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# On a root locus-based analysis of the limiting zeros of plants of nominal order at most two under FROH-discretization

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***Abstract:*** An analysis about the limiting zeros of a plant of nominal order at most two under FROH-discretization is presented. The continuous transfer function is decomposed as a sum of a nominal plant plus a disturbance transfer function of any order. A root-locus based analysis is used with the aim of finding the limiting discrete zeros. Numerical simulations show the influence of the FROH gain β parameter on the stability of the discretization zeros.

***Keywords:*** FROH-discretization, limiting zeros, root locus

### 1-Introduction

It is well known that unstable zeros limit the performance of many control techniques based on the cancellation of process zeros. In the last years, much work has been done in this field trying to discuss the stability of zeros of discrete-time systems obtained from a continuous plant [1,4,5,7,8]. Most studies about this topic are focused on the ZOH (zero–order hold) because of its simplicity. However, the use of a fractional-order hold (FROH) may give stable discrete zeros in some cases when the ZOH cannot do it [7]. Moreover, the stability conditions and properties of limiting zeros for sufficiently small sampling periods were discussed in [2,3,5]. Later, the properties of the limiting zeros of multivariable discrete-time plants have been studied as well [6]. In this paper, we present an analysis about the limiting zeros of a continuous plant under FROH-discretization. The continuous plant consists of a nominal plant of order at most two plus a disturbance plant of any order in a parallel connection. The study is based on a root locus analysis, which allows us to present the results for the limiting cases, i.e., for sufficiently small or large values of the sampling period and for small or large amounts of disturbance. Through numerical simulations, we will show how an appropriate choice of the FROH gain $\beta$ parameter may improve the stability of the discrete zeros.

### 2-The continuous and discrete transfer function under β-FROH

The transfer function of a continuous plant can be written as,

$$G(s) = G_0(s) + \varepsilon \cdot G'(s) = \frac{N_0(s)}{D_0(s)} + \varepsilon \frac{N'(s)}{D'(s)} \qquad (1)$$

where, $G_0(s)$ denotes the continuous nominal plant of order at most two, $G'(s)$ represents a parametrical disturbance of any order and the scalar $\varepsilon$ may take either a large or small value. The polynomial degrees may be written as $n_c = \deg(D_0) \le 2$, $m_c = \deg(N_0) \le 2$, $n_c' = \deg(D')$ and $m_c' = \deg(N')$. Using a $\beta$-FROH, the discrete transfer function is calculated as follows,

$$H(z) = Z[h_\beta(s) \cdot G(s)] = H_0(z) + \varepsilon H'(z) = \frac{B_0(z)}{A_0(z)} + \varepsilon \frac{B'(z)}{A'(z)} \quad (2)$$

where, $$h_\beta(s) = \left(1 - \beta e^{-sT} + \frac{\beta(1-e^{-sT})}{Ts}\right)\frac{1-e^{-sT}}{s} \qquad (3)$$

is the transfer function of a β-FROH, T the sampling time, Z the Z-transform and the polynomial degrees are $n_0 = \deg(A_0) \le 3$, $m_0 = \deg(B_0) \le 3$, $n' = \deg(A')$ and $m' = \deg(B')$. Note that when $\beta = 1$, the FROH becomes a first order hold (FOH) and if $\beta = 0$, then the zero order hold (ZOH) is obtained. $H(z)$ may be calculated just using ZOH in the following way,

$$H(z) = \frac{z-\beta}{z} Z[h_0(s) \cdot G(s)] + \frac{\beta(z-1)}{Tz} Z[h_0(s) \cdot \frac{G(s)}{s}] \quad (4)$$

where, $h_0(s) = \frac{1-e^{-sT}}{s}$ is the transfer function of a ZOH.

The zeros of the discrete transfer function are the roots of $B_0(z)A'(z) + \varepsilon A_0(z)B'(z) = 0$. Since the disturbance takes limiting values, a root-locus analysis can be used to discuss the location of the discrete zeros. This means that when $|\varepsilon|$ is sufficiently small, we can assure that the discrete zeros are close to the roots of $B_0(z)A'(z)$. On the other hand, if $|\varepsilon|$ takes a large value then they are close to the roots of $A_0(z)B'(z)$. Note that if the products of polynomials $B_0(z)A'(z)$ and $A_0(z)B'(z)$ do not have the same degree, then it is necessary to add as many zeros diverging to infinite to the polynomial product of smaller degree as the order difference. For the study of the location of discrete zeros, we consider the different combinations in the degrees of the polynomial of the continuous nominal plant. The study is done for the limiting cases of $T$ and $\varepsilon$ tending to zero or infinity, which are included in the following set of conditions.

