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Direct Synthesis of First Order Controllers from Frequency Response Measurements

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Abstract— **This paper gives a constructive algorithm to obtain the entire set of stabilizing first order controllers for a given single-input single-output linear time invariant system. Unlike earlier results, a mathematical model such as the transfer function or a state space model is not required. Instead, only the frequency response (Nyquist-Bode data) and knowledge of the number of RHP poles of the plant are utilized to solve the problem. The method is of practical importance especially when mathematical models are not available or identification is not desirable. We also show that the method can be extended to include various performance requirements such as guaranteed gain and phase margins, and guaranteed H_∞ margin. An example is given for illustration.**

I. INTRODUCTION

In control system design, a plant to be controlled can be described analytically and/or non-analytically. Analytical descriptions include transfer functions and state space models. A typical non-analytical description is the frequency response of the plant. A certain set of rules that describes the behaviour of the system is also viewed as a non-analytical description of the plant.

In classical control design, a single controller such as PID or phase lead/lag is designed by loop shaping, from the frequency domain data (free of analytical model) or transfer function of the plant to be controlled. In modern and post modern control approaches, an optimal controller of high order is designed with respect to certain performance measures such as H_∞, H_2, and ℓ_1 based on analytical plant model. On the other hand, fuzzy-neural control provides model free approaches to design a controller, but they lack any guarantee of stability or performance [1].

There has been recent interest in the design of low order and fixed structure controllers [2], [3], [4], [5], a problem area generally ignored in the post 1960 control literature. Recently, a technique to characterize the entire set of first order stabilizing controllers for a given LTI plant has been developed [6]. Subsequently, the result was extended to solve the problem of determining the first order stabilizing controller set that satisfies given H_∞ performance requirements [7]. The results are based on the plant transfer function provided. However, in practice, there are many situations where such precise information is unavailable or is difficult to obtain. On the other hand, it is often the case that the frequency response of the plant can easily be measured experimentally. In fact, frequency response measurements of the plant is an essential part of controller design in classical control design such as the classical methods of designing PID controllers and their tuning rules (see [8], [9], [10]).

The present paper solves the problem of characterizing the entire set of first order stabilizing controllers for a given plant. Unlike the solution obtained in [6], our primary assumption is that there is no mathematical or analytical model available. The only information about the plant that is required to establish the solution proposed here are: (1) a reasonable range of the frequency response data (Nyquist-Bode data) representing the plant and (2) the number of RHP poles of the plant. From these, we give here a constructive algorithm to solve the problem. Once the stabilizing first order controller set is obtained, we also show how a subset can be found that satisfies additional performance requirements such as guaranteed gain and phase margins, and guaranteed H_∞ margin. Our solution avoids identifying the plant transfer function or state space model and this has implications for robustness which will be explored elsewhere.

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II. PRELIMINARIES

Consider the feedback configuration with an LTI plant and a first order controller as shown in Fig. 1.

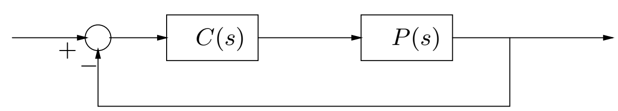

Fig. 1. A unity feedback system

Let the plant and the controller be

$$P(s) = \frac{N(s)}{D(s)} \quad \text{and} \quad C(s) = \frac{x_1 s + x_2}{s + x_3} \tag{1}$$

We make the following standing assumptions throughout the paper.

Assumption 1

1. The plant is stabilizable, that is, the polynomials $N(s)$ and $D(s)$ are coprime.
2. The plant has no $j\omega$ poles.
3. We assume that the *only* information available to the designer is:
a. Knowledge of the frequency response magnitude and phase, i.e.,

$$P(j\omega) = P_r(\omega) + jP_i(\omega), \quad \text{for } \omega \in [0, \infty). \tag{2}$$

b. Knowledge of the *number* of plant RHP poles, p_r.

