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Randomized and Deterministic Algorithms for Stabilization with Fixed Order Controllers

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***Abstract*— In this paper, we study fixed order stabilizing controllers for single-input single-output plants. Following previous research, the controller parameters are classified into two types: (computationally) tractable and intractable parameters. First, we propose to use randomized algorithms to find the intractable parameters. Then, we present a deterministic method to compute the values of tractable parameters. This technique is based on matrix inversion and it is shown to be superior (from the computational complexity point of view) to existing methods based on linear programming.**

I. INTRODUCTION

In recent years, research within systems and control focused on difficult problems such as fixed order output feedback design. This problem is known to have computational complexity difficulties similar to static output feedback design, see [1]. These difficulties come from NP-hardness, see [2], and are essentially unavoidable with an approach fully deterministic.

Even though PID design is a well-established topic, see [3], various innovative methods for designing low order controllers have been proposed in the last few years. These methods generally make use of a geometric characterization of the set of stabilizing PID gains. Subsequently, the controller is determined either with the aid of graphical methods, see e.g., [4], [5], [6] or by means of linear programming, see e.g. [7].

In this paper, we study a general method to characterize a fixed order stabilizing controller of a single-input single-output plant. This approach follows previous research, see [8], and it is based on the idea of splitting the controller parameters into two categories, the so-called (deterministically) *tractable* and *intractable* parameters. A precise definition of these parameters is given in Section II.

The first part of the paper, see Section III, deals with an approach based on randomized algorithms to determine the intractable parameters. Two specific randomized algorithms, based on the Chernoff Bound and related results, are presented. Subsequently, in Sections IV and V, we present a method which makes use of matrix inversions to compute a so-called "marginal stabilizer." A marginal stabilizer has the property of placing some of the roots of the closed-loop polynomial into the open left half plane and the remaining roots on the imaginary axis. A detailed analysis is carried on showing that this method enjoys polynomial-time complexity. Analytic and numerical comparisons with other methods based on linear programming are discussed demonstrating the superiority of this approach. Once a marginal stabilizer is determined as described above, we then proceed to compute a fixed order stabilizing controller. This latter step is simply performed by means of a combination of bisection and sensitivity methods and therefore it can be efficiently executed.

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II. PRELIMINARIES AND NOTATION

We now introduce the notation used in this paper. Consider a single-input single-output strictly proper plant of the form

$$P(s) = \frac{N_P(s)}{D_P(s)} \tag{1}$$

where $N_P(s)$ and $D_P(s)$ are numerator and denominator plant polynomials of order n_N and n_D, respectively. We study a fixed order controller of the form

$$C(s) = \frac{N_C(s)}{D_C(s)}$$

where $N_C(s)$ and $D_C(s)$ are numerator and denominator controller polynomials, respectively. Without loss of generality, we rewrite $C(s)$ as

$$C(s) = \frac{X(s^2) + sY(s^2)}{Z(s^2) + sW(s^2)} \tag{2}$$

where $X(s^2)$, $Y(s^2)$, $Z(s^2)$ and $W(s^2)$ are polynomials containing only even powers of s. These polynomials are of the form

$$\begin{aligned} X(s^2) &= \theta_0 + \theta_2 s^2 + \theta_4 s^4 + \cdots + \theta_{n_X} s^{n_X}; \\ Y(s^2) &= \eta_0 + \eta_2 s^2 + \eta_4 s^4 + \cdots + \eta_{n_Y} s^{n_Y}; \\ Z(s^2) &= \alpha_0 + \alpha_2 s^2 + \alpha_4 s^4 + \cdots + \alpha_{n_Z} s^{n_Z}; \\ W(s^2) &= \beta_0 + \beta_2 s^2 + \beta_4 s^4 + \cdots + \beta_{n_W} s^{n_W} \end{aligned} \tag{3}$$

and their orders in s are denoted by n_X, n_Y, n_Z and n_W, respectively.

