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Robust identification over networks

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***Abstract*— We study a robust identification scheme where the output of a plant is sent over a communication channel to a remote identifier. In particular, we are interested in using the channel efficiently in terms of bandwidth and consider the design of coarse quantizers. The problem is formulated under an H^∞ criterion in the discrete-time domain. By allowing feedback in the communication, we employ a class of quantizers called logarithmic quantizers that has been proposed in the context of stabilization with limited information.**

I. Introduction

In digital control systems, issues related to quantization inevitably arise and have long been studied. Recently, the wide use of networks for the communication in control systems [1], [8] has provided new motivation and viewpoints for this line of studies. It is indeed important in designing such systems to find the amount of bandwidth required to meet given control specifications. A more recent approach is to consider efficient communication schemes suitable for the purpose of control so as to reduce the bandwidth usage in a systematic way. Much work has been done on stabilization problems where the minimum data rate is established (see, e.g., [4], [5], [7]–[10], [12], [14]).

The objective of this paper is to extend the approaches on quantization to an identification problem. Here, the general problem setup is that, given an unknown plant, we would like to identify its parameters at a remote location by sending measured data over a network. The measurements from the plant are quantized (and coded) to be transmitted over a channel and hence may be only a coarse approximation. It is received by the identifier where real-time estimation of the parameters as well as the states is made.

An example where such a setup arises is when micro sensors are used to monitor a remote system. Such sensors may have limited capacity for computation to carry out identification and also limited power for communication. Identification hence takes place at a base station where more resources in terms of computation and power are available.

The problem of quantized identification was first considered in [13]. There, an optimal design is attained in a least-squares setup to minimize the parameter estimation error. In [6], we considered a robust parameter identification problem in continuous time and, in particular, proposed the use of the so-called logarithmic quantizers under a feedback communication scheme. Such quantizers have been studied in quadratic stabilization contexts [4], [5], [8] and are known to be the coarsest in some sense.

In this paper, we extend our previous results in [6] to the discrete-time domain. This approach is based on the H^∞ identification schemes developed in [3], [11] and the problem is formulated as a worst-case noise attenuation one. Our goal is to design an identifier and a quantizer to estimate the unknown parameters in an l^2 setting. To use the channel efficiently, in particular, we employ feedback in communication so that the estimation generated in the identifier can be sent to the sensor side; having in mind the micro sensor example described above, we allow this communication to take place without data rate constraints. We will see that, with this structure, the choice of logarithmic quantizers is an effective one.

Furthermore, it is shown that there is a trade-off between the performance in identification and the required communication rate. This means that finer quantization results in better identification, but requires more bandwidth. Under our scheme, certain parameters in the identifier determine this trade-off and will remain as design parameters.

The motivation to study the discrete-time case is due to the limitation in the continuous-time formulation in [6]. There, the notion of sampling is introduced only implicitly, and sampling is assumed to occur whenever the quantized signal takes a different value. This may not be realistic from the bandwidth viewpoint.

This paper is organized as follows: In Section II, we formulate the problem of H^∞ robust identification over networks. In Section III, we extend the results in [3], [11] to the discrete-time case. This forms the basis for solving the problem with quantizers, which is presented in Section IV. We present a numerical example in Section V and then conclude the paper with some remarks.

II. Problem formulation

In this section, we introduce the problem of robust identification over a network.

Consider the system shown in Fig.1. The plant is SISO and its parameters are unknown. It is assumed to be linear time-invariant and to have a transfer function of the form

$$P(z) = \frac{b_m z^m + \cdots + b_1 z + b_0}{z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0}.$$

We assume that the orders m and n are given and satisfy $m < n$. The coefficients $a_{n-1}, \dots, a_0, b_m, \dots, b_0 \in \mathbb{R}$ are to be identified.

We study this plant by writing it in the following linearly parameterized state-space equation with disturbance and

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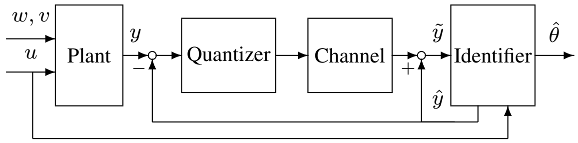

Fig. 1. Parameter identification over a network

measurement noise:

$$x_{k+1} = A_2 x_k + A_1(Cx_k, u_k)\theta + w_k, \qquad y_k = Cx_k + v_k. \tag{1}$$

