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Stability of Discrete Fuzzy Systems with Nontriangular Membership Functions

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***Abstract*[1]---** We present new results for the stability analysis of discrete dynamic fuzzy systems with nontriangular membership functions. We use arguments similar to those applied to the stability analysis of systems with triangular membership functions, including: the concept of fuzzy positive definite and fuzzy negative definite functions. We use a fuzzy Lyapunov function candidate with triangular membership functions to derive the new stability conditions for systems with nontriangular membership functions. We apply our results to two numerical examples.

I. Introduction

The stability analysis of fuzzy systems has been the subject of extensive research (see the extensive review of [6]). Many of these results are only applicable to the stability analysis of a particular controller but some provide general stability conditions.

The best known stability condition for fuzzy systems is the common Lyapunov condition of Tanaka and Sugeno [7]. The use of these conditions was greatly simplified by the possibility of testing them using linear matrix inequalities (LMIs) [8]. However, these conditions are only sufficient and can be restrictive in some situations.

Another Lyapunov-like fuzzy analysis and synthesis approach was developed by Margaliot and Langholz [3]. Their approach is the "computing with words" version of Lyapunov stability analysis. The approach is very promising especially for systems with low-order dynamics whose behavior is known qualitatively rather than described by a mathematical model.

Another Lyapunov-like approach uses fuzzy Lyapunov function candidates analogously to traditional Lyapunov stability analysis [4], [5]. If the function is fuzzy positive definite and its changes along the system trajectories are fuzzy negative definite, then we conclude asymptotic stability. If the changes are negative semidefinite we conclude stability in the sense of Lyapunov. However, the results of [4] and [5] are only applicable to systems with triangular membership functions.

In this paper, we use the concept of fuzzy positive definite function and fuzzy negative functions to obtain general fuzzy Lyapunov stability results for systems with nontriangular membership functions. However, we restrict the membership functions for the definite functions used in stability testing to the triangular type. We obtain stability results for Takagi-Sugeno-Kang (TSK) discrete dynamic systems as well as a stability result for Mamdani systems.

The paper is organized as follows. In Section II, we state the main properties of the fuzzy systems examined in the paper. In Section III, we review the definitions of fuzzy definite functions and derive our main stability results. In Section IV, we apply our methodology to a simple example. Section V summarizes and comments on our main results.

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II. Fuzzy Systems and Their Properties

For a systematic stability analysis, we start by reviewing Sugeno's classification of fuzzy systems [6]. He classified fuzzy systems into three types. The first type is that first introduced by Mamdani with fuzzy rules of the form

$$R_{i_1 \ldots i_n} : IF\ \mathbf{x}\ is\ \mathbf{A}_{i_1 \ldots i_n}\ THEN\ y^{i_1 \ldots i_n}\ is\ H^{i_1 \ldots i_n}$$
$$\mathbf{x} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^T, \mathbf{A}_{i_1 \ldots i_n} = \begin{bmatrix} A_1^{i_1} & \cdots & A_n^{i_n} \end{bmatrix}^T, \quad (1)$$
$$i_j = 1, \ldots, N_j,\ j = 1, \ldots, n$$

where $A_j^{i_j}$ and $H^{i_1 \ldots i_n}$ are fuzzy sets. The second and third are called TSK systems. They typically have rules of the form (1) but with consequent in the form of a mathematical function. A popular form of TSK systems has constant consequents of the form

$$\mathbf{h}_{i_1 \ldots i_n} = \begin{bmatrix} h_1^{i_1 \ldots i_n} & \cdots & h_n^{i_1 \ldots i_n} \end{bmatrix}^T, \quad (2)$$
$$i_j = 1, \ldots, N_j,\ j = 1, \ldots, n$$

where $\mathbf{h}_{i_1 \ldots i_n}$ is a vector of singletons. A more general form includes (2) as a special case and has the consequents

$$\mathbf{f}_{i_1 \ldots i_n}(\mathbf{x}) = \begin{bmatrix} f_1^{i_1 \ldots i_n}(\mathbf{x}) & \cdots & f_n^{i_1 \ldots i_n}(\mathbf{x}) \end{bmatrix}^T, \quad (3)$$
$$i_j = 1, \ldots, N_j,\ j = 1, \ldots, n$$

where $\mathbf{f}_{i_1 \ldots i_n}(\mathbf{x})$ are functions of x_i, i = 1, …, n. Typically, the consequent functions are affine or linear but Sugeno's formulation includes more complex forms.

The fuzzy systems analyzed in this paper comprise four principal components:

a) A fuzzifier that maps to normal and complete fuzzy sets.
b) A complete fuzzy rule base of the form (1), with fuzzy, constant, linear or nonlinear consequents.
c) A fuzzy inference engine with the T-norm in fuzzy implication and product inference.
d) A weighted-average defuzzifier.

