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On optimal criteria for optimal set point tracking

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Abstract— **The settling time for the response to a set point change is minimized for linear quadratic tracking systems where the criterion function is modified so that little or no weight is put on parts of the transient of the output error. Different ways of modifying the criterion function are discussed and simulation studies show that the settling time can be substantially reduced for nonminimum phase systems.**

I. Introduction

Consider the problem of tracking a sudden set point change so that the settling time is minimized. In many cases it is of no interest how the new set point is reached, as long as an oscillatory behavior and a large overshoot are avoided. Given a criterion function to be minimized, a possible solution strategy is to give parts of the transient of the output error small or no weights. As a consequence, the output signal does not have to follow a reference signal that might be unnatural for the closed-loop system, and all effort can be put on reaching the new set point as quickly as possible.

This approach is considered in [1], where zero time weights are put on the transient part of the output error for criteria that are quadratic functions of the output error. For absolute error functions of the output error, the corresponding thing is done in [2]. In these cases, an unknown system is controlled by a controller of fixed structure, and the controller parameters are determined by minimizing the criteria with respect to them. Information about the gradients of the criteria is given by the iterative feedback tuning algorithm.

In this paper the approach of weighing parts of the transient of the output error is applied to optimal control of known linear systems using quadratic criteria functions. The theory for optimal linear quadratic tracking systems, see, e.g., [3] or [4], is first briefly summarized, and then the optimal choice of weight function for the purpose of minimizing the settling time for the response to a set point change is searched for, and different alternatives are investigated. The alternatives are tested on a nonminimum phase system in a numerical study, which shows that the settling time can be drastically reduced with the proper choice of weight function.

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II. Linear quadratic tracking systems

Consider the linear observable system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \tag{1}$$

$$y(t) = C(t)x(t), \tag{2}$$

where $x(t)$ is the state vector, $u(t)$ is the input signal and $y(t)$ is the output signal. The objective is to control the system so that the output signal tracks the reference signal $z(t)$, i.e., so that the output error

$$e(t) = z(t) - y(t) \tag{3}$$

is minimized, for $t \in [t_0, t_f]$. The criterion function is therefore defined as

$$J = \frac{1}{2}e^T(t_f)F(t_f)e(t_f) + \frac{1}{2}\int_{t_0}^{t_f}\{e^T(t)Q(t)e(t) + u^T(t)R(t)u(t)\}\mathrm{d}t, \tag{4}$$

where the weight function $Q(t)$ is of special interest in this paper and will be discussed in Sec. III. Given the system (1)–(2) and the criterion function (4), the optimal control is

$$u(t) = -K(t)x(t) + R^{-1}(t)B^T(t)g(t), \tag{5}$$

where the Kalman gain $K(t)$ is given by

$$K(t) = R^{-1}(t)B^T(t)P(t) \tag{6}$$

with $P(t)$ as the solution to the differential Riccati equation

$$\dot{P}(t) = -P(t)A(t) - A^T(t)P(t) + P(t)E(t)P(t) - V(t) \tag{7}$$

with

$$E(t) = B(t)R^{-1}(t)B^T(t), \tag{8}$$

$$V(t) = C^T(t)Q(t)C(t) \tag{9}$$

and the final condition

$$P(t_f) = C^T(t_f)F(t_f)C(t_f). \tag{10}$$

Moreover, $g(t)$ is the solution to

$$\dot{g}(t) = -(A(t) - E(t)P(t))^T g(t) - W(t)z(t) \tag{11}$$

with

$$W(t) = C^T(t)Q(t) \tag{12}$$

and the final condition

$$g(t_f) = C^T(t_f)F(t_f)z(t_f). \tag{13}$$

The optimal state trajectory is the solution to

$$\dot{x}(t) = (A(t) - E(t)P(t))x(t) + E(t)g(t) \tag{14}$$

with the initial condition

$$x(t_0) = x_0. \tag{15}$$

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III. The output error weight function

What is the optimal choice of the output error weight function $Q(t)$ in the criterion function (4) when the purpose is to minimize the settling time for a set point tracking? As argued in Sec. I, it can be advantageous to use little or no weight in the beginning of the transient of the output error. This situation is depicted in Fig. 1 where $Q(t) \equiv Q_{\min}$ for $t \in [t_0, t_a]$, $Q(t)$ is linearly increased from $Q_{\min}$ to $Q_{\max}$ for $t \in [t_a, t_b]$ and $Q(t) \equiv Q_{\max}$ for $t \in [t_b, t_f]$ (it is assumed that the transient part of the output error is, in time, greater than t_b). Consider the following choices for $Q(t)$:

1) If $Q_{\max} \neq Q_{\min}$ and $t_b \neq t_a \neq t_0$, the situation illustrated in Fig. 1 is given.
2) When $Q_{\max} = Q_{\min}$, equal weight is given for all t.
3) If $t_b = t_a$, the weight is changed from $Q_{\min}$ to $Q_{\max}$ in a step-like fashion.
4) When $t_a = t_0$, $Q(t)$ is increased linearly from $Q_{\min}$ to $Q_{\max}$ for $t \in [t_0, t_b]$ and is then equal to $Q_{\max}$.

