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Networked Control Systems with Intermittent Observation

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Abstract— **In this paper we consider a system whose controller is directly connected to the actuators and receives the observation data from the sensors through a network. Because of packet loss in the network, the controller does not always have access to the observation data. For this networked control system we prove that if the covariance of the error of the estimate of the state of the system is bounded and if the original (non-networked) control system is stabilizable, then the overall undisturbed feedback control system is mean square stablilizable.**

I. Introduction

Networked control systems are systems whose sensors, actuators, estimator units, and control units are connected through communication networks [1] [2]. This type of system has the advantage of greater flexibility with respect to traditional control systems. Also, it allows for reduced wiring, as well as a lower installation cost. It also permits greater agility in diagnosis and maintenance procedures. Examples of such systems can be seen in aircrafts or manufacturing plants.

In the present paper, we show that if a non-networked control system is stabalizable and if the covariance of error given by the Kalman filter with intermittent observation is bounded, then the undisturbed networked control system is mean square stabilizable.

II. Background

Consider the following discrete time, linear dynamical system

$$\begin{aligned} \mathbf{x}_{k+1} &= A\mathbf{x}_k + B\mathbf{u}_k + \mathbf{w}_k \\ \mathbf{y}_k &= C\mathbf{x}_k + \mathbf{v}_k \ , \end{aligned} \tag{1}$$

where $\mathbf{x}_k \in \mathcal{R}^n$ is the state of the system, $\mathbf{y}_k \in \mathcal{R}^m$ is the observation, $\mathbf{u}_k \in \mathcal{R}^d$ is the control, and $\mathbf{v}_k \in \mathcal{R}^m$ and $\mathbf{w}_k \in \mathcal{R}^n$ are independent discrete time white Gaussian processes. M, N, and S are the covariance matrices of $\mathbf{w}_k$, $\mathbf{v}_k$, and $\mathbf{x}_0$ (the initial condition), respectively, and we assume that $\mathbf{w}_k$ and $\mathbf{v}_k$ have zero means. A, B, and C are known matrices with proper dimensions. We consider the quadratic cost function

$$\frac{1}{N} E\left\{ \mathbf{x}_N^T Q \mathbf{x}_N + \sum_{k=0}^{N-1} (\mathbf{x}_k^T Q_k \mathbf{x}_k + \mathbf{u}_k^T R \mathbf{u}_k) \right\} \ , \tag{2}$$

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where R is a positive definite matrix and Q is a positive semidefinite matrix. For (A, B) controllable, $(A, Q^{1/2})$ observable where $Q = (Q^{1/2})^T Q^{1/2}$, (A, C) observable, and $(A, M^{1/2})$ controllable where $M = (M^{1/2})^T M^{1/2} > 0$ [1], it is well known that for $N \to \infty$ the optimal controller is given by [3]

$$\mathbf{u}_k^* = L\hat{\mathbf{x}}_k \ , \tag{3}$$

$$\begin{aligned} \hat{\mathbf{x}}_{k+1} = \ & (A + BL)\hat{\mathbf{x}}_k + \\ & \overline{\Sigma} C^T N^{-1} (\mathbf{y}_{k+1} - C(A + BL)\hat{\mathbf{x}}_k), \end{aligned} \tag{4}$$

$$L = -(R + B^T K B)^{-1} B^T K A \ , \tag{5}$$

$$\overline{\Sigma} = \Sigma - \Sigma C^T (C \Sigma C^T + N)^{-1} C \Sigma, \tag{6}$$

where K is the solution of the Algebraic Riccati Equation (ARE)

$$K = A^T (K - KB(R + B^T K B)^{-1} B^T K) A + Q, \tag{7}$$

and Σ is the solution of the ARE

$$\Sigma = A(\Sigma - \Sigma C^T (C \Sigma C^T + N)^{-1} C \Sigma) A^T + M. \tag{8}$$

It is also well known that the $2n$ dimensional system given in (1), (3), and (4) is stable.

Now consider system (1) whose feedback loop is closed through a communication network. Here we assume that a single controller uses the observation data $\mathbf{y}_k$, which it receives from the plant through the network, to generate the control command $\mathbf{u}_k$. This control command is sent directly to the plant and the actuator.