For TSK systems the output is then of the form

$$\mathbf{y}=\frac{\sum_{i_1=1}^{N_1}\cdots\sum_{i_n=1}^{N_n}\mathbf{f}_{i_1\ldots i_n}(\mathbf{x})\prod_{j=1}^{n}\mu_{A_j^{i_j}}\left(x_j(k)\right)}{\sum_{i_1=1}^{N_1}\cdots\sum_{i_n=1}^{N_n}\prod_{j=1}^{n}\mu_{A_j^{i_j}}\left(x_j(k)\right)} \quad \textbf{(4)}$$

Researchers have used a variety of membership functions with the most popular being triangular or Gaussian [8], [9], [10]. For Gaussian membership functions, we have

$$\prod_{j=1}^{n}\mu_{A_j^{i_j}}\left(x_j(k)\right)=\exp\left\{-\frac{1}{2}\sum_{j=1}^{n}\left(\frac{x_j(k)-m_{j,i_j}}{\sigma_{ji_j}}\right)^2\right\}$$

$$=\exp\left\{-\frac{1}{2}\Delta\mathbf{x}_{i_1\ldots i_n}(k)^T S_{i_1\ldots i_n}^{-1}\Delta\mathbf{x}_{i_1\ldots i_n}(k)\right\}$$

$$\Delta\mathbf{x}_{i_1\ldots i_n}(k)=\mathbf{x}(k)-\mathbf{m}_{i_1\ldots i_n}$$

$$\mathbf{x}(k)=\left[x_1(k)\quad x_2(k)\quad\cdots\quad x_n(k)\right]^T$$

$$\mathbf{m}_{i_1\ldots i_n}=\left[m_{1,i_1}\quad m_{2,i_2}\quad\cdots\quad m_{n,i_n}\right]^T$$

$$S_{i_1\ldots i_n}=diag\left\{\sigma_{1i_1},\cdots,\sigma_{ni_n}\right\} \quad \textbf{(5)}$$

We often assume that the membership functions at a given point are practically determined by memberships in two fuzzy sets only with other membership values negligible. This is typically the case for triangular membership functions but is also approximately true for Gaussian membership. Under this assumption, the output of (4) reduces to

$$\mathbf{y}=\frac{\sum_{i_1=l_1}^{l_1+1}\cdots\sum_{i_n=l_n}^{l_n+1}\mathbf{f}_{i_1\ldots i_n}(\mathbf{x})\prod_{j=1}^{n}\mu_{A_j^{i_j}}\left(x_j(k)\right)}{\sum_{i_1=l_1}^{l_1+1}\cdots\sum_{i_n=l_n}^{l_n+1}\prod_{j=1}^{n}\mu_{A_j^{i_j}}\left(x_j(k)\right)} \quad \textbf{(6)}$$

and the consistent membership functions satisfy

$$\mu_{A_j^{i_j}}(x_j)=1-\mu_{A_j^{i_j+1}}(x_j) \quad \textbf{(7)}$$

Under the condition (7), the denominator of the output (6) can be shown to be unity.
A fuzzy system is called *dynamic* if its output is determined from the history of its inputs. Dynamic TSK fuzzy systems use fuzzy rules of the form

$$R_{i_1i_2\ldots i_n}: IF\ \mathbf{x}(k)\ is\ \mathbf{A}_{i_1\cdots i_n}\ THEN\ \mathbf{x}_{i_1\ldots i_n}(k+1)=\mathbf{f}_{i_1\cdots i_n}\left[\mathbf{x}(k)\right]$$

$$\mathbf{x}(k)=\left[x_1(k)\quad x_2(k)\quad\cdots\quad x_n(k)\right]^T$$

$$\mathbf{x}_{i_1i_2\ldots i_n}=\left[x_1^{i_1}\quad x_2^{i_2}\quad\cdots\quad x_n^{i_n}\right]^T$$

$$\mathbf{A}_{i_1i_2\ldots i_n}=\left[A_1^{i_1}\quad A_2^{i_2}\quad\cdots\quad A_n^{i_n}\right]^T$$

$$\mathbf{f}_{i_1\cdots i_n}=\left[f_1^{i_1i_2\ldots i_n}\quad f_2^{i_1i_2\ldots i_n}\quad\cdots\quad f_n^{i_1i_2\ldots i_n}\right]^T$$

$$i_j=1,2,\ldots,N_j,\ j=1,2,\ldots,n \quad \textbf{(8)}$$

III. Preliminary Results

In this section, we state some results from [4] that are needed in the paper and generalize others. We only prove new results. In [4], it was shown that the output of a fuzzy system **x**(k) can be represented as a fuzzy system with triangular membership functions and constant consequents. The original fuzzy system can be Mamdani or TSK and need not have triangular membership functions. We state this result without proof.