We now formally define deterministically tractable and intractable parameters, see Section III for further discussions. Four different cases are considered in the definition, so that some flexibility is allowed in the design of fixed order stabilizing controllers. For example, the coefficients of $X(s^2)$ may be chosen as tractable parameters. Once this choice is made, the intractable parameters are set, so that they are the coefficients of $Y(s^2)$, $Z(s^2)$, $W(s^2)$. It will be shown later in Section V that a tractable parameter value can be found in a computationally efficient way once the values of intractable parameters are computed as described in Section III.

Definition 1: [Tractable and intractable parameters] Consider a controller $C(s)$ expressed in the form (2) with parameters given by (3). Then, one of the following conditions hold:

1) The tractable parameters are the coefficients of $X(s^2)$ and the intractable parameters are the coefficients of $Y(s^2)$, $Z(s^2)$, $W(s^2)$;
2) The tractable parameters are the coefficients of $Y(s^2)$ and the intractable parameters are the coefficients of $X(s^2)$, $Z(s^2)$, $W(s^2)$;
3) The tractable parameters are the coefficients of $Z(s^2)$ and the intractable parameters are the coefficients of $X(s^2)$, $Y(s^2)$, $W(s^2)$;
4) The tractable parameters are the coefficients of $W(s^2)$ and the intractable parameters are the coefficients of $X(s^2)$, $Y(s^2)$, $Z(s^2)$.

Let n_θ and n_μ be the number of tractable and intractable parameters, respectively. Let θ and μ be the vectors containing the values of the tractable parameters and those of the intractable parameters, respectively. For simplicity, in the rest of this paper, we consider only the first case in Definition 1. That is, the tractable parameters are the coefficients of $X(s^2)$ and the intractable parameters are the coefficients of $Y(s^2)$, $Z(s^2)$ and $W(s^2)$. This implies that

$$\begin{aligned} n_\theta &= \frac{n_X}{2} + 1; \\ n_\mu &= \frac{n_Y + n_Z + n_W}{2} + 3; \\ \theta &= \left[\begin{array}{ccc} \theta_0 & \cdots & \theta_{n_X} \end{array}\right]^T; \\ \mu &= \left[\begin{array}{ccccccccc} \eta_0 & \cdots & \eta_{n_Y} & \alpha_0 & \cdots & \alpha_{n_Z} & \beta_0 & \cdots & \beta_{n_W} \end{array}\right]^T. \end{aligned}$$

The other three cases can be treated in a similar manner.

The closed-loop polynomial $p(s)$ is given by the Diophantine equation

$$\begin{aligned} p(s) &= N_P(s)N_C(s) + D_P(s)D_C(s) \\ &= N_P(s)(X(s^2) + sY(s^2)) + D_P(s)(Z(s^2) + sW(s^2)) \end{aligned}$$

where the order of $p(s)$ is assumed to be fixed. That is, we consider the generic subset of parameters of the controller coefficients which does not change the order of the closed-loop polynomial. The main objective of this paper is stabilization. This means finding controller parameters so that the closed-loop polynomial $p(s)$ has all its roots in the open left half plane; i.e., it is stable. If this controller is determined, we call $C(s)$ a *stabilizing controller*.

III. Randomized Algorithms for Computing Intractable Controller Parameters

In Section IV, we will show that, if the values of intractable parameters are fixed, the set of tractable parameter values corresponding to stabilizing controllers enjoys some convexity property which can be exploited for efficient computation of the tractable parameter values. For the intractable parameter values, however, no convexity is known and their efficient computation is difficult deterministically. In order to overcome this difficulty, we consider in this section the use of randomized algorithms, which are known to be effective for many deterministically difficult problems within systems and control, see, e.g., [9].