Here, $\theta := [-a_{n-1} \cdots -a_0 \; b_m \cdots b_0]^T \in \mathbb{R}^r$ is the vector of unknown parameters with $r := n + m + 1$, $x_k \in \mathbb{R}^n$ is the state with the initial condition x_0, $u_k \in \mathbb{R}$ is the known bounded input, $w_k \in \mathbb{R}^n$ is the unknown disturbance. The measurement $y_k \in \mathbb{R}$ is corrupted by the noise v_k. The system matrices in (1) are given as follows:

$$\begin{aligned} A_1(z, u) &:= zA_{11} + uA_{12}, \\ A_2 &:= \begin{bmatrix} 0_{(n-1)\times 1} & I_{n-1} \\ 0 & 0_{1\times(n-1)} \end{bmatrix}, \\ C &:= \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}, \end{aligned} \tag{2}$$

where

$$\begin{aligned} A_{11} &:= \begin{bmatrix} I_n & 0_{n\times(m+1)} \end{bmatrix}, \\ A_{12} &:= \begin{bmatrix} 0_{(n-m-1)\times n} & 0_{(n-m-1)\times(m+1)} \\ 0_{(m+1)\times n} & I_{m+1} \end{bmatrix}. \end{aligned}$$

We assume that w and v are both in $l^2 \cap l^\infty$.

On the plant side, the measurement y_k is quantized and then transmitted over the channel to the identifier using a communication scheme which will be described shortly. The identifier receives the signal denoted by $\tilde{y}_k$ and the input u_k at each time k. It then computes the estimate $\hat{\theta}_k$ of the parameter θ as well as the estimate $\hat{x}_k$ of the state x_k. The identifier is formally given by

$$(\hat{\theta}_k, \hat{x}_k) = \delta(k, \tilde{y}_{[0,k]}, u_{[0,k]}),$$

where the shorthand notation $y_{[0,k]} := \{y_l\}_{l=0}^{k}$ is used and δ is assumed to be Lipschitz continuous in $\tilde{y}_{[0,k]}$ and $u_{[0,k]}$. Let the estimated output be $\hat{y}_k := C\hat{x}_k$.

The communication scheme makes use of a feedback link through which the estimated output $\hat{y}$ is sent to the plant side; then, the error $y - \hat{y}$ is quantized to be sent to the identifier. It is assumed that the channels are noiseless. Then, as the signal $\tilde{y}$ is received by the identifier, it is reasonable to use

$$\tilde{y}_k = \hat{y}_k + q(y_k - \hat{y}_k), \tag{3}$$

where $q : \mathbb{R} \to \mathbb{R}$ is the quantizer, which is a piecewise-constant function. Notice that $q(y_k - \hat{y}_k)$ is the signal sent over the channel while $\hat{y}_k$ is available locally at the identifier. Similar approaches in communication have been employed in a remote control problem [7] as well as in the quantized identification problem in continuous time [6].

This system setup may arise, as in the micro sensor example described in the Introduction, when the identifier is remotely located at a base station: The transmission of feedback information can be done over a channel with a wider bandwidth and more power. On the other hand, regarding the communication of the input u, there are several possibilities: (i) One may fix it prior to operation so that it is known to the identifier via some communication; this may work, e.g., when simple sinusoidal signals are used as in our example in Section V. (ii) The input may be determined at the base station and then transmitted to the plant via the wider channel.

Under the choice of $\tilde{y}$ in (3), we may employ the so-called logarithmic quantizers, which have been found useful in stabilization problems [4], [5], [8]. This quantization is fine around the origin, but becomes coarser further away from the origin. Since, in our scheme, it is the estimation error that is quantized, it makes sense to ask for more precise information when the signal is small. We will describe more later how this class of quantizers arises.

The logarithmic quantizer q is defined as follows: Take $\eta > 1$ and let $\rho := (\eta - 1)/(\eta + 1)$. Then, let

$$q(z) := \begin{cases} 0 & \text{if } z = 0, \\ \pm(1+\rho)\eta^j & \text{if } z \in \pm[\eta^j, \eta^{j+1}),\ j \in \mathbb{Z}. \end{cases} \tag{4}$$

We note that this is a sector-type discontinuous nonlinearity. In fact, it satisfies the following inequality:

$$|z - q(z)| \le \rho|z|. \tag{5}$$

The parameter ρ determines the size of the sector bound. It is hence regarded as the coarseness of the quantizer.

Finally, we make a slight modification in the plant model (1) and rewrite it as

$$x_{k+1} = A_2 x_k + A_1(y_k, u_k)\theta + w_k, \qquad y_k = Cx_k + v_k.$$

Here, the first argument Cx_k of A_1 in (1) is replaced by y_k. We do so with the understanding that w_k can be redefined to contain the effect of v_k. A result of this change is that now the time-varying matrix $A_1(y_k, u_k)$ is known.

