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On the Stability Robustness in a Closed-Loop Interconnection Under a Class of Operations Performed on the Controller: A Novel Characterization for Static Controllers

Federico Najson
Mechanical and Aerospace Engineering Department,
University of California at Irvine, 4200 Engineering Gateway,
Irvine, CA 92697-3975, USA

***Abstract*— A closed-loop stable interconnection of two linear time-invariant finite-dimensional systems (one the plant, the other the controller) is considered. We analyze the preservation of the stability in the closed-loop interconnection whenever the input and output signals of the controller are multiplied by time-variant gains, one the reciprocal of the other, and in addition the function that represents those gains belongs to a specific class of functions. An important consequence of that analysis (and the main motivation in considering the aforementioned seemingly artificial robust stability setting) is the new characterization for output static stabilizing controllers we present in this communication. Moreover, a new technical tool is also presented: The Principal Function of the Closed-Loop Stable Interconnection. This function provides with information relative to the property of being static, of a stabilizing controller.**

***Index Terms*— Linear systems theory, stability of linear systems, output static stabilization, output static feedback, robust stability.**

I. INTRODUCTION

The analysis of the preservation of the stability in a (stable) closed-loop interconnection of two linear time-invariant finite-dimensional systems (one the plant, the other the controller) is considered in this article. The preservation of the stability is analyzed when the input signal of the controller is multiplied by a time-variant gain, and the output signal of the controller is multiplied by another gain which is the reciprocal of the previous one. In such analysis, the function representing the aforementioned gains are assumed to belong to a given specific family of functions.

The main motivation for such analysis, which is also of independent interest in linear systems theory, is to obtain a new characterization for output static stabilizing controllers in terms of the above described robust stability setting.

The study presented in this article is intended to report an important part (which is self-contained and complete by itself) of a more complete study that (already appears in [11], and that) due to space limitation considerations will be fully included in a future publication.

The new aforementioned characterization is intended to be a first step in the process of developing of a new approach or formulation to address the important problem of output static stabilization. For a linear time-invariant finite-dimensional system, this problem is the one of finding out (efficiently) computable conditions for the existence of a feedback matrix that renders the closed-loop system stable; and also the problem of (efficiently) synthesizing feedback matrices having such a property. The problem of output static stabilization have been extensively studied by several researchers (see e.g., [1], [7], [8], [9], [10] and [12] and references therein) for the last, roughly, three decades. The importance of such a problem has been recognized by the control community: it is a basic problem at the heart of linear systems theory, and moreover it is (to our knowledge [2], [3], [4]) still open.

The stability analysis presented here will be in a general input-output framework (see e.g., [5], [14]) where signals belong to $\mathcal{L}_{pe}(0,+\infty)$, and where p is any natural number ($1 \leq p < +\infty$) or $+\infty$ (and, as usual, e is used to mean extended $\mathcal{L}_p$ space).

In the present discussion we will consider the plant, $\mathcal{S}_P : \mathcal{L}_{pe}(0,+\infty) \longrightarrow \mathcal{L}_{pe}(0,+\infty)$ to be a linear time-invariant finite-dimensional system and it will be represented here by a strictly proper (real rational) transfer function $G_P(s)$ or by any realization (C_P, A_P, B_P), stabilizable and detectable, of G_P.

The controller, $\mathcal{S}_K : \mathcal{L}_{pe}(0,+\infty) \longrightarrow \mathcal{L}_{pe}(0,+\infty)$ will be considered to be a linear time-invariant finite-dimensional system and it will be represented by a (proper real rational) transfer function $G_K(s)$ or by any realization (C_K, A_K, B_K, D_K), stabilizable and detectable, of G_K.

Research carried out while the author was with the Mechanical and Aerospace Engineering Department, University of California at Los Angeles.

E-mail address: fnajson@ucla.edu

It will be assumed here that the following closed-loop interconnection is $\mathcal{L}_p$ stable (see e.g., [5], [14]):

$$y_1 = \mathcal{S}_P e_1 \,,\; e_1 = u_1 - y_2 \qquad (1)$$
$$y_2 = \mathcal{S}_K e_2 \,,\; e_2 = u_2 + y_1$$

Define the following family of functions, parameterized by $\eta \,:\, 0 < \eta \le 1$;
$\mathcal{F}(\eta) \overset{\text{def}}{=} \{\theta \in \mathcal{L}_\infty(0,+\infty) \,:\, \eta \le \theta(t) \le \eta^{-1} \text{ a.e.}\}$.