Let us say that an intractable parameter value μ is "admissible" when the set of stabilizing controllers is not empty for that value. Here, we present a randomized algorithm to find such a value μ. For this purpose, we introduce some probability distribution $\mathcal{P}$ into the set $\mathcal{M} \subseteq \mathbf{R}^{n_\mu}$, where we assume that all possible values μ belong to $\mathcal{M}$. Let ϵ and δ be any positive numbers less than unity and define

$$N_1 = \left\lceil \frac{\ln(1/\delta)}{\ln(1/(1-\epsilon))} \right\rceil$$

where $\ln$ denotes the natural logarithm. Now we propose the following algorithm.

Algorithm 1:

1. For $i := 1, \ldots, N_1$ do
   begin
2. Draw a sample $\mu^{(i)} \in \mathcal{M}$ according to $\mathcal{P}$;
3. If $\mu^{(i)}$ is admissible then return;
   end

The performance of this algorithm is guaranteed by the following theorem, which is an immediate consequence of the results in [10], [11]. Here, we let $\mathcal{A}$ denote the set of all admissible μ in $\mathcal{M}$ and $\mathcal{P}(\mathcal{A})$ its measure according to $\mathcal{P}$.

Theorem 1: Suppose that the measure $\mathcal{P}(\mathcal{A})$ is greater than ϵ. Then, the probability that no $\mu^{(i)}$, $i = 1, \ldots, N_1$, is admissible in Algorithm 1 is less than δ.

This theorem means that Algorithm 1 gives an admissible $\mu^{(i)}$ with confidence higher than $1 - \delta$. Note that the maximum number of samples N_1 depends only on ϵ and δ. As we will see in Section IV, Line 3 of the algorithm can be carried out in polynomial time. Although the complexity to execute Line 2 depends on $\mathcal{M}$ and $\mathcal{P}$, it is usually polynomial in the dimension n_μ. Moreover, this complexity is especially low in some cases: for example, in the case that $\mathcal{M}$ is box-shaped and $\mathcal{P}$ is a uniform distribution or that $\mathcal{M} = \mathbf{R}^m$ and $\mathcal{P}$ is Gaussian distribution. Such choices are used in many practical applications, see, e.g., [9] for further discussions.

Next, we present a randomized algorithm to evaluate the measure $\mathcal{P}(\mathcal{A})$. We choose positive numbers ϵ and δ to be smaller than unity and define

$$N_2 = \left\lceil \frac{1}{2\epsilon^2} \ln \frac{2}{\delta} \right\rceil.$$

Algorithm 2:

1. Set $N_s := 0$;
2. For $i := 1, \ldots, N_2$ do

begin

3. Draw a sample $\mu^{(i)} \in \mathcal{M}$ according to $\mathcal{P}$;
4. If $\mu^{(i)}$ is admissible then set $N_s := N_s + 1$;

end

Here, N_s counts the number of admissible $\mu^{(i)}$ among the N_2 samples. We have the following theorem based on the well-known Chernoff bound [9], [12].

Theorem 2: The probability that $|N_s/N_2 - \mathcal{P}(\mathcal{A})| > \epsilon$ holds is less than δ.

IV. Affine Characterization of Stability Boundaries for Tractable Parameters

We are interested in finding a fixed order stabilizing controller of the form given in (2) for a plant $P(s)$ defined in (1). More precisely, the objective is to determine controller parameters of the form (3) such that the closed-loop system corresponding to the feedback interconnection consisting of $P(s)$ and $C(s)$ is stable and the closed-loop polynomial has all its roots in the open left half plane. Following the previous developments, we work under the tractability assumption so that the intractable parameters are considered as fixed. In fact, these parameters have been chosen using the randomized algorithms developed in the previous section.

We now state without proof a preliminary result which gives a characterization of all fixed order stabilizing controllers satisfying Definition 1 of tractable parameters.

Lemma 1: Suppose that an intractable parameter vector μ is selected according to Algorithm 1. Then, the set of all tractable parameter vectors θ that gives a stabilizing controller $C(s, \theta)$ is either empty or is a union of a finite number of polyhedral sets.