We formulate the robust identification problem as a worst-case disturbance attenuation one in an l^2 setting. The performance index is given by

$$L(\delta, q) := \sup_{\substack{x_0,\theta \\ w_{[0,\infty]}, v_{[0,\infty]}}} \Sigma_{k=0}^{\infty} |\theta - \hat{\theta}_{k+1}|^2_{Q_{k+1}} \Big/ \left(\Sigma_{k=0}^{\infty}(|w_k|^2 + |v_k|^2) + |\theta - \bar{\theta}_0|^2_{Q_0} + |x_0 - \bar{x}_0|^2_{P_0}\right),$$

where $|\cdot|_R$ is the Euclidean norm weighted by the matrix R. The weights are assumed to satisfy $Q_{k+1} \ge 0$, $k \in \mathbb{Z}_+$, and $Q_0 > 0$. The vectors $\bar{x}_0 \in \mathbb{R}^n$ and $\bar{\theta}_0 \in \mathbb{R}^r$ represent initial guesses by the designer for x_0 and θ, respectively.

The quantized identification problem of this paper is as follows: Given a performance level $\gamma > 0$, design an identifier δ and a quantizer q, if they exist, so that the performance index satisfies $L(\delta, q) < \gamma$.

The difficulty in this identification problem lies in the information structure of the overall system. The quantizer has access only to the output y and not to the states in the identifier such as $\hat{\theta}$ and the estimated state $\hat{x}$. We will see that the feedback of $\hat{y}$ becomes helpful in reducing the information sent from the sensor to the remote identifier.

III. Discrete-time robust identification

In this section, we develop the robust identification scheme for the case without any quantization and solve the problem when $\tilde{y} \equiv y$. We extend the approach in [3], [11] for continuous-time systems to the discrete time. The results presented in this section form the basis for the next section, where the quantized identification problem is solved.

We follow a game theoretic approach and associate the original problem with a soft constrained differential game which has the following cost function:

$$\begin{aligned} &J_\gamma(\delta; x_0, \theta, w, v) \\ &\quad := \Sigma_{k=0}^{\infty} \left[|\theta - \hat{\theta}_{k+1}|^2_{Q_{k+1}} - \gamma^2 \left(|w_k|^2 + |v_k|^2 \right) \right] \\ &\qquad - \gamma^2 \left[|\theta - \bar{\theta}_0|^2_{Q_0} + |x_0 - \bar{x}_0|^2_{P_0} \right]. \end{aligned} \tag{6}$$

In this game, the identifier δ serves as the minimizer while the maximizer is the quadruple (x_0, θ, w, v).

The problem is then cast as an affine quadratic minimax controller design. For this purpose, we rewrite the system as follows. Let the extended state be $\xi := [\theta^T \ x^T]^T$. Since θ is a constant, the state equation becomes

$$\begin{aligned} \xi_{k+1} &= \bar{A}(y_k, u_k)\xi_k + \bar{D} w_k, \\ y_k &= \bar{C}\xi_k + v_k, \end{aligned} \tag{7}$$

where the new system matrices are given by

$$\begin{aligned} \bar{A}(y_k, u_k) &:= \begin{bmatrix} I_r & 0_{r\times n} \\ A_1(y_k, u_k) & A_2 \end{bmatrix}, \ \bar{D} := \begin{bmatrix} 0_{r\times n} \\ I_n \end{bmatrix}, \\ \bar{C} &:= [0_{1\times r} \ C]. \end{aligned} \tag{8}$$

For simplification, we sometimes use $\bar{A}_k$ for $\bar{A}(y_k, u_k)$.

For this system, the cost in (6) can be expressed as

$$\begin{aligned} &J_\gamma(\delta; \xi_0, w, v) \\ &\quad = \Sigma_{k=0}^{\infty} \left[|\xi_{k+1} - \hat{\xi}_{k+1}|^2_{\bar{Q}_{k+1}} - \gamma^2 \left(|w_k|^2 + |v_k|^2 \right) \right] \\ &\qquad - \gamma^2 |\xi_0 - \bar{\xi}_0|^2_{\bar{Q}_0}, \end{aligned}$$

where the weights are defined as $\bar{Q}_{k+1} := \mathrm{diag}(Q_{k+1}, 0)$ for $k \in \mathbb{Z}_+$ and $\bar{Q}_0 := \mathrm{diag}(Q_0, P_0)$, the estimated state is $\hat{\xi} := [\hat{\theta}^T \ \hat{x}^T]^T$, and its initial value is $\bar{\xi}_0 := [\bar{\theta}_0^T \ \bar{x}_0^T]^T$.