We can explain now, the robust stability problem under consideration. In the present discussion we will address the issue of existence of
η, $0 < \eta \le 1$, and $\theta \in \mathcal{F}(\eta)$ having the property that the following closed-loop interconnection,

$$y_1 = \mathcal{S}_P e_1 \,,\; e_1 = u_1 - y_2 \qquad (2)$$
$$y_2 = \theta^{-1}\mathcal{S}_K \theta e_2 \,,\; e_2 = u_2 + y_1$$

is not $\mathcal{L}_p$ stable.

It is obvious to see that in case $\mathcal{S}_K$ is an output static stabilizing controller for $\mathcal{S}_P$ then, $\forall\, \eta \in (0,1]$, and $\forall\, \theta \in \mathcal{F}(\eta)$, the associated closed-loop interconnection (2) will remain stable. Now, an important question is the following one: Is the converse of the above assertion, true? That is, if $\mathcal{S}_K$ is a (stabilizing) controller with the property that, $\forall\, \eta \in (0,1]$, and $\forall\, \theta \in \mathcal{F}(\eta)$, the associated closed-loop interconnection (2) remain stable, does the above property imply that $\mathcal{S}_K$ is a static (stabilizing) controller?
In section 3 we prove that, in the case in which the plant $\mathcal{S}_P$ is a single-input single-output system, the above converse statement holds. For the general case in which the plant $\mathcal{S}_P$ is a general multi-input multi-output system, we announce that, a weaker but also positive result (from the point of view of the output static stabilization problem) is valid regarding that converse; but such treatment is not included in the present article. (A complete, in detail, proof of such an assertion can be found in [11] and, as aforementioned, due to space limitations it will be included in a future publication.) Section 2 is devoted to prove some technical lemmas needed for the proof of the aforementioned result. Also in that section, the main technical instrument used in this work is introduced: The Principal Function of the Closed-Loop Stable Interconnection (1). Summary and concluding remarks are presented in section 4.

The notation used through the paper is standard. Just to clarify; for a matrix $A \in \mathcal{C}^{n\times n}$, we denote by $\rho\{\,A\,\}$ its spectral radius.

II. The Principal Function of the Closed-Loop Stable Interconnection

Lets define the following matrix value functions,
$$\mathcal{A}(\eta) \overset{\text{def}}{=} \begin{pmatrix} A_1 & -\eta^{-1}B_P C_K \\ \eta B_K C_P & A_K \end{pmatrix} ,\, 0 < \eta \le 1\,;$$
where $A_1 = (A_P - B_P D_K C_P)$.

We also define ϕ_{11}, ϕ_{12}, ϕ_{21}, ϕ_{22} in the following manner:
$$\begin{pmatrix} \phi_{11}(t) & \phi_{12}(t) \\ \phi_{21}(t) & \phi_{22}(t) \end{pmatrix} \overset{\text{def}}{=} e^{t\mathcal{A}(1)} \,,\; t \in \mathcal{R}^+\,;$$
where the above matrix partition matches that of $\mathcal{A}(1)$.

Remark 1: Since the above closed-loop interconnection is ($\mathcal{L}_p$) stable and the realizations of G_P, and G_K are stabilizable and detectable then, it follows (see e.g., [14], [6]) that the matrix $\mathcal{A}(1)$ is Hurwitz. It is also illuminating to observe that the following property is valid:
$\text{char}_{\mathcal{A}(1)}(\lambda) = \det(\lambda I - \mathcal{A}(1)) = \det(\lambda I - \mathcal{A}(\eta)) = \text{char}_{\mathcal{A}(\eta)}(\lambda)$, $0 < \eta \le 1$.
The following result can be easily verify to hold.

Fact 1: The following identities hold:
$$e^{t\mathcal{A}(\eta)} = \begin{pmatrix} \phi_{11}(t) & \eta^{-1}\phi_{12}(t) \\ \eta\phi_{21}(t) & \phi_{22}(t) \end{pmatrix} ,\; t \in \mathcal{R}^+ \,,\; 0 < \eta \le 1\,;$$

with $\phi_{11}(t) = E(tA_1, tA_K, -tB_P C_K, tB_K C_P)$,
$\phi_{22}(t) = E(tA_K, tA_1, tB_K C_P, -tB_P C_K)$,
$\phi_{12}(t) = F(tA_1, tA_K, -tB_P C_K, tB_K C_P)$,
$\phi_{21}(t) = F(tA_K, tA_1, tB_K C_P, -tB_P C_K)$,

where $E(tA_1, tA_K, -tB_P C_K, tB_K C_P) =$
$\mathcal{L}^{-1}\{(sI - A_1)^{-1}[I + B_P C_K (sI - A_K)^{-1} B_K C_P (sI - A_1)^{-1}]^{-1}\}(t)$, and

$F(tA_1, tA_K, -tB_P C_K, tB_K C_P) =$
$-\mathcal{L}^{-1}\{(sI - A_1)^{-1}[I + B_P C_K (sI - A_K)^{-1} B_K C_P (sI - A_1)^{-1}]^{-1} B_P C_K (sI - A_K)^{-1}\}(t)$.