This result studies stabilization properties in controller coefficients space for tractable parameters. Since we consider the case that the coefficients of $Y(s^2)$, $Z(s^2)$ and $W(s^2)$ are fixed, the result says that the set of all tractable stabilizing controllers parameters of the form

$$C(s, \theta) = \frac{\theta_0 + \theta_2 s^2 + \cdots + \theta_{n_X} s^{n_X} + sY(s^2)}{Z(s^2) + sW(s^2)}$$

is a finite union of polyhedral sets, provided that a stabilizing controller exists. In other words, the "tractable parameters" of $X(s^2)$ enjoy a polyhedral property which may be exploited in the development of computational methods.

Remark 1: Lemma 1 is an extension of earlier results available in the literature for the special case of PID controllers, see references [3] and [7] where the set of all stabilizing PID controllers with fixed proportional gain is shown to be a finite union of convex polygons. In these references, a linear programming approach for the design of PID is also proposed, even though linear programming is not strictly necessary since only two design parameters are involved and therefore graphical methods can be used. Subsequent papers along the same direction, and also addressing PID design or lead-lag compensators, are e.g. [4], [5] and [6]. In particular, in [4] a characterization of closed-loop polynomials with a certain even-odd structure is studied by means of the parameter space approach, see e.g. [13]. However, the characterization obtained in [4] is used only for analyzing PID stabilizing controllers and no attempt is made to handle more general classes of controllers. Finally, we recall that a linear programming approach based on a generalization of the Hermite-Biehler theorem is developed in [14] for synthesis with fixed structure controllers.

We now state, without proof, an extension of Lemma 1 from stability to the case when a fixed number of roots in the closed left half plane are considered. This result may have some system and control interpretation in terms of invariant roots within a specified region of the complex plane of a polynomial affected by parametric perturbations, see e.g. [15]. However, the interest here is purely technical since Theorem 4 in Section V is based on this result. A special case of this result is when a fixed number of roots lie on the imaginary axis and all the remaining roots lie in the open left half plane.

Corollary 1: Suppose that an intractable parameter vector μ is selected according to Algorithm 1. Then, the set of all tractable parameter vectors θ such that the corresponding closed-loop polynomial $p(s, \theta)$ has a fixed number of roots in the closed left half plane is either empty or is a union of a finite number of polyhedral sets.

In order to derive efficient deterministic algorithms to handle the tractable parameters, the first step is to study the so-called critical (or singular) frequencies. Consider the closed-loop polynomial

$$p(s) = p_0(s) + p_1(s)X(s^2)$$

where

$$p_0(s) \doteq sN(s)Y(s^2) + D(s)(Z(s^2) + sW(s^2));$$
$$p_1(s) \doteq N(s).$$

For $s = j\omega$, we write

$$p_0(j\omega) = R_0(\omega^2) + j\omega I_0(\omega^2);$$
$$p_1(j\omega) = R_1(\omega^2) + j\omega I_1(\omega^2).$$

The set of critical frequencies is given by

$$\Omega = \{\omega_1, \omega_2, \ldots, \omega_{n_f}\}$$

where the set Ω consists of the solutions of the polynomial equation

$$I_0(\omega_i^2)R_1(\omega_i^2) - I_1(\omega_i^2)R_0(\omega_i^2) = 0$$

for $\omega_i \in (0, \infty)$, and may contain 0 and/or ∞ in some cases. To evaluate the order of this polynomial, we have the following lemma, which follows from direct computations.