Our approach for the identifier design is based on the so-called cost-to-come function: At time $k = 0$, it is given by $W_\gamma(0, \xi) := -\gamma^2 |\xi - \bar{\xi}_0|^2_{\bar{Q}_0}$, and at time $k+1$, given the past and current measurement $y_{[0,k]}$, the input $u_{[0,k]}$, and the estimate $\hat{\xi}_{[0,k]}$, the cost-to-come function $W_\gamma(k, \xi)$ is defined by

$$\begin{aligned} &W_\gamma(k+1, \xi) \\ &:= \max_{\substack{\xi_0, w_{[0,k]}, v_{[0,k]} \\ y_l = \bar{C}\xi_l + v_l,\, l=0,\dots,k \\ \xi_{k+1} = \xi}} \Sigma_{l=0}^{k} \Big\{ |\xi_{l+1} - \hat{\xi}_{l+1}|^2_{\bar{Q}_{l+1}} \\ &\qquad - \gamma^2 [|w_l|^2 + |v_l|^2] \Big\} - \gamma^2 |\xi_0 - \bar{\xi}_0|^2_{\bar{Q}_0}, \end{aligned} \tag{9}$$

where the maximum is over the triple $(\xi_0, w_{[0,k]}, v_{[0,k]})$ such that the trajectory $\xi_{[0,k]}$ generated yields an output coinciding with $y_{[0,k]}$ and the state at time $k+1$ is equal to ξ; we say that such a triple is consistent with $y_{[0,k]}$ and yields $\xi_{k+1} = \xi$.

This function gives the worst-case cost at a given time based on the measurements so far. The use of cost-to-come function has been found useful in optimal control, especially in nonlinear H^∞ control and robust identification [2].

We have a few preliminary results on such functions.

Lemma 3.1: The cost-to-come function $W_\gamma(k+1, \xi)$ can be written in a recursive manner as follows:

$$\begin{aligned} W_\gamma(k+1, \xi) = \max_{\substack{\zeta, w, v \\ \xi = \bar{A}_k\zeta + \bar{D}w \\ y_k = \bar{C}\zeta + v}} \Big\{ &W_\gamma(k, \zeta) + |\xi - \hat{\xi}_{k+1}|_{\bar{Q}_{k+1}} \\ &- \gamma^2 [|w|^2 + |v|^2] \Big\} \end{aligned}$$

for $k \in \mathbb{Z}_+$. Moreover, $W(k, \xi)$ has an equivalent form:

$$W_\gamma(k, \xi) = -\gamma^2 |\xi - \lambda_k|^2_{\Sigma_k} + m_k,$$

where $\Sigma_k \in \mathbb{R}^{(r+n)\times(r+n)}$ and $\lambda_k \in \mathbb{R}^{r+n}$ are given by

$$Z_{k+1} = \bar{A}_k \left(\Sigma_k + \bar{C}^T\bar{C} \right)^{-1} \bar{A}_k^T + \bar{D}\bar{D}^T, \tag{10}$$

$$\Sigma_{k+1} = Z_{k+1}^{-1} - \gamma^{-2}\bar{Q}_{k+1}, \quad \Sigma_0 = \bar{Q}_0, \tag{11}$$

$$\begin{aligned} \bar{\xi}_{k+1} &= \bar{A}_k\bar{\xi}_k + \bar{A}_k(\Sigma_k + \bar{C}^T\bar{C})^{-1}\bar{C}^T(y_k - \bar{C}\lambda_k), \\ \lambda_{k+1} &= \left(I - \gamma^{-2} Z_{k+1}\bar{Q}_{k+1} \right)^{-1} \\ &\quad \cdot \left(\bar{\xi}_{k+1} - \gamma^{-2} Z_{k+1}\bar{Q}_{k+1}\hat{\xi}_{k+1} \right), \quad \lambda_0 = \bar{\xi}_0, \\ m_{k+1} &= m_k - \gamma^2/(1 + \bar{C}\Sigma_k^{-1}\bar{C}^T)|y_k - \bar{C}\lambda_k| \\ &\quad + |\bar{\xi}_{k+1} - \hat{\xi}_{k+1}|_{\bar{Q}_{k+1}(I - \gamma^{-2}Z_{k+1}\bar{Q}_{k+1})}, \quad m_0 = 0. \end{aligned}$$

□

Now, partition the matrix Σ_k as

$$\Sigma_k = \begin{bmatrix} \Sigma_{k,1} & \Sigma_{k,2} \\ \Sigma_{k,2}^T & \Sigma_{k,3} \end{bmatrix},$$

where $\Sigma_{k,1}$ is an $(r \times r)$-matrix. The following lemma will be useful in our development.

Lemma 3.2: The update laws for the submatrices in Σ_k are given as follows:

$$\begin{aligned} \Sigma_{k+1,1} &= \Sigma_{k,1} - \Sigma_{k,2} \left(\Sigma_{k,3} + C^TC \right)^{-1} \Sigma_{k,2}^T - \gamma^{-2} Q_{k+1} \\ &\quad - \Sigma_{k+1,2} \left[A_{1,k} - A_2 \left(\Sigma_{k,3} + C^TC \right)^{-1} \Sigma_{k,2}^T \right], \\ \Sigma_{k+1,2} &= - \left[A_{1,k}^T - \Sigma_{k,2} \left(\Sigma_{k,3} + C^TC \right)^{-1} A_2^T \right] \Sigma_{k+1,3}, \\ \Sigma_{k+1,3} &= I - A_2 \left(\Sigma_{k,3} + I \right)^{-1} A_2^T, \end{aligned}$$

where $\Sigma_{0,1} = Q_0$, $\Sigma_{0,2} = 0_{r\times n}$, and $\Sigma_{0,3} = P_0$. □