We introduce in the next definition one of the main technical instruments of the theory developed in the present work.

Definition 1: Given the ($\mathcal{L}_p$) stable closed-loop interconnection described by (1), we will associate, to such an interconnection, the following function:
$\mathcal{I}^{(1)}(\cdot\,,\,\cdot) \overset{\text{def}}{=} \rho\{\,\phi_{21}(\cdot)\phi_{12}(\cdot)\,\} : \mathcal{R}^+ \times \mathcal{R}^+ \longrightarrow \mathcal{R}^+$.
The function $\mathcal{I}^{(1)}$ will be named, principal function of the closed-loop stable interconnection (1). (When understood from context, the name of function of the closed-loop interconnection may also be used.)

The next two lemmas show that the above function is in fact well defined. In addition, some simple properties of that function are also included.

Lemma 1: Consider the ($\mathcal{L}_p$) stable closed-loop interconnection described by (1). The associated function $\rho\{\,\phi_{21}(\cdot)\phi_{12}(\cdot)\,\} : \mathcal{R}^+ \times \mathcal{R}^+ \longrightarrow \mathcal{R}^+$ satisfies the following properties:

- It is independent on the specific realizations of G_P and G_K (i.e., it only depends on S_P and S_K).
- $\rho\{\,\phi_{21}(t_2)\phi_{12}(0)\,\} = \rho\{\,\phi_{21}(0)\phi_{12}(t_1)\,\} = 0$, $t_1, t_2 \in \mathcal{R}^+$.
- $\lim_{t\to+\infty} \rho\{\,\phi_{21}(t_2)\phi_{12}(t)\,\} = \lim_{t\to+\infty} \rho\{\,\phi_{21}(t)\phi_{12}(t_1)\,\} = 0$, $t_1, t_2 \in \mathcal{R}^+$.

Proof: Let (C_P, A_P, B_P) and (C_K, A_K, B_K, D_K) be stabilizable and detectable realizations of G_P and G_K respectively.
The first property follows from the following facts.
First, the above function does not change when similarity transformations are applied to the above realizations.
It follows from Fact 1 that when:
$(C_P, A_P, B_P) \longrightarrow (C_P T_1, T_1^{-1} A_P T_1, T_1^{-1} B_P)$, and
$(C_K, A_K, B_K, D_K) \longrightarrow$
$(C_K T_2, T_2^{-1} A_K T_2, T_2^{-1} B_K, D_K)$,
then,
$\phi_{12}(t_1) \longrightarrow T_1^{-1}\phi_{12}(t_1)T_2$, and
$\phi_{21}(t_2) \longrightarrow T_2^{-1}\phi_{21}(t_2)T_1$, which implies that,
$\phi_{21}(t_2)\phi_{12}(t_1) \longrightarrow T_2^{-1}\phi_{21}(t_2)\phi_{12}(t_1)T_2$.
Second, the function under consideration does not change whenever the above realization of G_K is substituted by the following one:
$((C_K \quad C_a), \begin{pmatrix} A_K & A_b \\ 0 & A_a \end{pmatrix}, \begin{pmatrix} B_K \\ 0 \end{pmatrix}, D_K)$.
This follows from Fact 1 by noticing that when the above substitution is performed, then
$\phi_{12}(t_1) \longrightarrow \begin{pmatrix} \phi_{12}(t_1) & \phi_x(t_1) \end{pmatrix}$, and
$\phi_{21}(t_2) \longrightarrow \begin{pmatrix} \phi_{21}(t_2) \\ 0 \end{pmatrix}$.
Third, the function under consideration does not change whenever the above realization of G_K is substituted by the following one:
$((C_K \quad 0), \begin{pmatrix} A_K & 0 \\ A_b & A_a \end{pmatrix}, \begin{pmatrix} B_K \\ B_a \end{pmatrix}, D_K)$.
This follows from Fact 1 by noticing that when the above substitution is performed, then
$\phi_{12}(t_1) \longrightarrow \begin{pmatrix} \phi_{12}(t_1) & 0 \end{pmatrix}$, and
$\phi_{21}(t_2) \longrightarrow \begin{pmatrix} \phi_{21}(t_2) \\ \phi_y(t_2) \end{pmatrix}$.
Finally, it is also easy to see that substitution of the above realization of G_P by,
$((C_P \quad C_a), \begin{pmatrix} A_P & A_b \\ 0 & A_a \end{pmatrix}, \begin{pmatrix} B_P \\ 0 \end{pmatrix})$, or by
$((C_P \quad 0), \begin{pmatrix} A_P & 0 \\ A_b & A_a \end{pmatrix}, \begin{pmatrix} B_P \\ B_a \end{pmatrix})$,
will not modify the above defined function.

