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On Observability and Reachability in a Class of Discrete-Time Switched Linear Systems

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***Abstract*— We study the class of switched linear systems with constant dynamics, i.e., with switching in only the measurement or control matrices (which correspond to plants with several sensory and actuation modes), and establish that the length of the shortest mode path achieving observability or reachability, if one exists, is at most the square of the dimension of the state space. This not only proves that observability/reachability is decidable for such systems, but also gives a direct way of finding the observable/reachable mode path.**

I. Result

In this paper we are concerned with systems of the form

$$\begin{array}{l} x_{k+1} = Ax_k + B(\theta_k)u_k \\ y_k = C(\theta_k)x_k \end{array} \tag{1}$$

where x_k, y_k and u_k are in $\mathbb{R}^n$, $\mathbb{R}^p$ and $\mathbb{R}^m$, respectively, where A, $C(\cdot)$ and $B(\cdot)$ are real matrices of compatible dimensions, and where the discrete *mode* $\theta_k \in \bar{s} \triangleq \{1, \ldots, s\}$, so that $B(\theta_k) \in \{B(1), \ldots, B(s)\}$ and $C(\theta_k) \in \{C(1), \ldots, C(s)\}$, which are fixed sets of matrices. We ask whether there exists a *path*, i.e., a mode string $\theta \triangleq \theta_1 \cdots \theta_N$ over $\bar{s}$, such that the matrix

$$\mathcal{C}(\theta) \triangleq \begin{pmatrix} B(\theta_N) & AB(\theta_{N-1}) & \ldots & A^{N-1}B(\theta_1) \end{pmatrix}$$

has full (row) rank. We refer to such paths as *reachable paths*, since the reachable set of (1) under such paths is the whole state space. By duality, we end up also considering the equivalent problem of asking whether there exists an *observable path*, i.e., a path $\theta = \theta_1 \cdots \theta_N$ such that

$$\mathcal{O}(\theta) \triangleq \begin{pmatrix} C(\theta_1) \\ \vdots \\ C(\theta_N)A^{N-1} \end{pmatrix}$$

has full (column) rank, making it possible to observe the initial state x_1 in (1) under the path θ. We will refer to the length of the minimum-length observable (resp. reachable) path of some system as its *index* of observability (resp. reachability). The purpose of this paper is to establish the following result.

Main Result. *If (1) is observable (resp. reachable) then its index of observability (resp. reachability) is at most n^2.* $\diamondsuit$

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This result clearly implies that reachability (resp. observability) is decidable in our class of switched linear systems, since establishes that it suffices to check whether there exists a reachable (resp. observable) path of length n^2. After giving a short overview of the background to this problem in Section II, we prove our Main Result in Section III.

II. Background

We first note that most of the existing literature is focused on the reachability problem, and we will report it as such. It should also be stressed that our systems (1) form a special class of switched linear systems

$$x_{k+1} = A(\theta_k)x_k + B(\theta_k)u_k, \tag{2}$$

where the A matrix also undergoes switching, so that $A(\theta_k) \in \{A(1), \ldots, A(s)\}$.

The problem of determining the existence of a reachable path for (2) has eluded researchers for a long time, and, to the best of our knowledge, the only efforts in studying the existence of reachable paths for (2) can be found in [2], [3], [4], [5], [6]. First, in a series of papers [2], [3], [6], Stanford and Conner studied the structure of the reachable set of switched linear systems, and showed that it was a subspace of $\mathbb{R}^n$ in *reversible* (i.e., with invertible $A(\theta_k)$ matrices) systems [6], and, in general, a union of arbitrarily many maximal subspaces in [3]. They also, in parallel, tried to find upper bounds on the index of reachability depending only on s, n and m. Unfortunately, they only obtained partial results for special classes, i.e., for special combinations of n, m and s and for classes of switched linear systems with rank constraints on the parameters $B(i)$ and $A(i)$. In short, it is still unknown whether the indexes of reachability are bounded, even for the class of reversible systems. However, the existence of a reachable path in reversible systems has recently been shown to be decidable in [4], [5], although not constructively (no upper bounds on the indexes were given). More precisely, it was shown that a reversible system was reachable if and only if the minimal subspace invariant with respect to every $A(i)$, $1 \leq i \leq s$, and containing $\mathrm{Im}(B(1)) + \cdots + \mathrm{Im}(B(s))$, equaled the whole state space $\mathbb{R}^n$. This problem is clearly decidable since the condition above can easily be recast as a finite matrix rank problem (see [4]). Therefore, since A in (1) is not necessarily invertible, this paper is the first to establish the decidability of reachability and observability for our class of systems. Note, however, that the general problem remains wide open.