| | $m_c = 2, n_c = 0$ | $m_c = 2, n_c = 1$ | $m_c = 2, n_c = 2$ | $m_c = 1, n_c = 0$ | $m_c = 1, n_c = 1$ |
|---|---|---|---|---|---|
| $T \to 0, \varepsilon \to 0$ | A1 | A2 | A3 | A4 | A5 |
| $T \to \infty, \varepsilon \to 0$ | B1 | B2 | B3 | B4 | B5 |
| $T \to 0, \varepsilon \to \infty$ | C1 | C2 | C3 | C4 | C5 |
| $T \to \infty, \varepsilon \to \infty$ | D1 | D2 | D3 | D4 | D5 |

Fig.1: Table of conditions

### *3-Main results about limiting zeros*

In this section a study about the stability of limiting zeros in (2) is presented. In order to show the results in a clear way, they are presented by using a set of different tables. Each of them corresponds to a group of conditions of Figure 1. Table A shows the limiting discrete zeros for conditions A1-A5 with $n'_c > m'_c$ and table A' for $n'_c = m'_c$. The number of unstable poles in the disturbance transfer function $H'(z)$ is denoted by $n'_u$. For each condition, there are as many intrinsic discrete zeros, $z_i$, as zeros in the nominal continuous plant, the remaining discrete zeros (if any) being discretization zeros. The intrinsic zeros tend to unity when T tends to zero while they maintain the same stability condition as the continuous zeros they come from. In the same way, Table B (at the end of manuscript) and B' are presented including constraints B1-B5. In table B, $G_0(s)$ is written as $G_s(s)+G_u(s)$, where $G_s(s)$ has the stable poles and $G_u(s)$ the unstable ones. The integer $n_u$ denotes the number of unstable poles in the nominal transfer function.

| | |
|---|---|
| A1 | • $-1 \le \beta < 0$: $n'_c - n'_u + 2$ stable and $n'_u$ unstable<br>• $\beta = 0$: $n'_c - n'_u$ stable, $n'_u$ unstable, and<br>one in $\begin{cases} z = -1^{+}\ (stable) & if \sum poles\ of\ G_0(s) < 0 \\ z = -1^{-}\ (unstable) & if \sum poles\ of\ G_0(s) > 0 \\ z = -1 & if \sum poles\ of\ G_0(s) = 0 \end{cases}$<br>• $0 < \beta \le 1$: $n'_c - n'_u + 1$ stable and $n'_u + 1$ unstable |
| A2 | • $\beta \ne 0$: $n'_c - n'_u + 1$ stable, $n'_u$ unstable and 1 in $z = z_i$<br>• $\beta = 0$: $n'_c - n'_u$ stable, $n'_u$ unstable and 1 in $z = z_i$ |
| A3 | • $\beta \ne 0$: $n'_c - n'_u + 1$ stable, $n'_u$ unstable and 2 in $z = z_i$<br>• $\beta = 0$: $n'_c - n'_u$ stable, $n'_u$ unstable and 2 in $z = z_i$ |
| A4 | • $\beta \ne 0$: $n'_c - n'_u + 1$ stable and $n'_u$ unstable<br>• $\beta = 0$: $n'_c - n'_u$ stable and $n'_u$ unstable |
| A5 | • $\beta \ne 0$: $n'_c - n'_u + 1$ stable, $n'_u$ unstable and 1 in $z = z_i$<br>• $\beta = 0$: $n'_c - n'_u$ stable, $n'_u$ unstable and 1 in $z = z_i$ |

Table A: Conditions A1-A5 with $n'_c > m'_c$

| | |
|---|---|
| A1,A2,A4 | Same zeros as in Table A, but adding one diverging zero |
| A3,A4 | Same zeros as in Table A |

Table A': Conditions A1-A5 with $n'_c = m'_c$

| | |
|---|---|
| B1,B2,B4 | Same zeros as in Table B, but adding one diverging zero |
| B3,B4 | Same zeros as in Table B |

Table B': Conditions B1-B5 with $n'_c = m'_c$

For the rest of conditions C and D, the discrete zeros tend to the roots of $A_0(z)B'(z)$. In this case, such a detailed study about stability of zeros as in previous cases for the zeros coming from $B'(z)$ is not direct, since the order of the disturbance might be unknown. For those zeros see [5]. Thus, table below related to conditions C and D presents a more general analysis. In conditions C3, C5, D3 and D5 with $n'_c > m'_c$ (Tables C and D), a diverging zero has been added because of $A_0(z)B'(z)$ having an order lower than $B_0(z)A'(z)$.