The observation data from the sensors are sent to the controller/estimator in the form of data packets. Due to the network congestion, these packets may get dropped or delayed. The sensor data are delay sensitive and old measurements are often discarded. Therefore, the estimator (Kalman Filter) in (4) does not always receives the observation $\mathbf{y}_k$. We define the arrival of the observation packet at time k as an i.i.d. random variable γ_k with probability distribution $p_{\gamma_k}(1) = \lambda$ for successfully received packets and $p_{\gamma_k}(0) = 1 - \lambda$ for lost packets. Conditioned on $\gamma_0^k = \{\gamma_0, \gamma_1, \cdots, \gamma_k\}$, it is known that $\mathbf{x}_k$ and $\mathbf{y}_k$ are Gaussian [4]. In a recent paper Sinopoli et.al. [5] showed that the the estimator has the following form

$$\begin{aligned} \hat{\mathbf{x}}_{k+1} = \ & (A + BL)\hat{\mathbf{x}}_k + \\ & \gamma_{k+1} \Gamma_{k+1} (\mathbf{y}_{k+1} - C(A + BL)\hat{\mathbf{x}}_k), \end{aligned} \tag{9}$$

where Γ_k is given as follows

$$\Gamma_{k+1} \quad = \quad \Sigma_{k+1} C^T (C \Sigma_{k+1} C^t + N)^{-1} \tag{10}$$

[1] M positive definite is not necessary but we use it in the next section to provide a less tedious proof for the main result.

$$\Sigma_{k+1} = A\Sigma_k A^T + M - \gamma_k A\Sigma_k C^T(C\Sigma_k C^T + N)^{-1} C\Sigma_k A^T, \quad (11)$$

where Σ_k is the covariance of the error of the estimate at time k conditioned on γ_0^k. Note that since $\{\gamma_k\}$ is a random sequence, the difference equation (11) generates a sequence of random matrices. In [5] it was shown that if (A, C) is observable and $(A, M^{1/2})$ is controllable, and if there exist matrices X and Y such that[2]

$$X > (1-\bar{\gamma})\left(AXA^T + M\right) + \bar{\gamma}\left((A+YC)X(A+YC)^T + M + YNY^T\right) \quad (12)$$

then P_k, which is generated by the the Riccati Like Equation

$$P_{k+1} = APA^T + M - \bar{\gamma}A P_k C^T(C\Sigma_k C^T + N)^{-1} C P_k A^T, \quad (13)$$

converges to a positive definite matrix $\bar{P}$ for any initial condition P_0 and we have $E\Sigma_k \leq P_k, \quad \forall k$.

III. Main Result

Using equations (1), (3), and (9) the $2n$ dimensional feedback control system can be expressed as follows

$$\begin{pmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{pmatrix}_{k+1} = F_K \begin{pmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{pmatrix}_k + H_k \begin{pmatrix} \mathbf{w_k} \\ \mathbf{v}_k \end{pmatrix} \quad (14)$$

where F_k is

$$F_k = \begin{pmatrix} A & BL \\ \gamma_{k+1}\Gamma_{k+1} CA & A + BL - \gamma_{k+1}\Gamma_{k+1} CA \end{pmatrix}, \quad (15)$$

and

$$H_k = \begin{pmatrix} I & 0 \\ \gamma_{k+1}\Gamma_{k+1} C & \gamma_{k+1}\Gamma_{k+1} \end{pmatrix}. \quad (16)$$

Our main claim in this paper is presented in the following theorem

Theorem 1: For system (1) if (A, B) is controllable and $(A, Q^{1/2})$ is observable, and if $E\Sigma_k$ is bounded for all k, then the undisturbed $2n$ dimensional system in (14) is mean square stable.

Proof: The system in in (14) is undisturbed if $\mathbf{w_k}$ and $\mathbf{v}_k$ are set to zero. The undisturbed system is mean square stable if

$$\lim_{k\to\infty} E[\|\mathbf{x}'_k\|^2 + \|\hat{\mathbf{x}}'_k\|^2] = 0, \quad (17)$$

where

$$\begin{pmatrix} \mathbf{x}' \\ \hat{\mathbf{x}}' \end{pmatrix}_{k+1} = F_K \begin{pmatrix} \mathbf{x}' \\ \hat{\mathbf{x}}' \end{pmatrix}_k \quad (18)$$

First we consider the dynamics of $\mathbf{e}_k = \mathbf{x}_k - \hat{\mathbf{x}}_k$

$$\begin{aligned} \mathbf{e}_{k+1} &= (I - \gamma_{k+1}\Gamma_{k+1}C)A\mathbf{e}_k + (I - \gamma_{k+1}\Gamma_{k+1}C)\mathbf{w}_k - \gamma_{k+1}\Gamma_{k+1}\mathbf{v}_k \\ &= G_k\mathbf{e}_k + J_k\mathbf{w}_k - \gamma_{k+1}\Gamma_{k+1}\mathbf{v}_k. \end{aligned} \quad (19)$$