Lemma 1: If (i) **x** is the input to a fuzzy system, and (ii) $\mu_{A_j^{i_j}}\left(x_j\right)$ is the membership function of the triangular fuzzy set $A_j^{i_j}$, then

$$\mathbf{x}=\sum_{l_1=i_1}^{i_1+1}\sum_{l_2=i_2}^{i_2+1}\cdots\sum_{l_n=i_n}^{i_n+1}\mathbf{e}_{l_1l_2\ldots l_n}\prod_{j=1}^{n}\mu_{A_j^{i_j}}\left(x_j\right) \quad \textbf{(9)}$$

where $\mathbf{e}_{i_1i_2\ldots i_n}=\left[e_1^{i_1}\quad e_2^{i_2}\quad\cdots\quad e_n^{i_n}\right]^T$ are the set centers.

In our stability analysis we require the concept of the absolute value of a fuzzy system [4].

Definition 1: Let ***S*** be a fuzzy system, then the absolute value **|** ***S*** | is a system whose output is the absolute value of the output of ***S.***

The next lemma relates the absolute value fuzzy system |***S*** | to a system with consequents equal to the absolute values of the consequents of ***S***. A special case of this result for systems with triangular membership functions was presented in [4].

Lemma 2: If ***S*** and | ***S*** | are fuzzy systems, with the same fuzzy sets defined over the same domain, and rule bases given by:

$$\begin{aligned}R_{i_1i_2\ldots i_n}^1 &: IF\ \mathbf{x}\ is\ \mathbf{A}_{i_1i_2\ldots i_n}\ THEN\ \mathbf{y}_{i_1i_2\ldots i_n}^1=\mathbf{f}_{i_1\ldots i_n}(\mathbf{x})\\ R_{i_1i_2\ldots i_n}^2 &: IF\ \mathbf{x}\ is\ \mathbf{A}_{i_1i_2\ldots i_n}\ THEN\ \mathbf{y}_{i_1i_2\ldots i_n}^2=\left|\mathbf{f}_{i_1\ldots i_n}(\mathbf{x})\right|\end{aligned} \quad \textbf{(10)}$$

respectively, then the outputs of the two systems are related by

$$\left|\mathbf{y}^1\right|\le\mathbf{y}^2 \quad \textbf{(11)}$$

Proof: Using Definition 1 and (4), the output of |***S***| can be written as

$$\left|\mathbf{y}^1\right| = \left|\frac{\sum_{i_1=1}^{N_1}\cdots\sum_{i_n=1}^{N_n}\mathbf{f}_{i_1\ldots i_n}(\mathbf{x})\prod_{j=1}^{n}\mu_{A_j^{i_j}}\left(x_j(k)\right)}{\sum_{i_1=1}^{N_1}\cdots\sum_{i_n=1}^{N_n}\prod_{j=1}^{n}\mu_{A_j^{i_j}}\left(x_j(k)\right)}\right|$$

$$\le \frac{\sum_{i_1=1}^{N_1}\cdots\sum_{i_n=1}^{N_n}\left|\mathbf{f}_{i_1\ldots i_n}(\mathbf{x})\right|\prod_{j=1}^{n}\mu_{A_j^{i_j}}\left(x_j(k)\right)}{\sum_{i_1=1}^{N_1}\cdots\sum_{i_n=1}^{N_n}\prod_{j=1}^{n}\mu_{A_j^{i_j}}\left(x_j(k)\right)}$$

$$\equiv \mathbf{y}^2$$

•

The following lemma from [4] governing the absolute value vector is used in the proof of our stability conditions and is stated without proof.

Lemma 3: If (i) **x** is the input to a fuzzy system, (ii) $\mu_{A_j^{i_j}}\left(x_j\right)$ is the membership function of the triangular fuzzy set $A_j^{i_j}$, and (iii) $|\mathbf{x}| = \left[|x_1| \quad |x_2| \quad \cdots \quad |x_n|\right]^T$, then

$$|\mathbf{x}| = \sum_{l_1=i_1}^{i_1+1}\sum_{l_2=i_2}^{i_2+1}\cdots\sum_{l_n=i_n}^{i_n+1}\left|\mathbf{e}_{l_1 l_2\ldots l_n}\right|\prod_{j=1}^{n}\mu_{A_j^{l_j}}(x_j) \qquad \textbf{(12)}$$

The combination of fuzzy systems can be equivalent to a single fuzzy system. This idea plays an important role in modeling and stability analysis. One such combination is the sum/difference system for a set of fuzzy systems.

