aligned}&V_{H+\tau_{ca}}(x_k, \tilde{\pi}_k) - V_H(x_k, \pi_k) \\ &\le \|x_{k+H|k}\|^2_{U_{\tau_{ca}}^{P_0}} - \|x_{k+H|k}\|^2_{P_0} \le \left(\bar{\lambda}\left(U_{\tau_{ca}}^{P_0} P_0^{-1}\right) - 1\right)\|x_{k+H|k}\|^2_{P_0} \\ &\le \max\left[0, \bar{\lambda}\left(U_{\tau_{ca}}^{P_0} P_0^{-1}\right) - 1\right] \kappa_{H-(j+1)}(H-(j+1))\|x_k\|^2_{U_H^{P_0} - L_j^{Q}} \\ &\le \|x_k\|_{\Psi_{H,\tau_{ca}}^j}\end{aligned} \quad (14)$$

Hence, (12) and (14) provide the bounds to establish (10), which guarantees asymptotic stability. □

The preceding theorem is most useful as a design guide for the developed NCS control strategy. The parameters involved in designing a stable controller are Q, R, H, μ, and P_0. This will determine the worst-case delay τ_{ca} that the controller is able to handle. Some other possible parameters that can influence the closed-loop stability of the proposed strategy are various input, state and rate constraints, which may be included. As discussed in Section IV, the constraints are formulated into the LMI (7), which modify the upper bounds of the finite horizon quadratic cost U_j accordingly. It should be noted that the computational speed of the MPC optimization is also a factor in deciding the prediction horizon H.

IV. NUMERICAL EXAMPLE

This section illustrates the use of the main stability theorem by employing the model of an electro-hydraulic positioning servo system, as in the previous experiments of the authors [1,2]. The effect of various design parameters, particularly, the desired worst-case delay, the length of the prediction horizon, the terminal cost, and constraints on the stability boundaries of the closed-loop system, is evaluated.

The system is modeled to consist of one input and four states. The system is identified using pseudo-random-binary-sequence (PRBS) inputs to obtain a normalized state-space model, given as:

$$A = \begin{bmatrix} 1.06730 & -0.10423 & 0.13156 & 0.41143 \\ -0.02484 & 0.89404 & -1.51910 & 0.45326 \\ -0.00123 & 0.23267 & 1.15180 & -0.94321 \\ 0.30104 & 0.18348 & 0.19487 & 0.28778 \end{bmatrix}$$ and

$B = [0.00138 \quad 0.01027 \quad -0.00862 \quad -0.02929]^T$ with the sampling interval $h = 10\text{ms}$.

The weighting matrices are set as $Q = \text{diag}(10, 1, 0.001, 0.001)$ and $R = 10$. Here, μ is set at 0.90 while the stability boundaries are considered with various values of terminal weights P_0 as fractions of the open-loop infinite weight U_∞ obtained by solving the discrete Lyapunov equation: $U_\infty = Q + A^T U_\infty A$.

State constraints of the form $x^T T x \le \eta$ are enforced by augmenting the LMIs:

$$\begin{bmatrix} \gamma_i T + L_{i-1}^{Q} & I \\ I & S_0 \end{bmatrix} \ge 0 \quad (15)$$

for $0 \le i \le H-1$ to the base LMIs in (7) to compute U_j. Input constraints of the form $|u_{k+i|k}| \le u_{\lim}$ for $0 \le i \le H-1$ are included by augmenting the LMIs in (7) with the following two LMIs:

$$\begin{bmatrix} \boldsymbol{N}_i & \boldsymbol{Z}_i \\ \boldsymbol{Z}_i^T & \boldsymbol{S}_i \end{bmatrix} \geq \boldsymbol{0} \tag{16}$$

$$2u_{\text{lim}}^2 \delta_i \geq \eta \delta_i^2 N_{jj} + \gamma_i u_{\text{lim}}^2 \tag{17}$$

for $0 \leq i \leq H-1$, where $\boldsymbol{N}_i$ are arbitrary and symmetric matrices, and δ_i are arbitrary constants in the vicinity of γ_i.

Although Theorem 2 states that stability is guaranteed for "any" number of $j \geq 1$ that renders (10) to be positive definite, a stability metric is introduced to determine the relative degree of stability of a particular setting. This relative degree of stability is given as the ratio of the number of j to the length of the prediction horizon minus one.