| | | |
|---|---|---|
| Tables C'-D' | C1-C5,D1-D5 | $n - n_u$ stable, $n_u$ unstable and zeros of B' |
| Tables C-D | C1,C2,C4<br>D1,D2,D4 | $n - n_u$ stable, $n_u$ unstable and zeros of B' |
| | C3,C5<br>D3,D5 | $n - n_u$ stable, $n_u + 1$ unstable, with at least one diverging,and zeros of B' |

Tables C, D ( $n'_c > m'_c$ ), C' and D' ( $n'_c = m'_c$ )

### *4-Simulations results*

In this section some simulations are presented for the following second order unstable nominal plant plus a second order stable disturbance.

$$G(s) = \frac{s+1}{(s-2)(s-3)} + \varepsilon \frac{1}{(s+2)(s+5)} \qquad (5)$$

For sufficiently small values of $|\varepsilon|$, some of the discrete zeros of the plant tend to the discretization zeros of the nominal part while the remaining ones tend to the discrete poles of the disturbance transfer function, which are stable since the disturbance is stable. Thus, in the analysis of the location of the nominal discrete zeros, they are stable for small sampling periods (see condition A.2 in Table A). However, one of the nominal zeros becomes unstable as the sampling period increases (see condition B2, $n_u$=2, in Table B). For control purposes it is useful to know the range of the sampling period that locates all the discrete zeros inside the stable area. Fig.2 shows the maximum admitted value of the sampling period compatible with the discrete zeros being stable for different values of $\beta$ and ε. A proper election of the value of $\beta$ may let the use of a larger sampling period maintaining the discrete zeros stable. Note that it is useful to know both range of $\beta$ and T for obtaining stable discrete zeros. Also, it is useful to have some idea about the relative stability of the discrete zeros for each T. Fig.3 shows the optimum value of $\beta$ for each $T \in [0,2]$ in the sense of obtaining the worst discrete zero of smallest absolute value. Fig. 4 displays the magnitude of the worst zero for the optimum value of $\beta$ corresponding to each T. ε = 0.001 has been chosen in this simulation.


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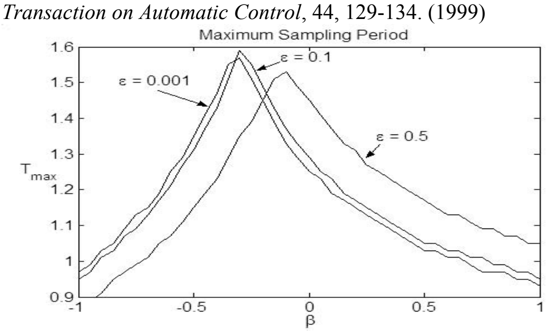


Fig.2: $T_{max}$ for different values of $\beta$ and $\varepsilon$

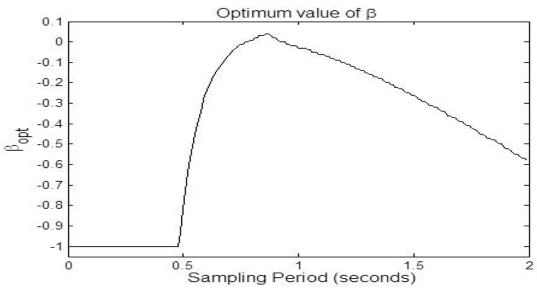


Fig.3: Optimum $\beta$ for each $T \in [0,2]$

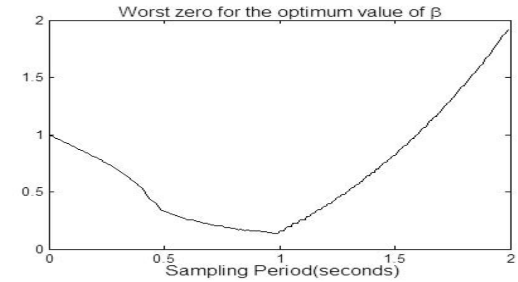


Fig.4: Absolute value of the worst zero for the optimum $\beta$ corresponding to each $T \in [0,2]$