Definition 2: Let $\boldsymbol{S}_j$, $j = 1, \ldots, m$, be fuzzy systems, with the same fuzzy sets defined over the same domain. Then their sum/difference $\boldsymbol{S}_1 \pm\cdots\pm \boldsymbol{S}_m$ is a system whose output is given by

$$\mathbf{y} \equiv \mathbf{y}^1 \pm\cdots\pm \mathbf{y}^m \qquad \textbf{(13)}$$

where $\mathbf{y}^j$ is the output of the j^{th} fuzzy system.

The next lemma allows us to represent the sum or difference of fuzzy TSK systems as a single equivalent fuzzy system. A version for constant consequents was derived in [4].

Lemma 4: Consider the sum or difference of L TSK fuzzy systems $\boldsymbol{S}_l$, $l = 1, \ldots, L$, with the following rule bases

$$R^l_{i_1\ldots i_n}: IF\ \mathbf{x}\ is\ \mathbf{A}_{i_1\ldots i_n}\ THEN\ \mathbf{y}^l_{i_1\ldots i_n} = \mathbf{f}^l_{i_1\ldots i_n}(\mathbf{x})$$
$$i_j = 1,\ldots,N_k,\ j = 1,\ldots,n, \qquad \textbf{(14)}$$
$$l = 1,\ldots,L$$

Then the parallel combination $\pm\, \boldsymbol{S}_1 \pm \boldsymbol{S}_2 \ldots \pm \boldsymbol{S}_L$ is equivalent to a single fuzzy system with inputs $\mathbf{x} \subset \mathbf{A}_{i_1\ldots i_n}$ and a rule base given by

$$R_{i_1\ldots i_n}: IF\ \mathbf{x}\ is\ \mathbf{A}_{i_1\ldots i_n}\ THEN\ \mathbf{y}_{i_1\ldots i_n} = \sum_{l=1}^{L}\pm\mathbf{f}^l_{i_1\ldots i_n}(\mathbf{x}) \qquad \textbf{(15)}$$

Proof: Using (4), we obtain the output

$$\mathbf{y} = \sum_{l=1}^{L}\pm\frac{\sum_{i_1=1}^{N_1}\cdots\sum_{i_n=1}^{N_n}\mathbf{f}^l_{i_1\ldots i_n}(\mathbf{x})\prod_{j=1}^{n}\mu_{A_j^{i_j}}\left(x_j(k)\right)}{\sum_{i_1=1}^{N_1}\cdots\sum_{i_n=1}^{N_n}\prod_{j=1}^{n}\mu_{A_j^{i_j}}\left(x_j(k)\right)}$$
$$= \frac{\sum_{l=1}^{L}\sum_{i_1=1}^{N_1}\cdots\sum_{i_n=1}^{N_n}\pm\mathbf{f}^l_{i_1\ldots i_n}(\mathbf{x})\prod_{j=1}^{n}\mu_{A_j^{i_j}}\left(x_j(k)\right)}{\sum_{i_1=1}^{N_1}\cdots\sum_{i_n=1}^{N_n}\prod_{j=1}^{n}\mu_{A_j^{i_j}}\left(x_j(k)\right)} \qquad \textbf{(16)}$$

Interchanging the summation order gives the output of the system of (15).

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In the next section we give new sufficient conditions for the stability of discrete dynamic fuzzy systems.

IV. Definite Functions and Stability Results

We present a stability analysis for fuzzy system that parallels traditional Lyapunov analysis of crisp systems. Our stability analysis requires the definition of definite functions that play an analogous role to that of definite functions in standard Lyapunov stability theory. All these functions are discrete TSK systems that comprise the four principal components of Section II, with corresponding constraints on their outputs. All are assumed to have constant consequents and triangular membership functions but are used in the stability analysis of systems with more general membership functions. The constraints on the outputs can be shown to be equivalent to constraints on the consequents as stated below [5].

Positive definite fuzzy function: A fuzzy function is positive definite if and only if

a) $\mathbf{h}_{i_1^* i_2^* \ldots i_n^*} = \mathbf{0}$ if and only if the vector of fuzzy sets $\mathbf{A}_{i_1^* i_2^* \ldots i_n^*}$ is centered at the origin.

b) $\mathbf{h}_{i_1 i_2 \ldots i_n} > \mathbf{0}$, $\forall\ i_j = 1,2,\ldots,N_j,\ i_j \ne i_j^*,\ j = 1,2,\ldots,n$, where $\mathbf{h}_{i_1 i_2\ldots i_n} > 0$ denotes $h_j^{i_1 i_2 \ldots i_n} > 0, j = 1,2,\cdots,n.$

Positive semi-definite fuzzy function: A fuzzy function is positive semi-definite if and only if:

$\mathbf{h}_{i_1 i_2 \ldots i_n} \ge \mathbf{0}$ for all $i_j = 1,2,\ldots,N_j,\ j = 1,2,\ldots,n.$