The second and third properties follow by noticing that $\phi_{12}(0) = 0$, $\phi_{21}(0) = 0$, and since the closed-loop interconnection is stable and the realizations of G_P, G_K are stabilizable and detectable then it follows that $\mathcal{A}(1)$ is Hurwitz. Therefore, $\lim_{t\to+\infty} e^{t\mathcal{A}(1)} = 0$.

■

Lemma 2: Consider the ($\mathcal{L}_p$) stable closed-loop interconnection described by (1). If $\mathcal{S}_P = 0$, or if $\mathcal{S}_K$ is a static system, then it follows that:
$\mathcal{I}^{(1)}(t_2, t_1) = 0$, $\forall\, t_1, t_2 \in \mathcal{R}^+$.

Proof: It is convenient to define,
$G_1(s) = C_P(sI - A_1)^{-1}B_P$, and
$G_2(s) = C_K(sI - A_K)^{-1}B_K$.
Since, $G_1(s) = G_P(s)[I + D_K G_P(s)]^{-1}$, then $G_P = 0$ implies $G_1 = 0$. Therefore, it follows from Fact 1 that, either if $\mathcal{S}_P = 0$, or if $\mathcal{S}_K$ is a static system (i.e., $G_2 = 0$) the following holds:
$\mathcal{L}\{\phi_{12}\}(s) = -(sI - A_1)^{-1}B_P C_K(sI - A_K)^{-1}$, and
$\mathcal{L}\{\phi_{21}\}(s) = (sI - A_K)^{-1}B_K C_P(sI - A_1)^{-1}$.

Lets assume now that $G_2 = 0$, then
$\phi_{12}(t_1)\phi_{21}(t_2) = -\frac{1}{4\pi^2}\int_{-\infty}^{+\infty}(j\omega I - A_1)^{-1}B_P C_K(j\omega I - A_K)^{-1}e^{j\omega t_1}d\omega$
$\int_{-\infty}^{+\infty}(j\nu I - A_K)^{-1}B_K C_P(j\nu I - A_1)^{-1}e^{j\nu t_2}d\nu =$
$-\frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(j\omega I - A_1)^{-1}B_P$
$C_K(j\omega I - A_K)^{-1}(j\nu I - A_K)^{-1}B_K C_P(j\nu I - A_1)^{-1}$
$e^{j\omega t_1}e^{j\nu t_2}d\omega d\nu = 0$, $t_1, t_2 \in \mathcal{R}^+$.
Where the above equalities hold, by invoking Fubini's theorem (and recalling that by assumption (see Remark 1) $\mathcal{A}(1)$ is Hurwitz), and due to the fact that by assumption $G_2 = 0$, which means that all the Markov parameters of G_2 are zero; that is,
$C_K A_K^i B_K = 0$, $i \in \mathcal{Z}^+$.
Therefore, since we can express $(sI - A_K)^{-1}$ in the form $(sI - A_K)^{-1} = \sum_{i=0}^{(n_K-1)} g_i(s)A_K^i$,
(where $g_i(s)$, $i = 0, \ldots, (n_K - 1)$, are strictly proper real rational functions), it follows that
$C_K(j\omega I - A_K)^{-1}(j\nu I - A_K)^{-1}B_K = 0$,
$\forall\,(\omega, \nu) \in \mathcal{R}^+ \times \mathcal{R}^+$.

Finally, it is easy to see, using the same reasoning as before, that whenever $G_P = 0$, then
$\phi_{21}(t_2)\phi_{12}(t_1) = 0$, $t_1, t_2 \in \mathcal{R}^+$.

■

The following technical lemma presents a sufficient condition for the destabilization of the closed-loop interconnection. Later, it will be proved that for some special case that condition is also necessary.

Lemma 3: If the ($\mathcal{L}_p$) stable closed-loop interconnection described by (1) is such that the following property is satisfied:

(Cds) $\exists\, t_1 > 0$, $t_2 > 0$: $\mathcal{I}^{(1)}(t_2, t_1) \neq 0$;

then, $\exists\, \eta : 0 < \eta \leq 1$, and $\exists\, \theta \in \mathcal{F}(\eta)$ with θ a periodic function of period $T_\theta = (t_1 + t_2)$ for which the closed-loop interconnection described by (2) is not $\mathcal{L}_p$ stable.