We note that from (11), the update laws for the submatrices of Z_{k+1}^{-1} can be obtained as well. Further, we observe in the lemma above that $\Sigma_{k,3}$ is independent of other elements of Σ_k. In particular, it has an equilibrium $\Sigma_3^* = \mathrm{diag}(1/n, 1/(n-1), \cdots, 1)$. These facts simplify the update laws of the matrices Z_k and Σ_k in (10) and (11), especially if we choose $\Sigma_{0,3} = \Sigma_3^*$.

We are now in a position to state the main result of this section.

Theorem 3.3: 1) The minimum performance level γ^* for the robust identification problem is given by

$$\gamma^* = \inf\{\gamma > 0 : \ \Sigma_k \geq 0 \text{ for all } k \text{ and all possible } y_{[0,\infty]}\}.$$

2) For each $\gamma > \gamma^*$, an identifier achieving this performance level is given by

$$\begin{aligned} &\hat{\xi}_{k+1} = \bar{A}_k \hat{\xi}_k + \bar{A}_k(\Sigma_k + \bar{C}^T\bar{C})^{-1}\bar{C}^T(y_k - \bar{C}\hat{\xi}_k), \\ &\begin{bmatrix} \hat{\theta}_k \\ \hat{x}_k \end{bmatrix} = \hat{\xi}_k, \end{aligned} \tag{12}$$

where the matrix Σ_k is obtained through (10) and (11), and the initial condition is $\hat{\xi}_0 = \bar{\xi}_0$. If, in addition, the persistency of excitation condition

$$\lim_{k\to\infty} \lambda_{\min}(\Sigma_k) = \infty \tag{13}$$

holds, then we have asymptotic perfect identification, that is, $\lim_{k\to\infty} \hat{\theta}_k = \theta$. □

The identifier given in the theorem is essentially obtained from (10) and (11), and we took $\hat{\xi}_k \equiv \bar{\xi}_k \equiv \lambda_k$. An identifier with a similar structure will be used in the quantized version of the problem in the next section.

In general, finding the minimum performance level γ^* is hard because of the difficulty in checking the condition $\Sigma_k \geq 0$ for all possible output y given in 1) of the theorem. We note that this is a natural consequence of the inverse design, which arises in adaptive robust control and regular (i.e., without quantization) worst-case identification; see also [3], [11]. However, there is one choice of weight functions Q_{k+1} for which we can obtain γ^* explicitly:

$$Q_{k+1} = (1 + (C\Sigma_{k,3}C^T)^{-1})^{-1}\Sigma_{k,2}\Sigma_{k,3}^{-1}C^T C\Sigma_{k,3}^{-1}\Sigma_{k,2}^T. \tag{14}$$

In this case, for $\gamma \geq 1$, $\Sigma_k \geq 0$ holds regardless of the response of the system, and hence $\gamma^* = 1$.

The results in this section show that an analogue of the approach in the continuous time can be developed at the general level presented in [11]. We note however that the formulae obtained here are overall more involved.

IV. Quantized Robust Identification

In this section, we extend the discrete-time robust identification scheme to the quantized input case. The main limitation introduced in this setup is that the identifier receives the quantized output $\tilde{y}$ defined in (3). The presence of the quantizer requires us to take an approach based on a variant of the cost-to-come function.

Let the quantization error be $e_k := y_k - \tilde{y}_k$. Following the notation in (7), we can write the extended plant with the state $\xi = [\theta^T \ x^T]^T$ as

$$\begin{aligned} \xi_{k+1} &= \left(\widetilde{A}_k + e_k F\right)\xi_k + \bar{D}w_k, \\ y_k &= \bar{C}\xi_k + v_k, \\ \tilde{y}_k &= \hat{y}_k + q(y_k - \hat{y}_k), \end{aligned}$$

where $\widetilde{A}_k := \bar{A}(\tilde{y}_k, u_k)$ and

$$F := \begin{bmatrix} 0_{r\times r} & 0_{r\times n} \\ -A_{11} & 0_{n\times n} \end{bmatrix}.$$

Note that the matrix $\widetilde{A}_k$ uses $\tilde{y}_k$ and not y_k. Thus, it is known to the identifier while $\bar{A}_k$ is not. Clearly, there is a relation $\widetilde{A}_k + e_k F = \bar{A}_k$ by (2) and (8).

For the quantized identification problem, it is difficult to find an explicit formula for the cost-to-come function as defined in (9). Instead, our approach is to find the so-called structured cost-to-come function, which bound the cost-to-come function from above. Such functions were introduced in [2] and are useful in practice because of their less stringent requirements.