Lemma 2: The number of critical frequencies n_f is bounded as

$$n_f \leq \frac{\deg p_0(s) \deg p_1(s) - \min \{\deg p_0(s), \deg p_1(s)\}}{2} + 2$$

where $\deg p_0(s)$ and $\deg p_1(s)$ are the degree of the polynomials $p_0(s)$ and $p_1(s)$ which are given by

$$\begin{aligned} &\deg p_0(s) \\ &= \max \{n_N + n_Y + 1, n_D + \max\{n_Z, n_W + 1\}\}; \\ &\deg p_1(s) = n_N. \end{aligned}$$

Indeed, the number of critical frequencies is a polynomial function of n_N, n_D, n_Y, n_Z and n_W. We also notice that n_f does not depend on n_X, i.e. n_θ. For each critical frequency $\omega_i \in \Omega$, we can determine a hyperplane which defines part of the stability boundary. This hyperplane has the form

$$\psi_0(\omega_i)\theta_0 + \psi_2(\omega_i)\theta_2 + \cdots + \psi_{n_X}(\omega_i)\theta_{n_X} = \nu(\omega_i) \quad (4)$$

where $\psi_0(\omega_i)$, $\psi_2(\omega_i)$, $\psi_4(\omega_i)$, ..., $\psi_{n_X}(\omega_i)$ and $\nu(\omega_i)$ are fixed coefficients. The entire stability boundary is given by the union of the hyperplanes obtained for all critical frequencies. Hence, specific stabilizing controllers and the set of all stabilizing controllers can be obtained, in principle, by solving a number of linear programs. However, since (4) is a linear equation, in order to study stabilizing regions, all possible combinations of the inequalities

$$\psi_0(\omega_i)\theta_0 + \psi_2(\omega_i)\theta_2 + \cdots + \psi_{n_X}(\omega_i)\theta_{n_X} \geq \nu(\omega_i)$$

and

$$\psi_0(\omega_i)\theta_0 + \psi_2(\omega_i)\theta_2 + \cdots + \psi_{n_X}(\omega_i)\theta_{n_X} \leq \nu(\omega_i)$$

for $\omega_i \in \Omega$, should be considered. In turn, this leads to an exponential number of linear programs.

To see this more precisely, let us introduce matrices

$$\Psi \doteq \begin{bmatrix} \psi_0(\omega_1) & \cdots & \psi_{n_X}(\omega_1) \\ \vdots & \ddots & \vdots \\ \psi_0(\omega_{n_f}) & \cdots & \psi_{n_X}(\omega_{n_f}) \end{bmatrix}, \ \nu \doteq \begin{bmatrix} \nu(\omega_1) \\ \vdots \\ \nu(\omega_{n_f}) \end{bmatrix}.$$

Note that $\Psi \in \mathbf{R}^{n_f \times n_\theta}$ and $\nu \in \mathbf{R}^{n_f}$. We also consider a diagonal matrix $S_i \in \mathbf{R}^{n_f \times n_f}$ each of which diagonal element is either -1 or 1. We see that the total number of such S_i is 2^{n_f}. Thus, all possible tractable parameter regions generated by the critical frequencies can be characterized as

$$\Psi\theta \geq S_i \nu, \qquad i = 1, 2, \ldots, 2^{n_f}.$$

We therefore conclude that the total number $N_{LP}(n_f)$ of required linear programs in the worst-case is given by

$$N_{LP}(n_f) = 2^{n_f}.$$

However, if the objective is to obtain a specific stabilizing controller (and not the entire set), a more efficient and direct procedure may be used, thus reducing the complexity of the algorithm and avoiding the combinatoric explosion in the number of linear programs. The first observation we make in this regard is the fact that the stability boundaries constitute a set of linear equations. Therefore, the issue analyzed in the next section is how to handle this set efficiently. In particular, the approach proposed deals with vertices, rather than inequalities, of stabilizing polyhedral sets.

V. Polynomial-Time Algorithms for Computing Tractable Controller Parameters

The proposed approach is divided into two steps. The first step is to compute so-called marginal stabilizers in polynomial-time. Once this stabilizer is determined, a fixed order controller can be subsequently determined. This second operation can be also efficiently performed with the aid of one-parameter optimization problem.