Negative definite fuzzy function: A fuzzy function is negative definite if and only if:

a) $\mathbf{h}_{i_1^* i_2^* \ldots i_n^*} = \mathbf{0}$ if and only if the vector of fuzzy sets $\mathbf{A}_{i_1^* i_2^* \ldots i_n^*}$ is centered at the origin.

b) $\mathbf{h}_{i_1 i_2 \ldots i_n} < \mathbf{0}\ \forall\ i_j = 1,2,\ldots,N_j,\ i_j \ne i_j^*,\ j = 1,2,\ldots,n.$

Negative semi-definite fuzzy function: A fuzzy function is negative semi-definite system if and only if:

$$\mathbf{h}_{i_1 i_2 \ldots i_n} \le \mathbf{0} \;\forall\; i_j = 1,2,\ldots,N_j,\; j = 1,2,\ldots,n. \qquad \textbf{(17)}$$

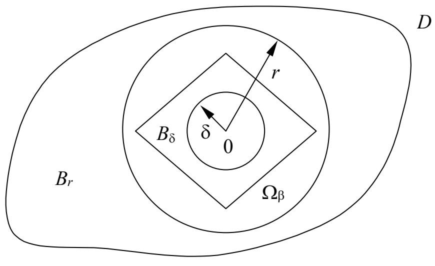

Figure 1 Geometric representation of Theorem 1.

We now introduce our new stability criterion for discrete TSK dynamic fuzzy systems with nontriangular membership functions.

Theorem 1: Let ***S*** be a discrete TSK dynamic fuzzy system. Consider the fuzzy Lyapunov function candidate $V(\mathbf{x}(k))$ with triangular membership functions and constant consequents defined by:

$$R_{i_1 i_2 \ldots i_n} : IF\ \mathbf{x}(k)\ is\ \mathbf{A}_{l_1 \ldots l_n}\ THEN\ \mathbf{x}_{l_1 \ldots l_n}(k+1) = \mathbf{b}^T \left|\mathbf{e}_{l_1 \ldots l_n}\right| \qquad \textbf{(18)}$$

with the difference $\Delta V(\mathbf{x}(k))$ defined by:

$$R_{i_1 i_2 \ldots i_n} : IF\ \mathbf{x}(k)\ is\ \mathbf{A}_{i_1 \ldots i_n}\ \&\ \mathbf{x}(k)\ is\ \mathbf{A}_{l_1 \ldots l_n}\ THEN$$
$$\mathbf{x}_{i_1 \ldots i_n l_1 \ldots l_n}(k+1) = \mathbf{b}^T \left(\left|\mathbf{f}_{i_1 \ldots i_n}(\mathbf{x})\right| - \left|\mathbf{e}_{l_1 \ldots l_n}\right| \right) \qquad \textbf{(19)}$$

where $\mathbf{b} = [b_1\ b_2\ \ldots\ b_n]^T$

1. If ∃ $\mathbf{b} > \mathbf{0}$ such that $\Delta V(\mathbf{x}(k))$ is a fuzzy negative semi-definite function, then ***S*** is stable in the sense of Lyapunov.
2. If ∃ $\mathbf{b} > \mathbf{0}$ such that $\Delta V(\mathbf{x}(k))$ is a fuzzy negative definite function, then ***f*** is asymptotically stable.

Proof: We can rewrite $V(\mathbf{x}(k))$ as

$$V(\mathbf{x}(k)) = \mathbf{b}^T \left|\mathbf{x}(k)\right| \qquad \textbf{(20)}$$

We obtain the fuzzy system $\Delta V(\mathbf{x}(k))$ as a difference using Lemma 4

$$\begin{aligned}\Delta V(\mathbf{x}(k)) &= V(\mathbf{x}(k+1)) - V(\mathbf{x}(k)) \\ &= \mathbf{b}^T \left(\left|\mathbf{x}(k+1)\right| - \left|\mathbf{x}(k)\right| \right)\end{aligned} \qquad \textbf{(21)}$$

Using (4) and Lemmas 1, 2, 3, we obtain

$$\Delta V(\mathbf{x}(k)) \le \frac{\sum_{i_1=1}^{N_1} \cdots \sum_{i_n=1}^{N_n} \mathbf{b}^T \left|\mathbf{f}_{i_1 \ldots i_n}(\mathbf{x})\right| \prod_{j=1}^{n} \mu^1_{A_j^{i_j}}\left(x_j(k)\right)}{\sum_{i_1=1}^{N_1} \cdots \sum_{i_n=1}^{N_n} \prod_{j=1}^{n} \mu^1_{A_j^{i_j}}\left(x_j(k)\right)}$$
$$- \sum_{l_1=i_1}^{i_1+1} \sum_{l_2=i_2}^{i_2+1} \cdots \sum_{l_n=i_n}^{i_n+1} \mathbf{b}^T \left|\mathbf{e}_{l_1 l_2 \ldots l_n}\right| \prod_{j=1}^{n} \mu^2_{A_j^{i_j}}\left(x_j(k)\right)$$