For a given identifier δ and a quantizer q, we say that a function $\tilde{W}_\gamma(k+1, \xi; \tilde{y}_{[0,k]}, u_{[0,k]})$ is a structured cost-to-come function if it satisfies the following two conditions:
1) For any given $(\xi, k+1, \tilde{y}_{[0,k]}, u_{[0,k]})$ and any (ζ, w, v) that are consistent with $(\xi, k+1, \tilde{y}_{[0,k]}, u_{[0,k]})$, that is, $\xi = \bar{A}_k\zeta + \bar{D}w$ and $y_k = \bar{C}\xi_k + v_k$, it holds that

$$\begin{aligned} &-\tilde{W}_\gamma(k+1, \xi) + \tilde{W}_\gamma(k, \zeta) \\ &\qquad + |\xi - \hat{\xi}_{k+1}|_{\bar{Q}_{k+1}} - \gamma^2[|w|^2 + |v|^2] \leq 0, \end{aligned}$$

where $\hat{\xi}_{[0,k+1]}$ is the estimation made by δ.
2) For any $\xi, \bar{\xi}_0$, $\tilde{W}_\gamma(0, \xi) \geq -\gamma^2|\xi - \bar{\xi}_0|^2_{\bar{Q}_0}$.

Based on these functions, a sufficient condition for the quantized identification problem can be obtained as shown in the following lemma.

Lemma 4.1: Given an identifier δ and a quantizer q, suppose that there is a structured cost-to-come function $\tilde{W}_\gamma(k, \xi)$ for the overall system and moreover that this function is nonpositive. Then, the quantized identification problem is solvable for the performance level γ. □

For the quantized problem, we use the identifier of the following form:

$$\begin{aligned} \widetilde{Z}_{k+1} &= \widetilde{A}_k\left(\Sigma_k + \bar{C}^T\bar{C}\right)^{-1}\widetilde{A}_k^T + \bar{D}\bar{D}^T, \\ \Sigma_{k+1} &= \widetilde{Z}_{k+1}^{-1} - \gamma^{-2}\bar{Q}_{k+1} - \nu I, \quad \Sigma_0 = \bar{Q}_0, \\ \hat{\xi}_{k+1} &= \widetilde{A}_k\hat{\xi}_k + \widetilde{A}_k(\Sigma_k + \bar{C}^T\bar{C})^{-1}\bar{C}^T(\tilde{y}_k - \bar{C}\hat{\xi}_k), \end{aligned} \tag{15}$$

where $\hat{\xi}_0 = \bar{\xi}_0$ and $\nu > 0$. Its output is $[\hat{\theta}_k^T \ \hat{x}_k^T]^T = \hat{\xi}_k$.

The main differences between this identifier and the one in Theorem 3.3 (12) are that (i) $\bar{A}_k$ is replaced with $\widetilde{A}_k$ and that (ii) there is an additional negative term $-\nu I$ in (15). This term $-\nu I$ clearly determines the growth rate of Σ_k. As shown in the persistency of excitation condition in Theorem 3.3, this growth rate dominates the rate in identification. Hence, smaller ν implies that Σ_k becomes large faster, which in turn implies that faster identification is achieved.

In view of the remark following Lemma 3.2, we take the initial value $\Sigma_{0,3}$ to be a solution of the equation:

$$\Sigma_{0,3} = I - A_2(\Sigma_{0,3} + I)^{-1}A_2^T - \nu I. \tag{16}$$

In this case, $\Sigma_{k,3}$ remains constant. We assume that ν is small enough that $\Sigma_{0,3}$ is positive definite. It can be shown that $\Sigma_{0,3}$ is a diagonal matrix; for later use, we introduce the notation $\mathrm{diag}(\sigma_1, \dots, \sigma_n) := \Sigma_{0,3}$. Also, let $\Phi_k := [\Sigma_{k,1} - \Sigma_{k,2}(\Sigma_{0,3} + C^T C)^{-1}\Sigma_{k,2}^T]^{-1}$.

Finally, we follow the approach in the previous section, and use a structured cost-to-come function given by

$$\tilde{W}_\gamma(k, \xi) = -\gamma^2 |\xi - \lambda_k|^2_{\Sigma_k}.$$

We now present the design of the quantizer:

1) Take positive numbers $\bar{\Sigma}_2, \bar{\Phi}, \bar{a}, \bar{q}, \nu > 0$, and $\mu \in (0, \nu)$.

2) Let

$$\bar{e} \in \left(0, \frac{1}{c}\left(-\bar{\Sigma}_2 + \sqrt{\bar{\Sigma}_2^2 + c\mu}\right)\right],$$

where $c := |(1/\mu)(\Sigma_{0,3} + \nu I)^2 - (\Sigma_{0,3} + \nu I)|$.