It is beyond the scope of this paper to provide an exhaustive literature survey on observability and controllability in switched and hybrid systems, which has become plethoric. However, closely related problems include the continuous-time version of the problem we consider in this paper, which was shown to be decidable in [5], [7], and the problem of determining the existence of an integer N such that *every* path of length N is reachable, which is the "universal" version of the "existential" problem considered here. This problem has recently been shown to be decidable in [5] and [1], where, furthermore, the indexes of *pathwise controllability* were shown to be bounded by numbers depending only on s and n.

III. Proof

We will prove the observability version of the Main Result, since it makes the presentation more straightforward. We first state, without proof, the following immediate corollary to the Cayley-Hamilton Theorem.

Lemma 1. *If $A \in \mathbb{R}^{n\times n}$, then whenever $i \geq n$, there exists a set of real numbers $\{\alpha_i\}_{j=1}^n$ such that $A^i = \sum_{j=1}^n \alpha_j A^{j-1}$.* $\diamondsuit$

Next, in order to ease the discussion, let us establish some notation. Let $\mathcal{R}(M)$ denote the row range space of a matrix M and $\rho(M)$ its rank. $|\theta|$ denotes the length of θ. θ^1 and θ^2 being paths of length N_1 and N_2 respectively, $\theta^1\theta^2$ denotes their concatenation $\theta_1^1\theta_2^1\cdots\theta_{N_1}^1\theta_1^2\theta_2^2\cdots\theta_{N_2}^2$. Furthermore, $\theta^{(q)}$ is the path θ concatenated with itself $q-1$ times. Given a path θ, $\theta_{[i,j]}$ is its substring (or infix) $\theta_i\theta_{i+1}\cdots\theta_j$. By convention, we let $\theta_{[i,i-1]} = \epsilon$, the null string, for all $1 \leq i \leq |\theta|$. Finally, let $\mathcal{O}(\theta)_{[i,j]}$ denote the submatrix of $\mathcal{O}(\theta)$ constituted by rows i through j of $\mathcal{O}(\theta)$:

$$\mathcal{O}(\theta)_{[i,j]} \triangleq \begin{pmatrix} C(\theta_i)A^{i-1} \\ \vdots \\ C(\theta_j)A^{j-1} \end{pmatrix},$$

and note that $\mathcal{O}(\theta)_{[i,j]} = \mathcal{O}(\theta_{[i,j]})A^{i-1}$, for all i and j such that $1 \leq i \leq j \leq |\theta|$. Let $\mu_k(\theta) \triangleq \rho(\mathcal{O}(\theta_{[1,k]}))$, and note that $k \mapsto \mu_k(\theta)$ is nondecreasing with at most n jumps. Let $N(\theta)$ be the number of jumps of $\mu_k(\theta)$, and let $i_l(\theta)$ be the time of the l^{th} jump of $\mu_k(\theta)$. To simplify matters in the sequel, we let $i_0(\theta) \triangleq 0$. Finally, let $I(\theta) \triangleq \{l \mid i_{l+1}(\theta) - i_l(\theta) > n\}$ index the set of consecutive jumps spaced by more than n time instants. If $I(\theta) \neq \emptyset$, let $q(\theta) \triangleq \min I(\theta)$ denote the position (not the time) of the first jump preceding the jump following it by more than n time units. Note that $I(\theta) \subset \{1,\ldots,n\}$ and $q(\theta) \leq n$ for any θ, since $\mu_k(\theta)$ has at most n jumps.

Lemma 2. *If θ is observable and $I(\theta) \neq \emptyset$, then, letting $l = q(\theta)$,*

$$\theta' = \theta_{[1,i_l(\theta)]}\theta_{i_{l+1}(\theta)}^{(n)}\theta_{[i_l(\theta)+n+1,|\theta|]}$$

is observable with either $I(\theta') = \emptyset$ or $q(\theta') > q(\theta)$. $\diamondsuit$