A. Computation of Marginal Stabilizers

We first introduce the definition of marginal stabilizer formally.

Definition 2: A marginal stabilizer is a controller $C(s)$ having the property that the corresponding closed-loop polynomial $p(s)$ has a fixed number of roots on the imaginary axis and no roots in the open right half plane.

Following the discussion in the previous section, the fixed order marginal stabilization problem can be reduced to solving the equation (4). In order to have a marginal stabilizing controller, we construct a square matrix $\overline{\Psi}_i$ which consists of n_θ rows of the matrix Ψ and a vector $\overline{\nu}_i$ which consists of the corresponding n_θ elements of ν. The resulting square linear system is given by

$$\overline{\Psi}_i \theta = \overline{\nu}_i.$$

If $\overline{\Psi}_i$ is invertible (see Lemma 3 below), we can immediately solve the system of linear equations as

$$\theta^{(i)} = \overline{\Psi}_i^{-1} \overline{\nu}_i. \quad (5)$$

This $\theta^{(i)}$ gives a candidate marginal stabilizer. Then, we can check by means of the Routh test if $p(s, \theta^{(i)})$ has all its roots in the closed left half plane. If $\theta^{(i)}$ is actually a marginal stabilizer, then we proceed to find a stabilizing controller, as discussed in Subsection V-B. Otherwise, we repeat this process for another combination of rows of Ψ and corresponding elements of ν. The total number of square matrices $\overline{\Psi}_i$ and vectors $\overline{\nu}_i$ that needs to be computed with this procedure is given by

$$N_{MI}(n_f, n_\theta) = \frac{n_f!}{n_\theta!(n_f - n_\theta)!}$$

where $n!$ denotes the factorial of n.

Remark 2: For fixed i, $\theta^{(i)}$ can be immediately found by matrix inversion. This requires $O(n_\theta^3)$ operations, see e.g. [16].

In the above procedure, invertibility of the square matrix $\overline{\Psi}_i$ is required. Here we give a technical lemma which ensures the invertibility for all $i = 1, 2, \dots, N_{MI}(n_f, n_\theta)$.

Lemma 3: Suppose that $n_f \geq n_\theta$. Then, all square matrices $\overline{\Psi}_i$, $i = 1, 2, \dots, N_{MI}(n_f, n_\theta)$, which consist of n_θ rows of the matrix Ψ, are invertible.

The statement of this lemma follows from the structure of Ψ

$$\Psi = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & -\omega_2^2 & \omega_2^4 & \cdots & (-1)^{n_\theta - 1}\omega_2^{2(n_\theta-1)} \\ 1 & -\omega_3^2 & \omega_3^4 & \cdots & (-1)^{n_\theta - 1}\omega_3^{2(n_\theta-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & -\omega_{n_f-1}^2 & \omega_{n_f-1}^4 & \cdots & (-1)^{n_\theta - 1}\omega_{n_f-1}^{2(n_\theta-1)} \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

In this matrix, the critical frequencies 0 and ∞ are included as ω_1 and ω_{n_f}, and they correspond to the first row and the last row of Ψ. In fact, this realization has a structure similar to the Vandermonde matrix. Thus, any possible square matrix $\overline{\Psi}_i$ is invertible if all values ω_i are distinct, which is always satisfied in our context.

We now further elaborate on the computational complexity of the problem. To this end, using properties of the factorial, we compute

$$N_{MI}(n_f, n_\theta) = \frac{n_f(n_f-1)\cdots(n_f-n_\theta+1)}{n_\theta!}. \qquad (6)$$

Since we study a fixed order controller problem, we observe that the number of parameters n_θ is fixed. It turns out that, for fixed n_θ, $N_{MI}(n_f, n_\theta)$ is a polynomial function of n_f. Using Lemma 2, we conclude that $N_{MI}(n_f, n_\theta)$ is a polynomial function of n_N, n_D, n_Y, n_Z and n_W. We therefore see that this procedure is computationally more efficient to perform than solving a linear program because the number of linear programs $N_{LP}(n_f)$ that should be solved in the worst case is 2^{n_f}.