III. DETERMINATION OF ROOT INVARIANT REGIONS

The root invariant regions in the parameter space (x_1, x_2, x_3) can be determined using the following result.

Theorem 1 *(Characterization of Root Invariant Region) Given the frequency domain data $P(j\omega)$ of the plant, the following two straight lines and one curve given below in the $x_1 - x_2$ plane, for each fixed x_3 completely partitions the first order controller parameter space (x_1, x_2, x_3) such that each and every open region bounded by them consists of parameters that correspond to closed-loop systems with an invariant number of open LHP poles.*

$$x_3 + x_2 P(0) = 0 \tag{3}$$

$$\begin{cases} x_1(\omega) = \frac{1}{|P(j\omega)|^2}\left(\frac{P_i(\omega)}{\omega}x_3 - P_r(\omega)\right) \\ x_2(\omega) = -\frac{1}{|P(j\omega)|^2}\left(P_r(\omega)x_3 + \omega P_i(\omega)\right) \end{cases} \tag{4}$$

$$1 + P(\infty)x_2 = 0. \tag{5}$$

Proof: Consider the characteristic polynomial

$$\Pi(s) = (s + x_3)D(s) + (x_1 s + x_2)N(s) \tag{6}$$

and

$$\begin{aligned}\Pi(j\omega) &= (j\omega + x_3)\left(D_e(-\omega^2) + j\omega D_o(-\omega^2)\right] \\ &\quad +(j\omega x_1 + x_2)\left[N_e(-\omega^2) + j\omega N_o(-\omega^2)\right] \\ &= R(\omega) + j\omega I(\omega)\end{aligned} \tag{7}$$

where

$$\begin{aligned}R(\omega) &= -\omega^2 x_1 N_o(-\omega^2) + x_2 N_e(-\omega^2) \\ &\quad +x_3 D_e(-\omega^2) - \omega^2 D_o(-\omega^2)\end{aligned} \tag{8}$$

$$\begin{aligned}I(\omega) &= x_1 N_e(-\omega^2) + x_2 N_o(-\omega^2) \\ &\quad +x_3 D_o(-\omega^2) + D_e(-\omega^2)\end{aligned} \tag{9}$$

Using the Boundary Crossing Theorem [11], we have following three conditions.

(A) Real root crossing condition:

$$\Pi(0) = x_3 D(0) + x_2 N(0) = 0 \tag{10}$$

and since $D(0) \neq 0$ from Assumption 1, equivalently,

$$x_3 + x_2 P(0) = 0. \tag{11}$$

(B) Complex root crossing condition: By setting (8) and (9) to be zero, we have

$$\begin{bmatrix} -\omega^2 N_o(-\omega^2) & N_e(-\omega^2) \\ N_e(-\omega^2) & N_o(-\omega^2) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_3 D_e(-\omega^2) + \omega^2 D_o(-\omega^2) \\ -x_3 D_o(-\omega^2) - D_e(-\omega^2) \end{bmatrix}. \tag{12}$$

We now consider the case when $|A(\omega)| \neq 0$ for all $\omega > 0$. The case when $|A(\omega)| = 0$ will be discussed later. Then

$$|A(\omega)| = \omega^2 N_o^2(-\omega^2) + N_e^2(-\omega^2) > 0, \tag{13}$$

for all $\omega > 0$. Simplifying the notations, we write

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{|A(\omega)|}\begin{bmatrix} N_o & -N_e \\ -N_e & -\omega^2 N_o \end{bmatrix}\begin{bmatrix} -x_3 D_e + \omega^2 D_o \\ -x_3 D_o - D_e \end{bmatrix}. \tag{14}$$