$$\le \sum_{i_1=1}^{N_1} \cdots \sum_{i_n=1}^{N_n} \sum_{l_1=i_1}^{i_1+1} \cdots \sum_{l_n=i_n}^{i_n+1} \left\{ \mathbf{b}^T \left[\left|\mathbf{f}_{i_1 \ldots i_n}(\mathbf{x})\right| - \left|\mathbf{e}_{l_1 l_2 \ldots l_n}\right| \right] w_{i_1 \cdots i_n l_1 \cdots l_n} \right\} \Big/$$
$$\sum_{i_1=1}^{N_1} \cdots \sum_{i_n=1}^{N_n} \sum_{l_1=i_1}^{i_1+1} \cdots \sum_{l_n=i_n}^{i_n+1} w_{i_1 \cdots i_n l_1 \cdots l_n}$$
$$w_{i_1 \cdots i_n l_1 \cdots l_n} = \prod_{j=1}^{n} \prod_{p=1}^{n} \mu^1_{A_j^{i_j}}\left(x_j(k)\right) \mu^2_{A_p^{i_p}}\left(x_p(k)\right) \qquad \textbf{(22)}$$

Given $\varepsilon > 0$, choose $r \in (0, \varepsilon]$ such that

$$B_r = \left\{ \mathbf{x}(k) \in \mathbf{R}^n \mid \left\|\mathbf{x}(k)\right\| \le r \right\} \subset D$$

Let $\alpha = \min_{\|\mathbf{x}\|=r} V(\mathbf{x}(k))$. Then $\alpha > 0$ since it is the minimum of a positive continuous function over a compact set. Take $\beta \in (0, \alpha)$ and let

$$\Omega_\beta = \left\{ \mathbf{x}(k) \in B_r \mid V(\mathbf{x}(k)) \le \beta \right\}$$

Then Ω_β is entirely inside B_r (see [2]). Let $\Delta V(\mathbf{x}(k))$ be a negative-semi-definite system with constant consequents. Then

$$\Delta V(\mathbf{x}(k)) \le 0 \Rightarrow V(\mathbf{x}(k+1)) \le V(\mathbf{x}(k)) \le \beta,$$
$$k = 0,1,2,\cdots$$

Since $V(\mathbf{x}(k))$ is continuous and $V(\mathbf{0}) = 0$, there exist $\delta > 0$ such that

$$\left\|\mathbf{x}(k)\right\| \le \delta \Rightarrow V(\mathbf{x}(k)) \le \beta$$

Hence, we have $B_\delta \subset \Omega_\beta \subset B_r$ and

$$\begin{aligned}\mathbf{x}(k) \in B_\delta \Rightarrow \mathbf{x}(k) \in \Omega_\beta &\Rightarrow \mathbf{x}(k+1) \in \Omega_\beta \\ &\Rightarrow \mathbf{x}(k+1) \in B_r\end{aligned}$$

Therefore, $\left\|\mathbf{x}(k)\right\| \le \delta \Rightarrow \mathbf{x}(k+1) < r \le \varepsilon$ and $\mathbf{x} = \mathbf{0}$ is stable in the sense of Lyapunov.

Similarly, we can show that $\mathbf{x} = \mathbf{0}$ is stable in the sense of Lyapunov for the negative definite case. To establish asymptotic stability, we prove convergence to the origin. $V(\mathbf{x}(k))$ decreases continuously along the system trajectories and is lower bounded by zero $V(\mathbf{x}(k)) \to L \ge 0$ as $k \to \infty$.

We show that L is zero by contradiction. Let $L > 0$ and consider the set $\Omega_L = \left\{ \mathbf{x}(k) \mid V(\mathbf{x}(k)) \le c \right\}$

Select a ball $B_d \subset \Omega_L$, then the trajectories of the system remain outside B_d. Let

$$-\gamma(k) = \sup_{d \le \|\mathbf{x}(k)\| \le r} \Delta V(\mathbf{x}(k)) < 0$$

and consider the function

$$V(\mathbf{x}(k)) = V(\mathbf{0}) + \sum_{i=0}^{k} V(\mathbf{x}(i) \le V(\mathbf{0}) - \gamma(k+1)$$

which tends to $-\infty$ as $k \to \infty$. This contradicts the lower boundedness of $V(\mathbf{x}(k))$. •

Remark: Note that it is implicitly assumed in the proof that the fuzzy system has an equilibrium point at the origin. The Theorem is valid for other equilibrium points as demonstrated in Example 1.