Fig. 2(a) shows the stability surface at $\boldsymbol{P}_0 = 0.25\boldsymbol{U}_\infty$ with input constraint $u_{\text{lim}} = 1$ and the state constraint terms set to $\boldsymbol{T} = \text{diag}(1,1,1,1)$ and $\eta = 1$. The system is unstable with a short prediction horizon. When the length of the prediction horizon is increased beyond 11, the closed-loop stability is maintained. Stability can be achieved for a shorter prediction horizon with higher terminal weight, as shown in Fig. 2(b) with $\boldsymbol{P}_0 = \boldsymbol{U}_\infty$. However, for a particular stabilizing prediction horizon, the worst-case controller-to-actuator delay that can be handled is lower. Although intuitively, the stability increases as $\boldsymbol{P}_0 \to \boldsymbol{U}_\infty$, it is not the case in the presence of delay when the future control inputs in the buffered sequence are used, as seen before. This is because, as the terminal weight $\boldsymbol{P}_0$ increases beyond a certain value, the terminal cost tends to dominate the optimization (minimization) and leads to a solution that is more suboptimal. So, in order to maintain stability of the closed-loop NCS, $\boldsymbol{P}_0$ has to be an intermittent value relative to $\boldsymbol{U}_\infty$. This effect can be reduced by introducing an additional constraint on the minimum decay rate of the states to modify the speed of response of the closed-loop system. Fig. 2(c) shows the relative stability of the closed-loop system under various levels of terminal weights at a fixed prediction horizon of 25. Again, note the lower degree of stability when $\boldsymbol{P}_0 = \boldsymbol{U}_\infty$ compared to $\boldsymbol{P}_0 = 0.50\boldsymbol{U}_\infty$ and $\boldsymbol{P}_0 = 0.75\boldsymbol{U}_\infty$. Stability can only be maintained up to a controller-to-actuator delay of 18. This phenomenon can be overcome by introducing the state minimum decay rate constraint into the control optimization. The effect of terminal weights to the closed-loop stability at a required worst-case delay of 8 is shown in Fig. 2(d). Obviously, in order to maintain stability, a longer prediction horizon is required for lower terminal weights. The fluctuations in the plots is due to some numerical errors in the computation, particularly, the optimization of the LMIs. It should not be of any concern because Theorem 2 only requires $j \geq 1$ for guaranteed closed-loop asymptotic stability.

The state minimum decay rate constraint used here is of the form $\boldsymbol{x}_{k+i+1|k}^T \boldsymbol{U}_{H-(i+1)} \boldsymbol{x}_{k+i+1|k} \leq \rho^2 \boldsymbol{x}_{k+i|k}^T \boldsymbol{U}_{H-i} \boldsymbol{x}_{k+i|k}$ for $0 \leq i \leq H-1$ where $\rho \in (0,1)$ is the additional tuning parameter for specifying the speed of closed-loop response [8]. It can be shown that this constraint reduces to the following LMIs:

$$\begin{bmatrix} \rho^2 \boldsymbol{S}_i & \left(\boldsymbol{A}\boldsymbol{S}_i + \boldsymbol{B}\boldsymbol{Z}_i\right)^T \\ \boldsymbol{A}\boldsymbol{S}_i + \boldsymbol{B}\boldsymbol{Z}_i & \boldsymbol{S}_{i+1} \end{bmatrix} \geq \boldsymbol{0} \tag{18}$$

for $0 \leq i \leq H-1$. Then, (18) is added to the LMIs (7), (15), (16), and (17) given above to compute $\boldsymbol{U}_j$.

Fig. 3 shows the stability boundaries with $\rho = 0.99$ and all other parameters set as before. Fig. 3(a) shows the stability surface at $\boldsymbol{P}_0 = 0.25\boldsymbol{U}_\infty$ which is the same value used in Fig. 2(a). Comparing Fig. 3(a) with 2(a), it can be seen that, overall, the relative degree of stability has decreased by the introduction of the state minimum decay rate constraint. However, significant improvement is achieved when $\boldsymbol{P}_0 = \boldsymbol{U}_\infty$ as can be observed in Fig. 3(b) and 3(c). The drop in the level of worst-case delay with longer prediction horizon is eliminated, and the overall stability has improved.

V. CONCLUSION

Utilizing the second method of Lyapunov, an MPC-based network control system strategy that was developed by the authors using future control input sequence buffering technique, was shown to be feasible and asymptotically stable. Although, under non-nominal conditions, the pre-computed control actions are sub-optimal, they are still feasible, and therefore stabilizing. The developed theorem has been provided as a design criterion to set the parameters of the controller. Besides the transmission delay, the developed strategy has the capability of compensating for data losses, vacant sampling, and out-of-order data without any modification.