Proof: It will be proved that
$\exists\, \eta : 0 < \eta \leq 1$, and $\exists\, \theta \in \mathcal{F}(\eta)$ such that for the corresponding closed-loop interconnection described by (2) $\exists \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathcal{L}_p$ (for every p) for which the corresponding output $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \notin \mathcal{L}_p$ (for all p).

First, since by assumption the condition **(Cds)** is satisfied, then invoking Lemma 2 it follows that $\mathcal{S}_P \neq 0$ and $\mathcal{S}_K$ is not static. That implies the existence of minimal realizations (C_P, A_P, B_P), and (C_K, A_K, B_K, D_K), of G_P and G_K respectively. In the sequel in the present proof, minimal realizations of G_P and G_K will be used (in order to construct a destabilizing input signal

$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathcal{L}_p$).
We will use the following description of the closed-loop interconnection (under consideration) as a tool for the construction of a destabilizing input signal:

$$\begin{pmatrix} \dot{x}_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} A_1 & -\theta^{-1}(t)B_P C_K \\ \theta(t) B_K C_P & A_K \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} B_P & 0 \\ 0 & \theta(t) B_K \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix},$$
$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} C_P & 0 \\ 0 & \theta^{-1}(t) C_K \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix},$$
$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \; t \in \mathcal{R}^+ ; \quad (3)$$
where $\theta \in \mathcal{F}(\eta)$, for some $\eta \in (0\,,\,1]$.

It follows from Fact 1 that,

$$e^{t_2 \mathcal{A}(\eta^{-1})} e^{t_1 \mathcal{A}(\eta)} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix},$$
where $\alpha_{11} = \phi_{11}(t_2)\phi_{11}(t_1) + \eta^2 \phi_{12}(t_2)\phi_{21}(t_1)$,
$\alpha_{12} = \eta^{-1}\phi_{11}(t_2)\phi_{12}(t_1) + \eta\phi_{12}(t_2)\phi_{22}(t_1)$,
$\alpha_{21} = \eta^{-1}\phi_{21}(t_2)\phi_{11}(t_1) + \eta\phi_{22}(t_2)\phi_{21}(t_1)$,
$\alpha_{22} = \eta^{-2}\phi_{21}(t_2)\phi_{12}(t_1) + \phi_{22}(t_2)\phi_{22}(t_1)$.

Now,

$$\eta^2 \rho\{\, e^{t_2 \mathcal{A}(\eta^{-1})} e^{t_1 \mathcal{A}(\eta)} \,\} = \rho\{\, \eta^2 e^{t_2 \mathcal{A}(\eta^{-1})} e^{t_1 \mathcal{A}(\eta)} \,\} =$$

$$\rho\{ \begin{pmatrix} 0 & 0 \\ 0 & \phi_{21}(t_2)\phi_{12}(t_1) \end{pmatrix} + \eta M(\eta) \,\};$$
(where $M(\eta)$ is a polynomial in η)

therefore, it follows from the continuity of the function $\rho\{\cdot\}$ that,

$$\lim_{\eta \to 0^+} \eta^2 \rho\{\, e^{t_2 \mathcal{A}(\eta^{-1})} e^{t_1 \mathcal{A}(\eta)} \,\} = \rho\{ \begin{pmatrix} 0 & 0 \\ 0 & \phi_{21}(t_2)\phi_{12}(t_1) \end{pmatrix} \} = \rho\{\, \phi_{21}(t_2)\phi_{12}(t_1) \,\} = \mathcal{I}^{(1)}(t_2, t_1) > 0 .$$

That proves that for any η ($0 < \eta \leq 1$) small enough, $\rho\{\, e^{t_2 \mathcal{A}(\eta^{-1})} e^{t_1 \mathcal{A}(\eta)} \,\} \sim \eta^{-2} \mathcal{I}^{(1)}(t_2, t_1)$;
or in other words we have proved that for any η $(0 < \eta \leq 1)$ small enough the periodic function defined by:

$$\theta_\eta(t) = \begin{cases} \eta & , \quad 0 \leq t < t_1 \\ \eta^{-1} & , \quad t_1 \leq t < T \end{cases} ,$$
$\theta_\eta(t+T) = \theta_\eta(t)$ $(T = t_1 + t_2)$,

will convey to the satisfaction of $\rho\{\, \Phi_{\mathcal{A}(\theta_\eta)}(T,0) \,\} = \rho\{\, e^{t_2 \mathcal{A}(\eta^{-1})} e^{t_1 \mathcal{A}(\eta)} \,\} > 1$ (since, $\Phi_{\mathcal{A}(\theta_\eta)}(T,0) = e^{t_2 \mathcal{A}(\eta^{-1})} e^{t_1 \mathcal{A}(\eta)}$) which in turn will imply, as we prove next, that the closed-loop interconnection described by (2) is not $\mathcal{L}_p$ stable.