Since $\mathbf{w}_k$ and $\mathbf{v}_k$ are independent white Gaussian random sequences we have

$$\begin{aligned} E[\mathbf{e}_{k+1}^T\mathbf{e}_{k+1}] &= E[\mathbf{e}_k^T G_k \mathbf{e}_k] + E[\mathbf{w}_k^T J_k^T J_k \mathbf{w}_k] + E[\gamma_{k+1}^2 \mathbf{v}_k^T \Gamma_{k+1}^T \Gamma_{k+1} \mathbf{v}_k] \\ &\geq E[\mathbf{e}_k^T G_k \mathbf{e}_k] + E[\mathbf{w}_k^T J_k^T J_k \mathbf{w}_k] \\ &\geq E[\mathbf{e}_0^T (\prod_{i=0}^{k}(G_i))^T (\prod_{i=0}^{k}(G_i))\mathbf{e}_0] + \\ &\quad \sum_{j=0}^{k-1} E[\mathbf{w}_j^T J_j^T (\prod_{i=j}^{k-1}(G_i))^T (\prod_{i=j}^{k-1}(G_i)) J_j \mathbf{w}_j] \end{aligned} \quad (20)$$

We know $E[\Sigma_k]$ is bounded for all k, i.e. $E[\mathbf{e}_{k+1}^T\mathbf{e}_{k+1}]$ is bounded for all k. Therefore

$$\lim_{k\to\infty} E[\mathbf{w}_j^T J_j^T (\prod_{i=j}^{k-1}(G_i))^T (\prod_{i=j}^{k-1}(G_i)) J_j \mathbf{w}_j] = 0, \quad \forall j \quad (21)$$

Since $\mathbf{w}_j$ is a Gaussian random variable with a full rank covariance matrix then, $\forall \mathbf{x} \in \Re^n$ and $\forall j$

$$\lim_{k\to\infty} E[\mathbf{x}^T J_j^T (\prod_{i=j}^{k-1}(G_i))^T (\prod_{i=j}^{k-1}(G_i)) J_j \mathbf{x}] = 0.$$

From (19) we know that $G_k = J_j A$ therefore, $\forall \mathbf{x} \in \Re^n$ and $\forall j$

$$\lim_{k\to\infty} E[\mathbf{x}^T A^T J_j^T (\prod_{i=j}^{k-1}(G_i))^T (\prod_{i=j}^{k-1}(G_i)) J_j A \mathbf{x}] = \lim_{k\to\infty} E[\mathbf{x}^T (\prod_{i=j}^{k}(G_i))^T (\prod_{i=j}^{k}(G_i)) \mathbf{x}] = 0.$$

Therefore, we conclude that the dynamical system $\mathbf{e}'_{k+1} = G_k \mathbf{e}'_k$ is mean square stable. Now we should prove that the $2n$ dimensional system is mean square stable. We have

$$\begin{aligned} \mathbf{x}'_{k+1} &= A\mathbf{x}'_k + BL\hat{\mathbf{x}}'_k \\ &= (A+BL)\mathbf{x}'_k - BL(\mathbf{x}'_k - \hat{\mathbf{x}}'_k) \\ &= (A+BL)\mathbf{x}'_k - BL\mathbf{e}'_k. \end{aligned} \quad (22)$$

From the previous section we know that $(A + BL)$ is a stable matrix with all eigenvalues inside the unit circle and $\lim_{k\to\infty} E[e'^T_k e'_k] = 0$, therefore, $\lim_{k\to\infty} E[\mathbf{x}'^T_k \mathbf{x}'_k] = 0$ [3]. Also, we have that $E[\hat{\mathbf{x}}'^T_k \hat{\mathbf{x}}'_k] = E[(\mathbf{x}'_k - \mathbf{e}'_k)^T(\mathbf{x}'_k - \mathbf{e}'_k)] \leq 2E[\mathbf{x}'^T_k \mathbf{x}'_k] + 2E[e'^T_k e'_k]$. Therefore, we conclude that $\lim_{k\to\infty} E[\hat{\mathbf{x}}'^T_k \hat{\mathbf{x}}'_k] = 0$. **QED**

References

[1] G.C. Walsh and Y. Hong and L.G. Bushnell. Stability Analysis of Networked Control Systems. *IEEE Transactions on Control Systems Technology*, 10(3):438–46, May 2002.

[2] W. Zhang and M.S. Branicky and S.M. Phillips. Stability of Networked Control Systems. *IEEE Control System Magazine*, 21(1):84–99, February 2001.

[3] D.P. Bertsekas. *Dynamic Programing and Optimal Control*, volume 1. Athena Scientific, Belmont, MA, USA, 1995.

[4] H.F. Chen, P.R. Kumar, and J.H. van Schuppen. On Kalman Filtering for Conditionally Gaussian Systems with Random Matrices. *System and Control Letters*, 13(5):397–404, December 1989.

[5] B. Sinopoli and L. Schenato and M. Franceschetti and K. Poolla and M. Jordan and S. Sastry. "kalman filtering with intermittent observations". *Proceedings of IEEE Conference in Decison Control*, 2003.

[2] For matrices X_1 and X_2 we say $X_1 > X_2$ iff $X_1 - X_2$ is positive definite. $X_1 \geq X_2$ means that $X_1 - X_2$ is positive semi-definite.

[3] Note that a linear time invariant system with all poles strictly in inside the unit circle is exponentially stable.