Note that

$$|P(j\omega)| = \frac{|N_e + j\omega N_o|}{|D_e + j\omega D_o|}$$

$$|P(j\omega)|^2 = \frac{N_e^2 + \omega^2 N_o^2}{D_e^2 + \omega^2 D_o^2} = \frac{|A(\omega)|}{D_e^2 + \omega^2 D_o^2}$$

and

$$\begin{aligned}P(j\omega) &= \frac{N_e D_e + \omega^2 N_o D_o}{D_e^2 + \omega^2 D_o^2} + j\frac{\omega(N_o D_e - N_e D_o)}{D_e^2 + \omega^2 D_o^2} \\ &= P_r(\omega) + jP_i(\omega).\end{aligned}$$

Then we have

$$\begin{aligned}x_1(\omega) &= \frac{1}{|A(\omega)|} \\ &\quad\left[(N_o D_e - N_e D_o)x_3 - (N_e D_e + \omega^2 N_o D_o)\right] \\ &= \frac{1}{|P(j\omega)|^2}\left(\frac{N_o D_e - N_e D_o}{D_e^2 + \omega^2 D_o^2}x_3 \right.\\ &\qquad \left. - \frac{N_e D_e + \omega^2 N_o D_o}{D_e^2 + \omega^2 D_o^2}\right) \\ &= \frac{1}{|P(j\omega)|^2}\left(\frac{P_i(\omega)}{\omega}x_3 - P_r(\omega)\right)\end{aligned} \tag{15}$$

and

$$\begin{aligned}x_2(\omega) &= \frac{1}{|A(\omega)|} \\ &\quad\left[-(N_e D_e + \omega^2 N_o D_o)x_3 - \omega^2(N_o D_e - N_e D_o)\right] \\ &= \frac{1}{|P(j\omega)|^2}\left(-\frac{N_e D_e + \omega^2 N_o D_o}{D_e^2 + \omega^2 D_o^2}x_3 \right.\\ &\qquad \left. - \frac{\omega^2(N_o D_e - N_e D_o}{D_e^2 + \omega^2 D_o^2}\right) \\ &= -\frac{1}{|P(j\omega)|^2}\Big(P_r(\omega)x_3 + \omega P_i(\omega)\Big).\end{aligned} \tag{16}$$

(C) Degree dropping condition: Let the $\deg[D(s)] = n$ and $\deg[N(s)] \leq n$. Let us also denote the n^{th} order coefficient of $D(s)$ and $N(s)$ to be d_n and n_n if nonzero. Then the degree dropping condition is given by

$$d_n + x_2 n_n = 0, \tag{17}$$

equivalently,

$$1 + \frac{n_n}{d_n}x_2 = 1 + P(\infty)x_2 = 0. \tag{18}$$

Finally, consider the case when $|A(\omega)| = 0$ for some $\omega \neq 0$. Let

$$|A(\omega)| = \omega^2 N_o^2(-\omega^2) + N_e^2(-\omega^2) = 0 \tag{19}$$

for some $\omega \neq 0$. Since $N_o^2(-\omega^2), N_e^2(-\omega^2) \geq 0$, (19) holds if and only if

$$N_o(-\omega^2) = N_e(-\omega^2) = 0. \quad (20)$$

¿From (12), it follows that

$$-x_3 D_e(-\omega^2) + \omega^2 D_o(-\omega^2) = 0$$
$$-x_3 D_o(-\omega^2) - D_e(-\omega^2) = 0$$

and equivalently,

$$\omega^2 D_o^2(-\omega^2) + D_e^2(-\omega^2) = 0. \quad (21)$$

Since $D_o^2(-\omega^2), D_e^2(-\omega^2) \geq 0$, (21) holds if and only if

$$D_o(-\omega^2) - D_e(-\omega^2) = 0. \quad (22)$$

¿From (20) and (22), it follows that (19) has a solution for $\omega \neq 0$ if and only if $D(s)$ and $N(s)$ have a common factor $s^2 + \omega^2$ and this is ruled out by Assumption 1. ■

Once the first-order controller parameter space is partitioned into root invariant regions, it is necessary to select a point from each region and test the stability of the corresponding closed-loop system. The stability test can easily be done by plotting the Nyquist plot and from the knowledge of the number of RHP poles of the plant.