The following theorem provides conditions for the exponential stability of fuzzy systems with nontriangular membership functions. A version of the theorem for triangular membership functions was presented in [4].

Theorem 2: Let $\boldsymbol{S}$ be a dynamic TSK fuzzy system. $\boldsymbol{S}$ is exponentially stable with decay rate s_α if:

$$\left\|\mathbf{f}_{i_1\ldots i_n}(\mathbf{x}(k))\right\| \le \alpha_{i_1 i_2\ldots i_n}.\left\|\mathbf{x}(k)\right\| \tag{23}$$

where $0 < \alpha_{i_1 i_2 \ldots i_n} \le s_\alpha < 1, i_j = 1,\cdots,N_j, j = 1\cdots,n$

Proof: Assume that

$$\mathbf{x}(k+1) = \frac{\sum_{i_1=1}^{N_1}\cdots\sum_{i_n=1}^{N_n}\mathbf{f}_{i_1\ldots i_n}(\mathbf{x}(k))\prod_{j=1}^{n}\mu_{A_j^{i_j}}\left(x_j(k)\right)}{\sum_{i_1=1}^{N_1}\cdots\sum_{i_n=1}^{N_n}\prod_{j=1}^{n}\mu_{A_j^{i_j}}\left(x_j(k)\right)} \tag{24}$$

Using (4), (23), (24), we obtain

$$\left\|\mathbf{x}(k+1)\right\| \le \frac{\sum_{i_1=1}^{N_1}\cdots\sum_{i_n=1}^{N_n}\alpha_{i_1 i_2\ldots i_n}.\left\|\mathbf{x}(k)\right\|\prod_{j=1}^{n}\mu_{A_j^{i_j}}\left(x_j(k)\right)}{\sum_{i_1=1}^{N_1}\cdots\sum_{i_n=1}^{N_n}\prod_{j=1}^{n}\mu_{A_j^{i_j}}\left(x_j(k)\right)} \tag{25}$$

$$\le s_\alpha\left\|\mathbf{x}(k)\right\|$$

By induction, we have

$$\left\|\mathbf{x}(k+1)\right\| \le s_\alpha\left\|\mathbf{x}(k)\right\| \le s_\alpha{}^k\left\|\mathbf{x}(0)\right\| \tag{26}$$

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Theorem 2 is applicable to any TSK system that satisfies condition (23), even with nonlinear consequents. For a system with linear consequents of the form $A_{i_1 i_2\ldots i_n}\mathbf{x}(k)$, Theorem 2 reduces to an earlier result from [1]

$$\left\|A_{i_1 i_2\ldots i_n}\right\| \le \alpha_{i_1 i_2\ldots i_n} \tag{27}$$

Corollary: Let $\boldsymbol{f}$ be a Mamdani dynamic fuzzy system then $\boldsymbol{f}$ is exponentially stable if:

$$\left\|\mathbf{x}(k+1)\right\| \le \alpha_{i_1 i_2\ldots i_n}.\left\|\mathbf{x}(k)\right\|$$
$$\forall\mathbf{x}\in\mathbf{A}_{i_1\ldots i_n}, \forall\mathbf{x}(k+1)\in H^{i_1\cdots i_n} \tag{28}$$

where $0 < \alpha_{i_1 i_2 \ldots i_n} \le s_\alpha < 1, i_j = 1,\cdots,N_j, j = 1\cdots,n$

Proof: The corollary follows directly from the theorem since (26)-(28) are valid for $\max_{A_{i_1\cdots i_n}}\left\|\mathbf{x}(k)\right\|$ and $\min_{H_{i_1\cdots i_n}}\left\|\mathbf{x}(k+1)\right\|$

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V. Examples

We present two examples that demonstrate the stability analysis using the results of this paper. The first is a dynamic TSK system with constant consequents, while the second is a dynamic TSK system with linear consequents

Example 1: We consider the TSK system with Gaussian membership functions and constant consequents. The centers of the Gaussian fuzzy sets are given in Table 1 together with the corresponding consequent values. The value of σ used for all membership functions was 2. In this example, we have a second order system (n = 2) with each state variable having 5 fuzzy sets. Thus, there are 25 linear inequalities to be satisfied and one can use a variety of methods to solve this linear programming feasibility problem. An acceptable Lyapunov function is obtained with $b = [1, 1]^T$. Our simulation results of Figure 2 show that all the trajectories of the system converge to its equilibrium point at $[-1.25, 0.51]^T$ and verify that the system is asymptotically stable. Note that the equilibrium of the system is not at the origin.