| | | |
|---|---|---|
| B1 | $n_u = 0$ | - β = 0: n'$_c$ – n'$_u$ +1 stable and n'$_u$ unstable<br>- β ≠ -1 and β ≠ 0: n'$_c$ – n'$_u$ +1 stable and n'$_u$ unstable and one in z = β/(1+β) (i.e., stable if $\beta$ > -0.5)<br>- β = -1: n'$_c$ – n'$_u$ +1 stable and n'$_u$ +1 unstable with at least one diverging |
| | $n_u = 1$ | - β = 0: n'$_c$ – n'$_u$ stable, n'$_u$ unstable and one in pu/ps, where ps is the stable continuous pole in $D_0$ and pu the unstable one<br>- β ≠ 0: - Gu(0) ≠ 0: n'$_c$ – n'$_u$ stable, n'$_u$ unstable and roots of Gu(0)$z^2$ + (1+β)Gs(0) - βGs(0) = 0<br>- Gu(0) = 0, β = -1: n'$_c$ – n'$_u$ stable, n'$_u$ +1 unstable with at least one diverging and one in z = β/(1+β)<br>- Gu(0) = 0, β ≠ -1: n'$_c$ – n'$_u$ stable, n'$_u$ +2 unstable with at least two diverging |
| | $n_u = 2$ | - β = 0: n'$_c$ – n'$_u$ stable and n'$_u$ +1 unstable with at least one diverging<br>- β ≠ 0: n'$_c$ – n'$_u$ +1 stable and n'$_u$ +1 unstable with at least one diverging |
| B2 | $n_u = 0$ | - β = 0: n'$_c$ – n'$_u$ +1 stable and n'$_u$ unstable<br>- β ≠ -1 and β ≠ 0: n'$_c$ – n'$_u$ +1 stable, n'$_u$ unstable and one in z = β/(1+β)<br>- β = -1: n'$_c$ – n'$_u$ +1 stable and n'$_u$ +1 unstable with one diverging |
| | $n_u = 1$ | - β = 0: n'$_c$ – n'$_u$ stable, n'$_u$ unstable and one in pu (ps-zc)/ps(pu-zc) , where zc is the continuous zero in $N_0$, ps the stable continuous pole in $D_0$ and pu the unstable one.<br>- β ≠ 0: - Gu(0) ≠ 0: n'$_c$ – n'$_u$ stable, n'$_u$ unstable and roots of Gu(0) $z^2$ + (1+β)Gs(0) z - βGs(0)<br>- Gu(0) = 0, β ≠ -1: n'$_c$ – n'$_u$ stable, n'$_u$ +1 unstable with at least one diverging and one in z = β/(1+β)<br>- Gu(0) = 0, β = -1: n'$_c$ – n'$_u$ stable and n'$_u$ +2 unstable with at least two diverging |
| | $n_u = 2$ | - β = 0: n'$_c$ – n'$_u$ stable and n'$_u$ +1 unstable with at least one diverging<br>- β ≠ 0: n'$_c$ – n'$_u$ +1 stable and n'$_u$ +1 unstable with at least one diverging |
| B3 | $n_u = 0,1,2$ | - n'$_c$ - n'$_u$ stable, n'$_u$ unstable and roots of $(\lambda_1 z + \delta_1)(z-e^{-T\cdot p2}) + (\lambda_2 z + \delta_2)(z- e^{-T\, p1}) + b_2(z-e^{-T\, p1})(z-e^{-T\cdot p2})z = 0$<br>where, $\lambda_i = \alpha_i p_i^{-1}(1 - e^{-Tpi}) + \beta T^{-1} p_i^{-2}(p_i T - 1 + e^{-Tpi})$, $\delta_i = \alpha_i \beta T^{-1} p_i^{-2}(1- Tp_i - e^{-Tpi})$ with i = {1,2}.<br>Nominal continuous plant: $G_0(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^2 + a_1 s + a_0} = \frac{\alpha_1}{s + p_1} + \frac{\alpha_2}{s + p_2} + b_2$ |
| B4 | $n_u = 0$ | - β = 0: n'$_c$ – n'$_u$ stable and n'$_u$ unstable<br>- β ≠ -1 and β ≠ 0: n'$_c$ – n'$_u$ stable, n'$_u$ unstable and one in z = β/(1+β)<br>- β = -1: n'$_c$ – n'$_u$ stable and n'$_u$+ 1 unstable with one diverging |
| | $n_u = 1$ | - β = 0: n'$_c$ – n'$_u$ stable and n'$_u$ unstable<br>- β ≠ 0: n'$_c$ – n'$_u$ +1 stable and n'$_u$ unstable |
| B5 | $n_u = 0$ | - β = 0: n'$_c$ – n'$_u$ stable, n'$_u$ unstable and one in z = 1 – zc/pc<br>- β ≠ 0: n'$_c$ – n'$_u$ stable, n'$_u$ unstable and roots of $z^2$+ (β+1)(zc-pc) z+β(pc-zc) = 0 where zc is the zero in $N_o$ and pc the pole in $D_o$ |
| | $n_u = 1$ | - β = 0: n'$_c$ – n'$_u$ stable, n'$_u$+1 unstable with at least one diverging<br>- β ≠ 0: n'$_c$ – n'$_u$ +1 stable, n'$_u$+1 unstable with at least one diverging |

Table B: Conditions B1-B5 with $n_c' > m_c'$