Example 1 For illustration, we have collected the frequency domain (Nyquist-Bode) data of the stable plant used in [6] and refer to

$$\mathbf{P}(j\omega) = \{P(j\omega) : \omega \in (0, 10) \text{ sampled every } 0.01\}.$$

The Nyquist plot of the plant obtained is shown in Figs. 2 and 3.

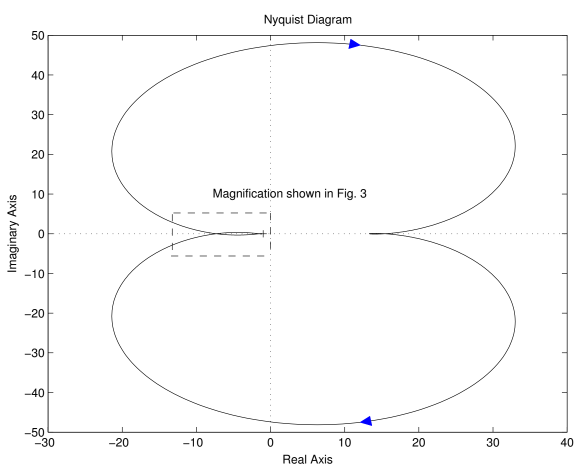

Fig. 2. Nyquist plot of $P(j\omega)$

From the data $\mathbf{P}(j\omega)$, we have $P(0) = 13.333$ and $P(\infty) = 0$. Then it is easy to see that the straight line (5) is not applicable. After fixing $x_3 = 0.2$, the data points representing the straight line in (3) and the curve in (4) are depicted in Fig. 4. By testing a point for each root invariant region, we obtained the stabilizing regions shown in Fig. 4 which are identical to those in the example in [6].

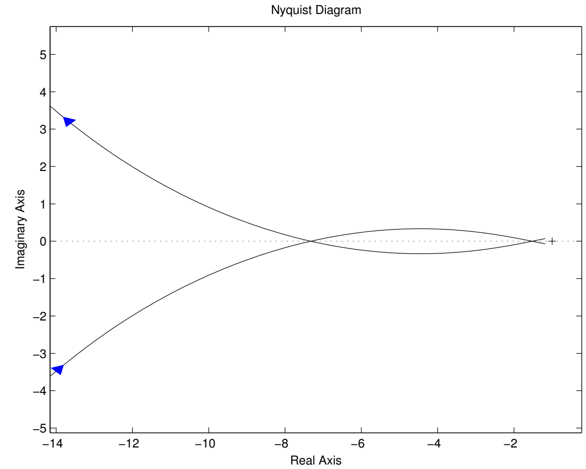

Fig. 3. Nyquist plot of $P(j\omega)$ (Area magnified)

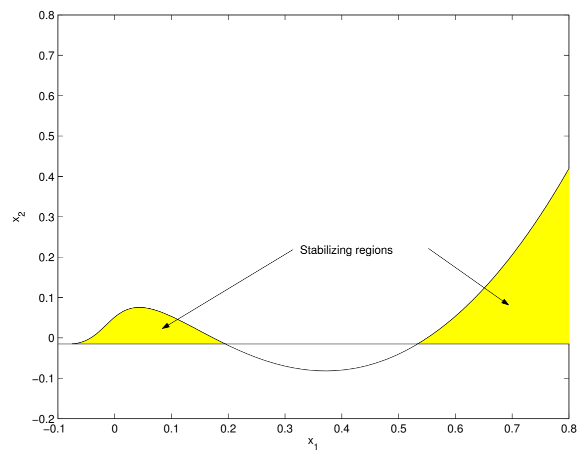

Fig. 4. Stabilizing regions for $x_3 = 0.2$

IV. STABILITY SUBSETS ACHIEVING PERFORMANCE SPECIFICATIONS

In control system design, additional requirements beyond closed-loop stability are imposed. These additional requirements include guaranteed gain and phase margin, and acceptable bounds on the H_∞ norm of one or more closed loop transfer function. Such a problem is equivalent to the simultaneous stabilization problem of the original plant $P(s)$ and one or more of the family of real or complex systems with transfer function $\mathbf{P}_c(s)$ whose "frequency responses" are defined as follows.