3) Take $\rho > 0$ small enough that

$$\rho \le \sqrt{(\nu - \mu)\underline{p}} \left| \begin{bmatrix} \bar{h}_0 \\ 0 \end{bmatrix} + \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} \begin{bmatrix} \bar{h}_1 \\ \bar{h}_2 \end{bmatrix} \right|^{-1},$$

where

$$\underline{p} := \left(1 + \frac{1}{\sigma_1} + \frac{1}{\sigma_1^2}\bar{\Sigma}_2^2\bar{\Phi}\right)^{-1},$$

$$\bar{h}_0 := \frac{1}{\sigma_1 + 1}\bar{\Sigma}_2\left(1 + \frac{1}{\sigma_2}\bar{\Sigma}_2^2\bar{\Phi}\right), \qquad \bar{h}_1 := \frac{1}{\sigma_1 + 1}\bar{\Sigma}_2\bar{\Phi},$$

$$\bar{h}_2 := \frac{1}{\sigma_1 + 1}\bar{\Sigma}_2\bar{\Phi}\left(\frac{\bar{\Sigma}_2}{\sigma_1 + \nu} + \bar{q}\right) + \bar{a},$$

$$x_1 := \left(\frac{1}{\sigma_1 + \nu} + \frac{1}{\sigma_2}\right)\bar{\Sigma}_2^2 + \mu, \quad x_2 := \bar{\Sigma}_2, \quad x_3 := 1 - \mu.$$

4) Let the quantizer q be a logarithmic one (as in (4)) satisfying

$$|z - q(z)| \le \min\{\rho|z|, \ \bar{e}\} \quad \text{for all } z \in [-\bar{q}, \bar{q}].$$

The following is the main result of the paper.

Theorem 4.2: For a scalar $\gamma > 0$, construct the identifier as in (15) and the quantizer as above. Suppose that Σ_k satisfies the following conditions:

1) $|\Sigma_{k,2}| \le \bar{\Sigma}_2$,
2) $|\Phi_k| \le \bar{\Phi}$,
3) $\Sigma_k > 0$

for all $k \in \mathbb{Z}_+$ and for all possible pairs $(\tilde{y}_{[0,\infty]}, u_{[0,\infty]})$ of output and input of the plant. Moreover, assume that $|\hat{a}_k| \le \bar{a}$ and $|y_k - \bar{C}\hat{\xi}_k| \le \bar{q}$ for all k. Then, the identifier achieves the performance level γ. Further, if in addition the persistency of excitation condition in (13) holds, then $\lim_{k\to\infty} \hat{\theta}_k = \theta$. □

We have several remarks regarding this result. The logarithmic quantizer seems to be a natural choice in our approach. This is mainly due to the quadratic cost arising in the robust identification problem. On the other hand, there is a trade-off between the performance in identification and the coarseness in the quantizer. This is determined by the parameter ν. While smaller ν means faster identification as we saw earlier, for the parameter ρ representing the coarseness of the quantizer, smaller ν implies smaller ρ.

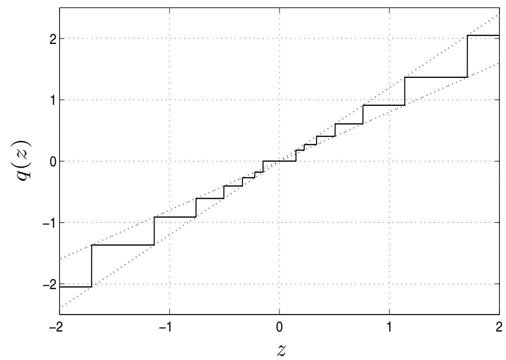

Fig. 2. Logarithmic quantizer with deadzone

Compared to the continuous-time counterpart in [6], the formulae obtained in this section are far more involved though the basic ideas used are very similar. We hence do not carry out some of the ideas in [6]. For example, it was shown there that under the simpler communication scheme where y is quantized directly (without feeding back $\hat{y}$) and $\tilde{y}_k = q(y_k)$ is used, we obtained uniform quantizers. This quantization is fine throughout the input space and hence requires high bandwidth in the communication. It is likely that similar results hold also in the discrete time.

V. Numerical example

In this section, we present a numerical study that illustrates the robustness and the use of logarithmic quantization in H^∞ identification. The objective is to show that the proposed scheme is robust against measurement noise introduced by very coarse logarithmic type quantizers; this is motivated by the observation in the last section. The performance of the scheme becomes clear by comparing it with a conventional least-squares one based on pre-filtering.

As the plant, we first chose the continuous-time, second-order system given by

$$P_c(s) = \frac{2}{s^2 + 3s + 4},$$

and then discretized it with zero-order hold using sampling period $T = 0.2$; hence the discrete-time plant is

$$P(z) = \frac{0.0327z + 0.0267}{z^2 - 1.43z + 0.549}.$$

The input we used is $u_k = 20\sin(0.4kT) + 10\sin(1.5kT) + 0.5\sin(4kT)$.