Here we formally state the computational complexity of $N_{LP}(n_f)$ and compare it with $N_{MI}(n_f, n_\theta)$.

Theorem 3: Suppose that $n_f \geq n_\theta$. Then,

$$N_{MI}(n_f, n_\theta) = O(n_f^{n_\theta}). \qquad (7)$$

Furthermore, for any $n_\theta \geq 0$,

$$N_{MI}(n_f, n_\theta) \leq N_{LP}(n_f) \qquad (8)$$

where equality is attained only if $n_f = 0$.

The equality (7) follows from (6). The inequality (8) is a direct consequence of a well-known identity

$$\sum_{r=0}^{n} \frac{n!}{r!(n-r)!} = 2^n.$$

which is derived from the so-called bimodal theorem.

Theorem 3 says that $N_{MI}(n_f, n_\theta)$ is always smaller than or equal to $N_{LP}(n_f)$. Some computations of $N_{MI}(n_f, 2)$, $N_{MI}(n_f, 4)$ and $N_{LP}(n_f)$ are given in Table I for a different number of n_f. From this table, we conclude that $N_{MI}(n_f, n_\theta)$ is actually much smaller than $N_{LP}(n_f)$.

TABLE I

COMPARISON OF $N_{LP}(n_f)$ AND $N_{MI}(n_f, n_\theta)$

n_f	8	16	32	64
$N_{MI}(n_f, 2)$	28	120	496	$2,016$
$N_{MI}(n_f, 4)$	70	$1,820$	$35,960$	6.3538×10^5
$N_{LP}(n_f)$	256	$65,536$	$4,294,967,296$	1.8447×10^{19}

We now state a result regarding stabilization properties of the controller parameters $\theta^{(i)}$.

Theorem 4: Let $p(s, \theta^{(i)})$ be the closed-loop polynomial corresponding to $\theta^{(i)}$. There exists a marginal stabilizer if and only if there exists $\theta^{(i)}, i = 1, 2, \dots, N_{MI}$, such that $p(s, \theta^{(i)})$ has no roots in the open right half plane.

Remark 3: The controller parameter vector $\theta^{(i)}$, if it exists, is a vertex of a polyhedron of stabilizing controllers. In this case, the n_θ rows of the corresponding matrix $\overline{\Psi}_i$ and the n_θ elements of $\overline{\nu}_i$ define some of the hyperplanes generating the boundary of a polyhedron of stabilizing controllers.

B. Computation of a Stabilizing Controller

We now address a subsequent crucial problem: given a marginal stabilizer, determine a fixed order stabilizing controller which places the roots of the closed-loop polynomial in the open left half plane. To this end, we consider the sensitivity of zeros of $p(s, \theta)$ against perturbation on θ. This kind of approach has been presented for the case of PID controllers in [17], and the method proposed here is an extension to the general case.

Suppose that θ is a marginally stabilizing parameter so that $p(s, \theta)$ has k, where $1 \leq k \leq \deg p(s, \theta)$, simple zeros on the imaginary axis and all the other zeros lie in the open left half plane. Let us consider one imaginary zero $j\omega_i$ and study how $j\omega_i$ moves when we perturb θ by $\Delta\theta$. Since $j\omega_i$ is simple, there exists an analytic function $z_i(\Delta\theta)$ in $\|\Delta\theta\| < \epsilon$ for some positive ϵ such that $z_i(0) = j\omega_i$ and $p(z_i(\Delta\theta), \theta + \Delta\theta) = 0$. By differentiating the last equality at $\Delta\theta = 0$, we have

$$\frac{\partial z_i}{\partial \theta_j} = -\Big(\frac{\partial p}{\partial s}\Big|_{s=j\omega_i}\Big)^{-1} \frac{\partial p}{\partial \theta_j}\Big|_{s=j\omega_i}.$$