Guaranteed Gain Margin Problem:

$$\mathbf{P}_c(j\omega) := \{KP(j\omega) \quad : K \in [1, K^*]\} \quad (23)$$

where $K^* \geq 1$ is the required gain margins. If conditional gain margin is required, it can be treated similarly.

Guaranteed Phase Margin Problem:

$$\mathbf{P}_c(j\omega) := \left\{e^{j\theta} P(j\omega) : \theta \in [0, \theta^*]\right\} \quad (24)$$

where $\theta^* \geq 0$ is the required phase margins.

Guaranteed H_∞ Margin Problem: For the case of the sensitivity function $S(s)$, that is,

$$\|W(s)S(s)\|_\infty < \gamma,$$

$$\mathbf{P}_c(j\omega) := \left\{ P(j\omega) \left[\frac{1}{1+\frac{1}{\gamma}e^{j\theta}W(j\omega)} \right] : \theta \in [0, 2\pi] \right\}. \quad (25)$$

For the case of the complementary sensitivity function $T(s)$, that is,

$$\|W(s)T(s)\|_\infty < \gamma,$$

$$\mathbf{P}_c(j\omega) := \left\{ P(j\omega) \left[1+\frac{1}{\gamma}e^{j\theta}W(j\omega) \right] : \theta \in [0, 2\pi] \right\}. \quad (26)$$

As seen above, the family $\mathbf{P}_c(s)$ is in general a complex family. Therefore, for $P_c(s, \theta)$ with a fixed θ, the conditions in (3) and (5) become respectively

$$x_3 + x_2 P_r^c(0) = 0, \quad x_3 + x_2 P_i^c(0) = 0 \quad (27)$$

and

$$1 + x_2 P_r^c(\infty) = 0, \quad 1 + x_2 P_i^c(\infty) = 0 \quad (28)$$

where

$$P_c(j\omega) := P_r^c(\omega) + jP_i^c(\omega), \omega \in [-\infty, *\infty].$$

Example 2 To verify the technique, we consider the example used in [7]. We collect

$$\mathbf{P}(j\omega) = \{P(j\omega) : \omega \in (-10, 10) \text{ sampled every } 0.01\}.$$

We also have the knowledge that the plant has one RHP pole. We first find the stabilizing region in the controller parameter space. As we did in the previous example, we let $x_3 = 2.5$. This is shown in Fig. 5.

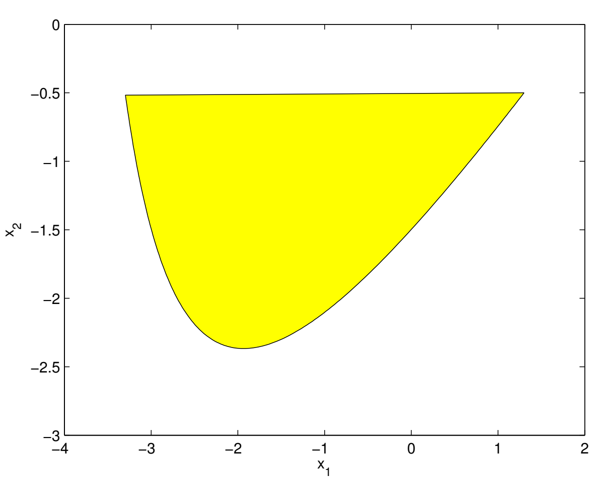

Fig. 5. Stabilizing regions for $x_3 = 2.5$

We now consider the problem of determining the entire set of first order stabilizing controllers satisfying the required closed-loop performance described by the requirement on the H_∞ norm of the weighted complementary sensitivity function:

$$\|W(j\omega)T(j\omega)\|_\infty < \gamma, \quad \text{for all } \omega$$

As shown above, this is equivalent to the problem of simultaneously stabilizing the complex family in (26) as well as the original plant $P(s)$. In this problem, we let $\gamma = 1$. On the top of the stabilizing region shown in Fig. 5, stabilizing sets for the complex plant families $\mathbf{P}_c(j\omega, \theta)$ for $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi$ are plotted in Fig. 6.