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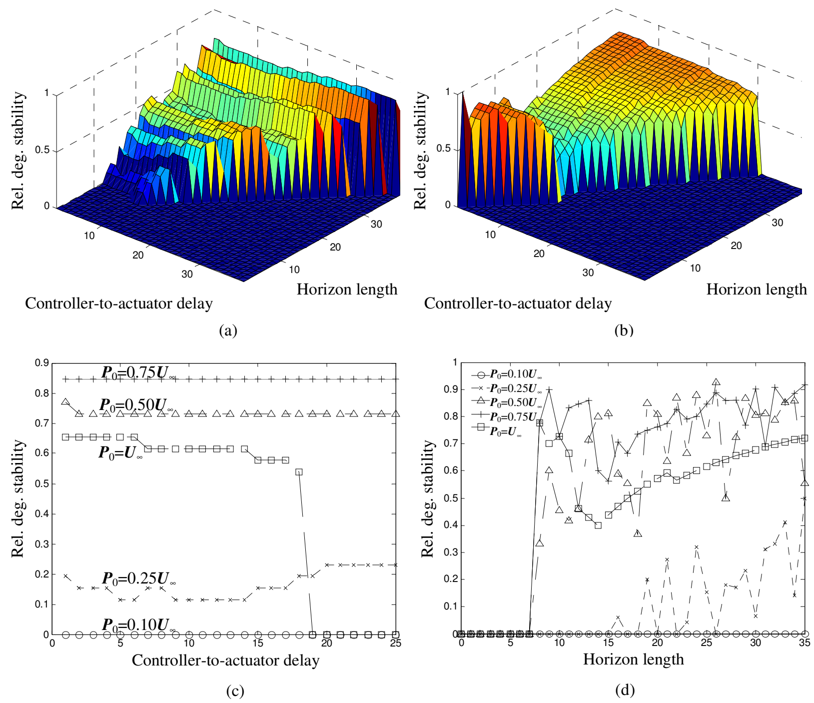

Fig. 2. Stability boundaries with $u_{\lim} = 1$. (a) Stability surface at $P_0 = 0.25U_\infty$; (b) Stability surface at $P_0 = U_\infty$; (c) Comparison of relative stability over different terminal weights at horizon length of 25; and (d) Comparison of relative stability over different terminal weights at $\tau_{ca} = 8$.

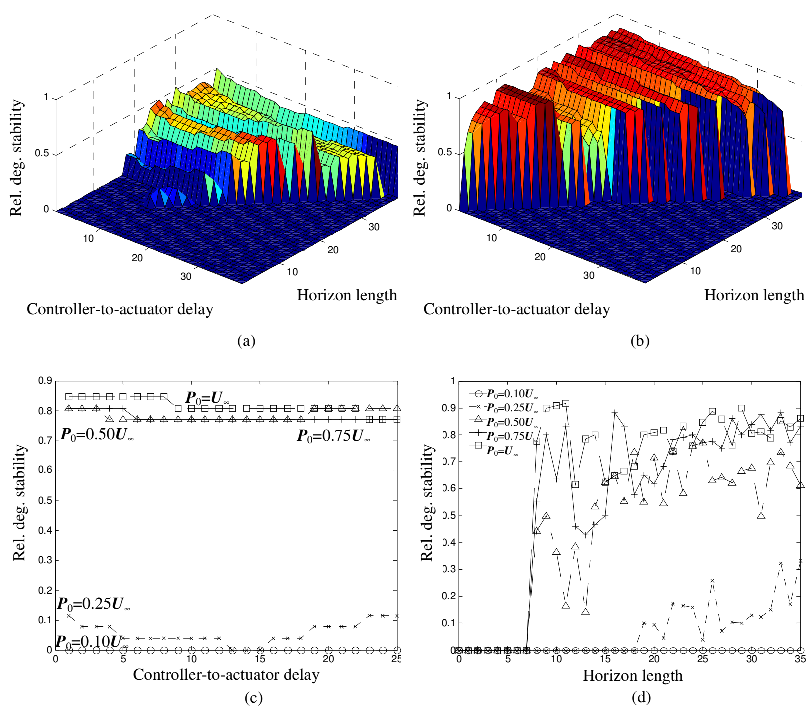

Fig. 3. Stability boundaries with $u_{\lim} = 1$ and $\rho = 0.99$. (a) Stability surface at $P_0 = 0.25U_\infty$; (b) Stability surface at $P_0 = U_\infty$; (c) Comparison of relative stability over different terminal weights at horizon length of 25; and (d) Comparison of relative stability over different terminal weights at $\tau_{ca} = 8$.