Chosen then, a small enough η such that $|\lambda_{max}| = \rho\{\, e^{t_2 \mathcal{A}(\eta^{-1})} e^{t_1 \mathcal{A}(\eta)} \,\} > 1$ (where λ_{max} is an eigenvalue of $e^{t_2 \mathcal{A}(\eta^{-1})} e^{t_1 \mathcal{A}(\eta)}$, satisfying the above identity), and chosen the above defined periodic function $\theta_\eta \in \mathcal{F}(\eta)$, it follows that the state-space description (3) of the closed-loop interconnection is controllable and observable during each semi-period (in which the function θ_η remains constant). That is a direct consequence of the fact that (C_P, A_P, B_P), and (C_K, A_K, B_K, D_K) are both controllable and observable.
Then, it follows that there exists a bounded function $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ having support only in $[0\,,\,T]$ ($\Longrightarrow$ $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathcal{L}_p$) such that it steer the state of (3) from zero to $\begin{pmatrix} x_1(T) \\ x_2(T) \end{pmatrix} = \xi_1 \; : \; \|\xi_1\| = 1$, and $\xi_1 \in \Lambda_{real}(\lambda_{max})$, where $\Lambda_{real}(\lambda_{max})$, is a given real eigen-subspace (of $e^{t_2 \mathcal{A}(\eta^{-1})} e^{t_1 \mathcal{A}(\eta)}$) associated to λ_{max} in the following manner:
let $w \neq 0$ be an eigenvector corresponding to λ_{max}, then $\Lambda_{real}(\lambda_{max}) = \mathcal{R}(\begin{pmatrix} \Re\{w\} & \Im\{w\} \end{pmatrix})$.
Therefore, for such an input signal it is satisfied that,

$\begin{pmatrix} x_1(kT) \\ x_2(kT) \end{pmatrix} = |\lambda_{max}|^{k-1}\xi_k \; : \; \|\xi_k\| \geq \gamma_\Lambda$, and $\xi_k \in \Lambda_{real}(\lambda_{max})$, $k = 1, \ldots$
(where γ_Λ satisfies, $0 < \gamma_\Lambda \leq 1$).

Now, since the state-space system (3) is observable on $[kT\,,\,kT + t_1]$, $k = 1, \ldots$, it follows that,

$$\left\| \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\|_{\mathcal{L}_p(kT\,,\,kT+t_1)} \geq |\lambda_{max}|^{k-1} \gamma_\Lambda \gamma_p ;$$

where, $\gamma_p \overset{\text{def}}{=}$
$$\min_{\begin{pmatrix} x_1(T) \\ x_2(T) \end{pmatrix} \in \{\xi : \|\xi\| = 1,\, \xi \in \Lambda_{real}(\lambda_{max})\}} \left\| \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\|_{\mathcal{L}_p(T, T+t_1)}$$

$\Longrightarrow \gamma_p > 0$.

Which implies that $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \notin \mathcal{L}_p$, and the proof is complete.

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Remark 2: It is important to mention that, as follows from the last result, whenever the destabilizing condition **(Cds)** is satisfied (or in other words, whenever the function of the closed-loop interconnection $\mathcal{I}^{(1)}$ is not identically zero), then a destabilizing function $\theta \in \mathcal{F}(\eta)$ (for some $\eta \in (0\,,\,1]$) can always be chosen to be periodic and piecewise constant as the one used in the proof of Lemma 3:
$\theta_\eta(t) = \begin{cases} \eta & , \quad 0 \leq t < t_1 \\ \eta^{-1} & , \quad t_1 \leq t < T \end{cases}$, $\theta_\eta(t+T) = \theta_\eta(t)$
$(T = t_1 + t_2,\, t_1 > 0,\, t_2 > 0)$.

III. A Characterization for Output Static Stabilizing Controllers

We present now the following result, which is valid for a closed-loop interconnection of single-input single-output (SISO) systems.

Theorem 1: Consider the ($\mathcal{L}_p$) stable closed-loop interconnection described by (1). Assume that $\mathcal{S}_P$ is a SISO system and moreover that $\mathcal{S}_P \neq 0$.
Under the above conditions, if $\mathcal{S}_K$ is not static then,
$\exists \eta \ : \ 0 < \eta \leq 1$, and $\exists \theta \in \mathcal{F}(\eta)$, with θ a periodic function, for which the closed-loop interconnection described by (2) is not $\mathcal{L}_p$ stable.