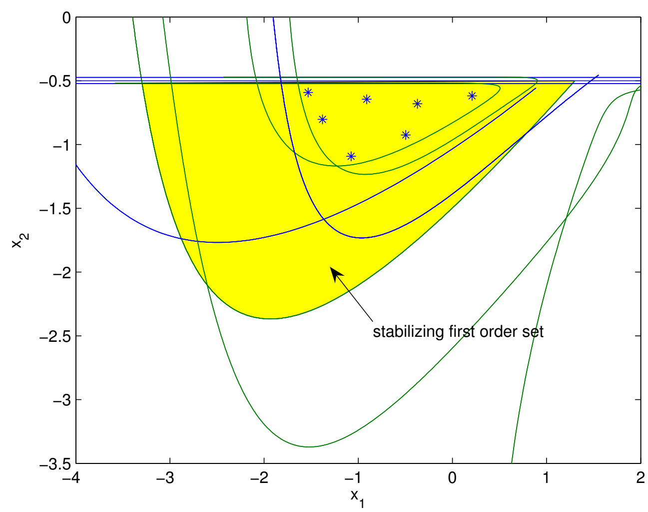

Fig. 6. First order controllers satisfying H_∞ performance

To verify, a number of points inside the performance region, construct the corresponding controllers, and Nyquist plots of $W(s)T(s)$ have been plotted as shown in Fig. 7. These points are shown as "*" in Fig. 6. We observe from the Nyquist plots in Fig. 7, every test set satisfies the H_∞ performance requirement.

V. CONCLUDING REMARKS

A new method of completely characterizing the entire stabilizing first order controller set for a given system is developed. As in [6], the complete set was obtained in (x_1, x_2) parameter space by analytically determining boundaries, for each fixed x_3. The difference from [6] is that we have developed the expression of the boundaries directly in terms of frequency response data $P(j\omega)$ of the plant rather than from parameters of the plant model (transfer function or state space). Thus, the procedure does not require identification or knowledge of the mathematical model of the plant. Instead, the frequency response of the plant and the knowledge of the number of RHP poles of the plant are the *only* information required for the design. This is the kind of design information used in

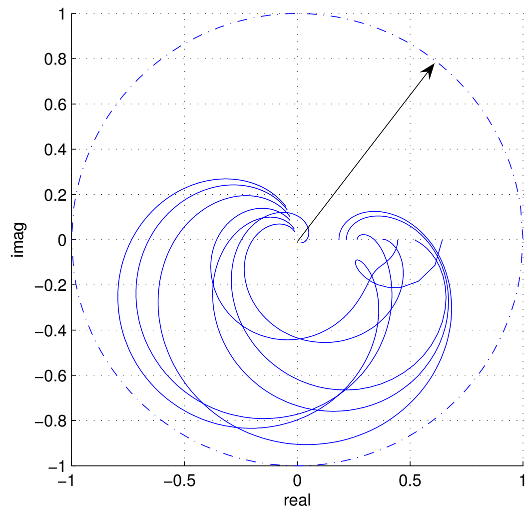

Fig. 7. Nyquist plots of $W(s)T(s)$ with selected controllers

classical control. However our results are distinct from classical approaches in that the complete set of solutions is found from this data, It is in that spirit that we have achieved synthesis rather than mere design. This allows the quasi-analytical determination of complete sets of solutions achieving multiple specifications [12] with a first order controller. These topics and the extension to more general fixed order structures are areas of research worth pursuing. The significant fact that synthesis can be done without model buildup or identification would generally appeal to industrial control designers. The implications on robustness and deviations from the assumptions are important topics of research.

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