Proof: First, $l \in I(\theta)$ implies that $i_{l+1}(\theta) - i_l(\theta) > n$, which, by Lemma 1, gives real numbers $\{\alpha_i\}_{j=1}^n$ such that $A^{i_{l+1}(\theta)-i_l(\theta)-1} = \sum_{j=1}^n \alpha_j A^{j-1}$, thus that $C(\theta_{i_{l+1}(\theta)})A^{i_{l+1}(\theta)-i_l(\theta)-1} = \sum_{j=1}^n \alpha_j C(\theta_{i_{l+1}(\theta)})A^{j-1}$, which, right-multiplied by $A^{i_l(\theta)}$ on both sides, gives $C(\theta_{i_{l+1}(\theta)})A^{i_{l+1}(\theta)-1} = \sum_{j=1}^n \alpha_j C(\theta_{i_{l+1}(\theta)})A^{i_l(\theta)+j-1}$, which is equivalent to $\mathcal{R}(C(\theta_{i_{l+1}(\theta)})A^{i_{l+1}(\theta)-1}) \subset \mathcal{R}(\mathcal{O}(\theta')_{[i_l(\theta)+1,i_l(\theta)+n]})$. By definition of θ', we furthermore have $\mathcal{R}(\mathcal{O}(\theta')_{[1,i_l(\theta)]}) = \mathcal{R}(\mathcal{O}(\theta)_{[1,i_l(\theta)]})$. Moreover, $\mu_k(\theta)$ having a jump at $i_{l+1}(\theta)$ implies that $\mathcal{R}(C(\theta_{i_{l+1}(\theta)})A^{i_{l+1}(\theta)-1}) \not\subset \mathcal{R}(\mathcal{O}(\theta)_{[1,i_l(\theta)]})$. Combining the last three claims, we get $\mathcal{R}(\mathcal{O}(\theta')_{[i_l(\theta)+1,i_l(\theta)+n]}) \not\subset \mathcal{R}(\mathcal{O}(\theta')_{[1,i_l(\theta)]})$, which, since $i_j(\theta') = i_j(\theta)$ for $j \leq l$ because $\theta'_{[1,i_l(\theta)]} = \theta'_{[1,i_l(\theta)]}$, implies that $i_{l+1}(\theta') - i_l(\theta') \leq n$, proving that either $I(\theta') = \emptyset$ or $q(\theta') > l = q(\theta)$.

It now remains to show that $\mathcal{R}(\mathcal{O}(\theta)) \subset \mathcal{R}(\mathcal{O}(\theta'))$, which, since θ is observable, resulting in $\rho(\mathcal{O}(\theta)) = n$, would imply that $\rho(\mathcal{O}(\theta')) = n$, hence that θ' is observable as well. To this end, we first note that, by definition of θ', $\theta_{[1,i_l(\theta)]} = \theta'_{[1,i_l(\theta)]}$ and $\theta_{[i_l(\theta)+n+1,|\theta|]} = \theta'_{[i_l(\theta)+n+1,|\theta'|]}$, which implies that $\mathcal{R}(\mathcal{O}(\theta)_{[1,i_l(\theta)]}) = \mathcal{R}(\mathcal{O}(\theta')_{[1,i_l(\theta)]}) \subset \mathcal{R}(\mathcal{O}(\theta'))$ and that $\mathcal{R}(\mathcal{O}(\theta)_{[i_l(\theta)+n+1,|\theta|]}) = \mathcal{R}(\mathcal{O}(\theta')_{[i_l(\theta)+n+1,|\theta'|]}) \subset \mathcal{R}(\mathcal{O}(\theta'))$. Therefore, all that remains to be shown is that $\mathcal{R}(\mathcal{O}(\theta)_{[i_l(\theta)+1,i_l(\theta)+n]}) \subset \mathcal{R}(\mathcal{O}(\theta'))$, which follows from $\mathcal{R}(\mathcal{O}(\theta)_{[i_l(\theta)+1,i_l(\theta)+n]}) \subset \mathcal{R}(\mathcal{O}(\theta)_{[1,i_l(\theta)]})$, (which, itself, is due to $i_{l+1}(\theta) - i_l(\theta) > n$) combined with $\mathcal{R}(\mathcal{O}(\theta)_{[1,i_l(\theta)]}) \subset \mathcal{R}(\mathcal{O}(\theta'))$. $\square$

Proof of Main Result: Let θ be observable. If $I(\theta) = \emptyset$, then we are done, since $i_{N(\theta)} \leq n^2$, and therefore $\theta_{[1,n^2]}$ is observable of length n^2. Otherwise, let θ' be given by lemma 2. Then θ' is observable, and furthermore, either $I(\theta')$ is empty, in which case we are done, or $q(\theta') > q(\theta)$, in which case we can, once again, invoke Lemma 2 and get an observable θ'' such that $q(\theta'') > q(\theta') > q(\theta)$ or $I(\theta'') = \emptyset$. Finally, since $q(\cdot)$ is bounded by n, the previous procedure will eventually end, yielding an observable path λ such that $I(\lambda) = \emptyset$, thus that $\lambda_{[1,n^2]}$ is observable. $\square$

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