Proof: Define,
$G_1(s) = C_P(sI - A_1)^{-1}B_P$, and
$G_2(s) = C_K(sI - A_K)^{-1}B_K$.

From Fact 1 it follows that:
$\mathcal{L}\{\phi_{12}\}(s) =$
$-(sI - A_1)^{-1}B_P[I + G_2(s)G_1(s)]^{-1}C_K(sI - A_K)^{-1}$,
and $\mathcal{L}\{\phi_{21}\}(s) =$
$(sI - A_K)^{-1}B_K[I + G_1(s)G_2(s)]^{-1}C_P(sI - A_1)^{-1}$.

Then,
trace$\{\mathcal{L}\{\phi_{21}\}(s)\mathcal{L}\{\phi_{12}\}(s)\} =$
$-[1 + G_1(s)G_2(s)]^{-2}C_P(sI - A_1)^{-2}B_P$
$C_K(sI - A_K)^{-2}B_K$.

Since by assumption $\mathcal{S}_K$ is not static, i.e. $G_2 \neq 0$, then it follows that $\tilde{G}_2(s) = C_K(sI - A_K)^{-2}B_K \not\equiv 0$.
The above last implication follows from the fact that the Markov parameters of G_2, $M^{(i)}_{G_2}$, and the Markov parameters of $\tilde{G}_2$, $M^{(i)}_{\tilde{G}_2}$, are related in the following manner:

$$M^{(i)}_{\tilde{G}_2} = \begin{cases} 0 & , \quad i = 0 \\ iM^{(i-1)}_{G_2} & , \quad i = 1, 2, \ldots \end{cases} ,$$

which implies that, $G_2 = 0 \iff \tilde{G}_2 = 0$.

Also recall that,
$G_1(s) = C_P(sI - A_1)^{-1}B_P = G_P(s)[I + D_K G_P(s)]^{-1}$.
Since by assumption $G_P \neq 0$, then $G_1 \neq 0$, which implies that
$\tilde{G}_1(s) = C_P(sI - A_1)^{-2}B_P \not\equiv 0$.

As a consequence of all that, it follows that trace$\{\mathcal{L}\{\phi_{21}\}(s)\mathcal{L}\{\phi_{12}\}(s)\} \not\equiv 0$.
The above result implies that the function,
$\int_0^t$ trace$\{\phi_{21}(t-\tau)\phi_{12}(\tau)\}\,d\tau$, $t \in \mathcal{R}^+$,
is not identically zero.
Therefore,
$\exists\, t_1 > 0, t_2 > 0$: trace$\{\phi_{21}(t_2)\phi_{12}(t_1)\} \neq 0$;
$\Longrightarrow \exists\, t_1 > 0, t_2 > 0$:
$\mathcal{I}^{(1)}(t_2, t_1) = \rho\{\phi_{21}(t_2)\phi_{12}(t_1)\} \neq 0$.

Now, invoking Lemma 3 the result follows. ■

After the above result we can make the following straightforward but important statements (in the next corollaries) regarding the problem under consideration. It was shown that, the principal function of the closed-loop stable interconnection provides with information, relative to the property of being static, of a stabilizing controller.

Corollary 1: Consider the ($\mathcal{L}_p$) stable closed-loop interconnection described by (1) under the additional assumptions that $\mathcal{S}_P$ is a SISO system and moreover $\mathcal{S}_P \neq 0$. Then, the function $\mathcal{I}^{(1)}$ being not identically zero, is a necessary and sufficient condition for the existence of $\eta \in (0, 1]$ and a function $\theta \in \mathcal{F}(\eta)$ with the property that the associated closed-loop interconnection (2) is not ($\mathcal{L}_p$) stable.

The next important result presents a new characterization for (SISO) output static stabilizing controllers. The output static stabilizing controllers are the unique output stabilizing controllers with the property of preserving the stability of the closed-loop interconnection with respect to the class of operations (under consideration) performed on the controller.

Corollary 2: Consider the ($\mathcal{L}_p$) stable closed-loop interconnection described by (1) under the additional assumptions that $\mathcal{S}_P$ is a SISO system and moreover $\mathcal{S}_P \neq 0$. Then, $\mathcal{S}_K$ is a static ($\mathcal{L}_p$) stabilizing controller
if and only if,
$\forall\, \eta \in (0, 1]$, and $\forall\, \theta \in \mathcal{F}(\eta)$, the associated closed-loop interconnection (2) remains ($\mathcal{L}_p$) stable.