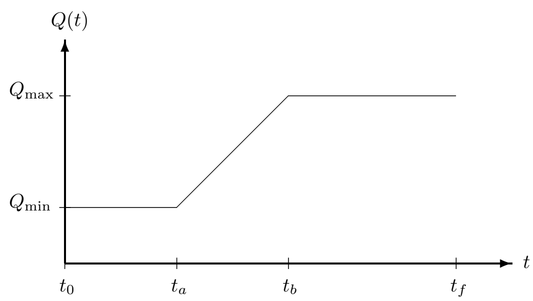

Fig. 1. The weight function $Q(t)$.

IV. A numerical study

It is desired to make the output signal from the nonminimum phase system with

$$A = \begin{bmatrix} -0.4 & 1 \\ -0.08 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ -5 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad (16)$$

in (1)–(2) follow a unit step change that occurs in the reference signal $z(t)$ at time $t = 0$. Here, $t_0 = 0$, $t_f = 1$, $x(0) = [0\ 0]^T$, $F(t_f) = 1$, and $R = 0.001$. Choices 1, 3 and 4 for $Q(t)$ described in Sec. III are considered with $Q_{\min} = 0$ and $Q_{\max} = 1$, and the specific t_a and t_b (only t_b for choice 4) that minimize the settling time t_s for each choice are given in Tab. I. Choice 2 with $Q_{\min} = Q_{\max} = 1$ is also considered for comparison. The resulting output and input signals are shown in Figs. 2 and 3, respectively. The graphs show that a constant weight function gives the longest settling time $t_s = 0.75$, and when weight functions with small or no weights for parts of the transient of the output error are used, the settling time is reduced. The shortest settling time $t_s = 0.33$ is achieved for the case when the function is quickly changed from $Q_{\min}$ to $Q_{\max}$ at $t = 0.23$. An interesting observation is that the time of the weight change is about the same as the shortest settling time achievable with little or no overshoot, and is also about the same as the size of the natural frequency of the open-loop system. It is also seen that the maximum absolute value of the input signal is largest for the case when the weight function is constant over time. Therefore, using weight functions with small or no weights for parts of the transient of the output error does not mean that large input signals have to be generated.

TABLE I

The optimal t_a, t_b and t_s for different $Q(t)$.

choice	$Q_{\min}$	$Q_{\max}$	t_a	t_b	t_s
1	0	1	0.22	0.23	0.33
2	1	1	0	0	0.75
3	0	1	0.23	0.23	0.33
4	0	1	0	0.52	0.51

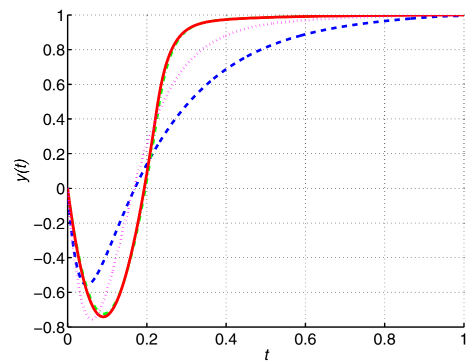

Fig. 2. The output signal $y(t)$ for the four different choices of weight function $Q(t)$ described in Tab. I. Choice 1: solid, choice 2: dashed, choice 3: dashdot, choice 4: dotted. Note that the output signals for choices 1 and 3 are almost identical.

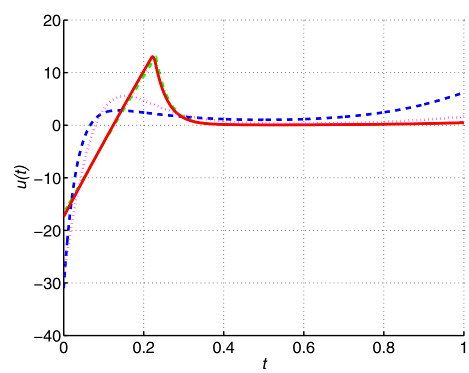

Fig. 3. The input signal $u(t)$ for the four different choices of weight function $Q(t)$ described in Tab. I. Choice 1: solid, choice 2: dashed, choice 3: dashdot, choice 4: dotted. Note that the input signals for choices 1 and 3 are almost identical.

References

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