In the proposed identifier, the weight in the cost was chosen as the one given in (14) and for its initial value $Q_0 = 0.01I$ was used. As discussed in Section III, with this cost function, the optimal performance level is $\gamma^* = 1$ when there is no quantization. In our design, the value $\gamma = 1.2$ was employed. Also, as the initial value for $\Sigma_{k,3}$, we used the one that satisfies (16); in this case, $\Sigma_{k,3}$ is constant.

Next, we constructed a standard, prefiltering-based identifier for comparison. The prefilter is $1/z^2$ while the initial

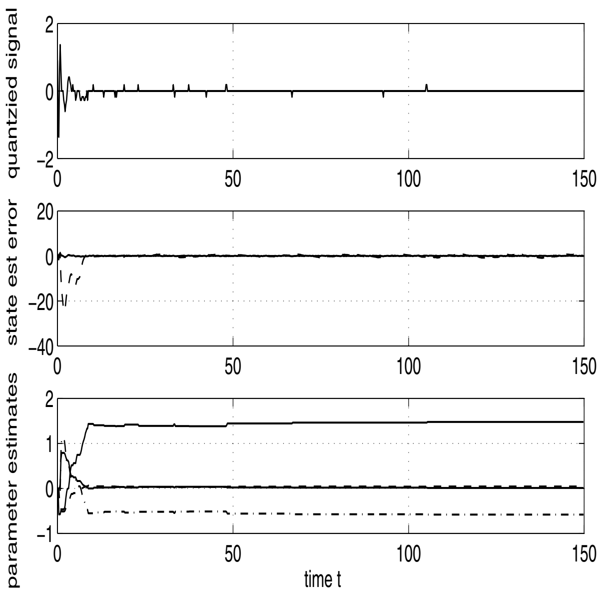

Fig. 3. Response of the proposed identifier

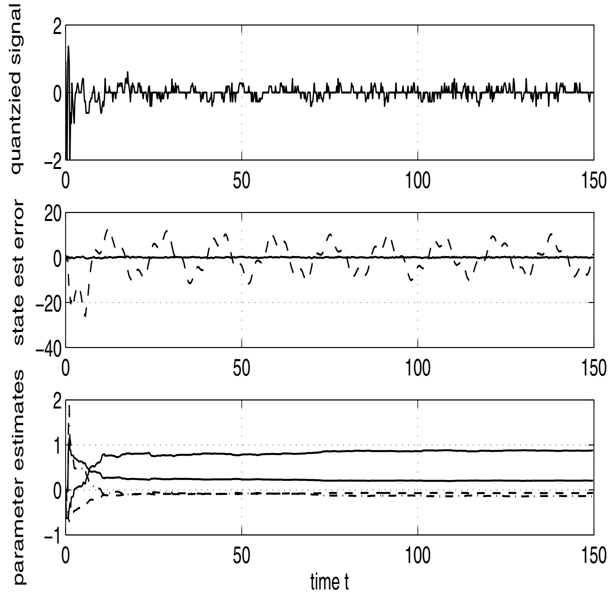

Fig. 4. Response of the prefilter-based identifier

condition for the covariance matrix is $20I$. These parameters were chosen so that its performance when no quantization is used in identification is similar to the one of our design.

For the quantizer, we chose a logarithmic one with some modification. Its graph is shown in Fig. 2. It has a deadzone around the origin in the interval $[-0.15, 0.15]$. Outside this region, it has a logarithmic characteristic; the dotted lines are the sector-type bound envelope given in (5), where the constant ρ is taken as 0.2. The initial condition for the plant was chosen to be $x_0 = [1\ -2]^T$ and those for the two identifiers were $\hat{x}_0 = [0\ 0]^T$. The disturbance w and the noise v were taken as Gaussian white noises with variances 0.04 and 0.02, respectively.

The responses of the proposed identifier and the conventional one are shown in Figs. 3 and 4, respectively. In both cases, the first plot shows the quantized signal $q(y_k - \hat{y}_k)$. the second one is the state estimation errors, and the third one is the parameter estimates. Overall, the response of the proposed scheme in Fig. 3 is good and is actually similar to that without quantization. In particular, we observe very good parameter identification in the third plot. Moreover, as shown in the first plot, after 20 seconds, the quantized signal takes zero most of the time; we may interpret that, after $t = 20$, little information is transmitted and hence little communication is needed. In contrast, the performance of the conventional scheme is degraded by quantization in Fig. 4. We note also that more communication is required in the sense described above.

VI. Conclusion

In this paper, we considered the design of a quantizer in a robust identification setup. By allowing feedback in the communication, we obtained a class of nonuniform quantizers, the so-called logarithmic quantizers. We observed that there is a trade-off between the performance in identification and the coarseness in the quantizer.

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