Table 1 Centers of fuzzy system in Example 1.

x_2 \ x_1		e_1^1	e_1^2	e_1^3	e_1^4	e_1^5
		−4	−2	0	2	4
e_2^1	−4	−0.76 0.3	−1.67 0.69	−2.44 0.98	−1.0 0.4	−1.48 0.62
e_2^2	−2	−0.97 0.38	−0.94 0.42	−1.88 0.79	−2.12 0.84	−1.47 0.62
e_2^3	0	−0.32 0.19	−1.35 0.57	0 0	−0.5 0.23	−2.35 0.93
e_2^4	2	−1.63 0.66	−1.49 0.63	−0.95 0.4	−1.54 0.65	0.88 0.37
e_2^5		−1.43 0.59	−1.76 0.72	−1.92 0.78	−0.34 0.19	−2.39 0.98

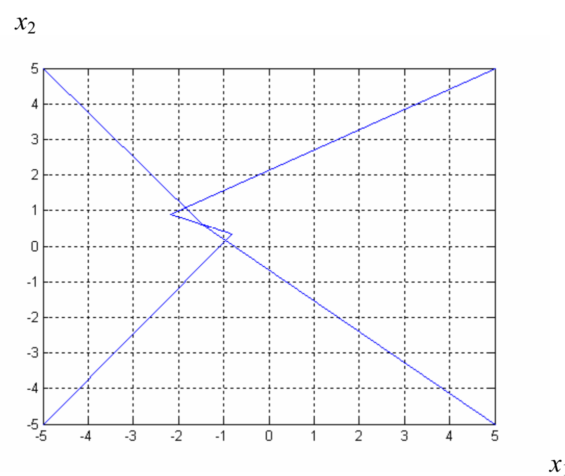

Figure 2. Simulation results for the fuzzy TSK system of Example 1.

Example 2: Consider the TSK system with linear consequents of Table 2. All the matrices in the table have column sums (norms) less than or equal to 0.9. Thus, the system satisfies the stability conditions of (23) and (27) and is therefore exponentially stable. The simulation results of Figure 3 confirm the stability of the system. The system trajectories converge to the origin after a few iterations for all tested initial conditions.

Table 2 Centers of fuzzy system in Example 2.

x_1 \ x_2		e_1^1	e_1^2	e_1^3
		-5	0	5
e_2^1	-5	$\begin{bmatrix} 0.2 & 0.2 \\ -0.6 & -0.7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$	$\begin{bmatrix} -0.2 & 0.4 \\ -0.5 & 0.2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$	$\begin{bmatrix} -0.3 & 0.4 \\ 0.4 & -0.3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
e_2^2	0	$\begin{bmatrix} -0.4 & 0.1 \\ -0.5 & -0.4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$	$\begin{bmatrix} -0.3 & -0.3 \\ 0.1 & -0.3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
e_2^3	5	$\begin{bmatrix} -0.5 & -0.3 \\ 0.3 & -0.2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$	$\begin{bmatrix} 0 & 0.4 \\ 0.7 & -0.2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$	$\begin{bmatrix} -0.2 & -0.1 \\ -0.1 & 0.8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

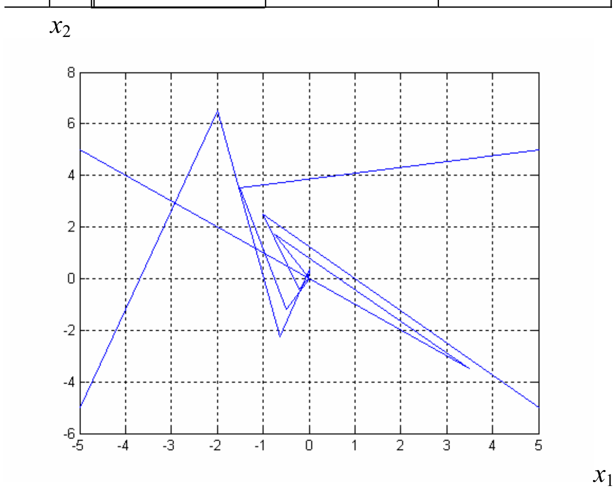

Figure 3. Simulation results for the fuzzy TSK system of Example 2.

VI. Conclusion

This paper provides a stability test for TSK fuzzy systems with nontriangular membership functions. It uses the concept of fuzzy positive definite and fuzzy negative definite functions to determine the stability of discrete TSK dynamic fuzzy systems. As in standard Lyapunov stability theory, we show that if a fuzzy positive definite function has fuzzy negative definite changes along the trajectories of a discrete TSK dynamic fuzzy system, then the system is asymptotically stable. Similar arguments allow us to derive conditions for stability in the sense of Lyapunov and for exponential stability. We also provide exponential stability results for TSK as well as for Mamdani fuzzy systems.

Future work will extend these results to continuous fuzzy systems and analyze the stability of forced systems.

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