Notice that the quantities on the right-hand side can be easily computed because the inverse of $(\partial p/\partial s)|_{s=j\omega_i}$ is just the reciprocal of a complex number. Since we want $\Delta\theta$ moving all the imaginary zeros inside the left half plane, we consider to solve

$$\begin{bmatrix} \mathrm{Re}\,\frac{\partial z_1}{\partial \theta_0} & \cdots & \mathrm{Re}\,\frac{\partial z_1}{\partial \theta_{n_z}} \\ \vdots & \ddots & \vdots \\ \mathrm{Re}\,\frac{\partial z_{n_z/2+1}}{\partial \theta_0} & \cdots & \mathrm{Re}\,\frac{\partial z_{n_z/2+1}}{\partial \theta_{n_z}} \end{bmatrix} \begin{bmatrix} \Delta\theta_0 \\ \vdots \\ \Delta\theta_{n_z} \end{bmatrix} = \begin{bmatrix} -1 \\ \vdots \\ -1 \end{bmatrix}. \qquad (9)$$

Under the assumption that the matrix on the left-hand side is invertible, we can immediately obtain the desired $\Delta\theta$.

After we obtain the desired $\Delta\theta$, we consider a parameter $\theta + \alpha\Delta\theta$ for a positive α. Although a small α gives a stabilizing controller, a large α can be used as well. One recommendable procedure is to use a bisection method for the parameter α.

Remark 4: The above procedure requires polynomial-time operations for n_θ because there is no combinatorial operation involved.

As we have seen, the proposed method may be used under the conditions that $p(s, \theta)$ has simple zeros on the imaginary axis and the matrix on the left-hand side of (9) is invertible. Notice that these conditions are generically satisfied. Otherwise, we may use a randomization based method as an alternative approach. In fact, as a consequence of Theorem 4, the following fact holds true: given a ball (for example ℓ_2) of radius $\epsilon > 0$ centered around $\theta^{(i)}$, then there exists a fixed order stabilizer $\theta^{(k)}$ within the ball. Using this observation, we can find $\theta^{(k)}$ using randomization. That is, we generate N points within the ball until we find a stabilizer. This procedure is guaranteed to converge because a stabilizer exists within the ball.

However, with this alternative approach, randomization is performed only within a "small" ball of radius ϵ; i.e. the search for a fixed order stabilizer is made only locally around a marginal stabilizer. The consequence is that the stabilizer found with this approach is close in some sense to a marginal stabilizer and it may be fragile. On the other hand, the deterministic procedure previously derived in (9) leads to finding stabilizers which may lie "deeply inside" the stability region and do not suffer from this drawback.

In closing this section, we summarize the proposed algorithm which looks for a stabilizing controller when an intractable parameter $\mu^{(i)}$ is determined according to Algorithm 1.

Algorithm 3:

```
1.  Construct Ψ and ν for given μ^(i);
2.  For j := 1, ..., N_MI(n_f, n_θ) do
    begin
3.      Compute θ^(j) according to (5);
4.      If θ^(j) gives a marginal stabilizer then
        begin
5.          Compute Δθ according to (9);
6.          If a stabilizing parameter θ^(j) + αΔθ is
                found then stop;
        end
    end
```

VI. Conclusion

In this paper, we studied fixed order stabilization of single-input single-output plants. To this end, we presented a polynomial-time algorithm based on randomized algorithms and matrix inversions. A detailed complexity analysis has been also performed.

Subsequent research will be carried on along several directions. In particular, we plan to extend the results of this paper to the design of controllers which guarantee an H_∞ bound on the sensitivity and complementary sensitivity functions. Another research direction is to extend the methods given here to uncertain plants $P(s, q)$, where q represents parametric uncertainty. Finally, we will also consider stabilization of plants $P(s, \tau)$ affected by a fixed delay $\tau > 0$.

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