As previously mentioned, an extension of the above characterization, valid for general MIMO systems (can be found in [11], however it) will not be included in the present article but in a future publication. It may be important, however, to clarify here that Theorem 1 does not apply for cases in which the plant, $\mathcal{S}_P$, is a general MIMO system, as the following example shows.

Example 1: Consider the stable closed-loop interconnection (1) where,
$G_P(s) = \begin{pmatrix} \frac{1}{(s+q_1)} & 0 \end{pmatrix}$, $G_K(s) = \begin{pmatrix} 0 \\ \frac{1}{(s+q_2)} \end{pmatrix}$,
$q_1 > 0$, $q_2 > 0$.
Notice that, since $C_P = 1$, $A_P = -q_1$, $B_P = \begin{pmatrix} 1 & 0 \end{pmatrix}$, and $C_K = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $A_K = -q_2$, $B_K = 1$,
are realizations for G_P and G_K respectively, then the closed-loop interconnection (2) can be described by

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} -q_1 & 0 \\ \theta(t) & -q_2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} B_P & 0 \\ 0 & \theta(t) \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix},$$
$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \theta^{-1}(t)C_K \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix},$$
$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ t \in \mathcal{R}^+.$$

Since for any given $\theta \in \mathcal{F}(\eta)$, (for any given $\eta \in (0, 1]$) the above time-dependent matrix $\mathcal{A}(\theta)$, is the A-matrix of an unforced time-variant linear system which is exponentially stable, then this implies (see, e.g. [14]) that, $\forall\,\eta \in (0\,,\,1]$ and $\forall\,\theta \in \mathcal{F}(\eta)$, the closed-loop interconnection (2) is always $\mathcal{L}_p$ stable.

IV. Summary and Concluding Remarks

A new class of stability robustness in a (stable) closed-loop interconnection of two linear time-invariant finite-dimensional systems (one the plant, the other the controller) was considered. We have analyzed the problem of preservation of the stability of the closed-loop system whenever a special class of operations is performed on the controller. It was proved, in this communication, that in case the plant is a SISO system, the unique class of stabilizing controllers with the property that the stability of the closed-loop system remains invariant, for all possible operations (under consideration) on the controller, is the class of static stabilizing controllers.

In case the closed-loop interconnection is not robust (in the sense considered in this work), the results presented here also provides with important qualitative information (Remark 2) regarding the nature of the destabilizing functions. It was also introduced, in this article, The Principal Function of the Closed-Loop Stable Interconnection (1). This novel function was proved to provide with information relative with the property of being static, of the stabilizing controller. It was proved here that, if $\mathcal{S}_K$ stabilizes a SISO system $\mathcal{S}_P \neq 0$ then, $\mathcal{S}_K$ is static $\iff \mathcal{I}^{(1)} \equiv 0 \iff$ $\forall\eta \in (0\,,\,1]$ and $\forall\theta \in \mathcal{F}(\eta)$ the closed-loop interconnection (2) is stable.

Notice that the following straightforward continuity result just follows from the definition of $\mathcal{I}^{(1)}$.

Fact 2: Consider the ($\mathcal{L}_p$) stable closed-loop interconnection described by (1) where the controller, $\mathcal{S}_K$, is assumed to be a static system.

Let $\{\mathcal{S}_{K_n}\}$ be a sequence of controllers having state-space representation $(C_{K_n}, A_{K_n}, B_{K_n}, D_{K_n})$ with the property that there exists a stabilizable and detectable state-space representation for $\mathcal{S}_K$, (C_K, A_K, B_K, D_K), for which the following is satisfied:

$\lim_{n\to+\infty} A_{K_n} = A_K$, $\lim_{n\to+\infty} B_{K_n} = B_K$,
$\lim_{n\to+\infty} C_{K_n} = C_K$, $\lim_{n\to+\infty} D_{K_n} = D_K$ [1].

Then, the following equality holds,

$$\lim_{n\to+\infty} \mathcal{I}^{(1)}_{(\mathcal{S}_{K_n},\mathcal{S}_P)}(t_2,t_1) = \mathcal{I}^{(1)}_{(\mathcal{S}_K,\mathcal{S}_P)}(t_2,t_1) = 0\,,\, \forall t_1,t_2 \in \mathcal{R}^+\,.$$

Research is presently being conducted to evaluate if a notion of 'close to static', on the set of the stabilizing controllers of a plant, can be devised based on the principal function of the closed-loop stable interconnection.

It is also subject of current research, the development of a methodology, based on the theory presented in this work, for the synthesis of stabilizing static controllers.

[1] We remark that under these conditions, $\lim_{n\to+\infty} \mathrm{dgraph}(G_{K_n}, G_K) = 0$, where dgraph is the graph metric in **M(R(s))** (see, e.g. [13]).

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