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Products of Row Stochastic Matrices and Their Applications to Cooperative Control for Autonomous Mobile Robots

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Abstract— **In this paper, cooperative control of dynamic systems is formulated as the problem of choosing a linear feedback control law of the systems' outputs and making the states of individual systems converge to the same steady state. As such, cooperative behavior of the overall system can be studied by investigating the convergence property of products of row stochastic matrices. Two new results on the convergence of matrix products are obtained, one on products of lower-triangular matrices and the other on products of lower-triangular matrices and general matrices. Neither of the two results requires that matrices be irreducible, and they can be used as the tools for the design and stability analysis of cooperative control. In particular, less-restrictive conditions on the design of cooperative control feedback matrices are established for a general class of MIMO dynamic systems of finite but arbitrary relative degree. The proposed design doesn't require either the directed robot sensor graph being irreducible or a fixed leader. An example is provided to illustrate the proposed design method and new results.**

I. Introduction

The study of cooperative control for multiple robots has been motivated by the flock behavior of animals. As early as 1980's, Reynolds introduced the first computer animation of flocking by using the local control strategies: cohesion, separation, and alignment [8]. In [9], a flocking model for a group of particles moving in the plane was proposed using the "neighboring rules" and verified through simulations. The theoretical explanation of this "nearest neighbor rules" was recently given in [5] by using graph theory. It is proved that all the agents' headings converge to a common value provided that the *undirected* sensor graphs for all agents are periodically strongly connected. The extension to the case of *directed* sensor graph was done in [11][6]. In our recent paper [7], we further extended those results to the case of dealing with a class of general MIMO dynamic systems of finite but arbitrary relative degree, and studied the coordination behavior of the overall system under leader-follower cooperative control while the *directed* sensor graphs always being not strongly connected.

In this paper, as a continuation of [7], we address the leaderless cooperative control problem by finding a set of less-restrictive conditions on the connectivity requirements among robots. Our study starts with exploiting the new convergence properties of the product of row stochastic matrices [10]. First, for a class of row stochastic matrices in the lower-triangular structure, we find an easy-to-check condition on its convergence and give an elegant proof (theorem 2.1). Then, by presenting one important result on the property of row stochastic matrices (lemma 2.4), an important convergence result on the product of two combined sequences of row stochastic matrices are established (theorem 2.2). In particular, a necessary and sufficient condition is given for row stochastic matrices in the lower-triangular structure (corollary 2.1), which has direct application to find the less-restrictive conditions on leaderless cooperative control. Third, by using the new convergence results on row stochastic matrices, we consider a group of MIMO dynamic systems of finite but arbitrary relative degree and propose a new guideline for the design of cooperative controls. The obtained less-restrictive conditions are easy-to-check and don't require the irreducibility of directed sensor graphs. Moreover, no fixed leader is required during the motion of robots. It is proved that the overall system will converge to the same steady state. The contribution of this paper is two-fold: first, we obtain some new results on the convergence of row stochastic matrices (theorem 2.1, lemma 2.4, theorem 2.2, and corollary 2.1), which provide a rigorous framework for the stability analysis of cooperative control systems; second, we find a set of less-restrictive conditions on the connectivity requirements among robots, and establish a guideline for the design of leaderless cooperative control. The proposed method is applicable to a general class of systems which can be transformed into the canonical model given in the paper.

Throughout the paper, the following notations and definitions are used. Let $\mathbf{1}_p$ be the p-dimensional column vector with all its elements being 1, and $\mathbf{J}_{r_1\times r_2} \in \Re^{r_1\times r_2}$ be a matrix whose elements are all 1. A nonnegative matrix has all entries nonnegative. A square real matrix is row stochastic if it is nonnegative and its row sums all equal 1. For a row stochastic matrix E, define $\delta(E) = \max_j \max_{i_1,i_2} |E_{i_1 j} - E_{i_2 j}|$, which measures how different the rows of E are. Also, define $\lambda(E) = 1 - \min_{i_1,i_2} \sum_j \min(E_{i_1 j}, E_{i_2 j})$.

Given a sequence of nonnegative matrix $E(k)$, $E(k) \succ 0, k = 0, 1, \cdots$, means that, there is a sub-sequence $\{l_v, v = 1, \cdots, \infty\}$ of $\{0, 1, 2, \cdots, \infty\}$ such that $\lim_{v\to\infty} l_v = +\infty$ and $E(l_v) \neq 0$, that is, there exists at least one element $E_{ij}(l_v) \geq \epsilon$ for $\epsilon > 0$. A non-negative matrix E is said to be *reducible* if the set of its indices, $\mathcal{I} \triangleq \{1, 2, \cdots, n\}$, can be divided into two disjoint nonempty sets $\mathcal{S} \triangleq \{i_1, i_2, \cdots, i_\mu\}$ and $\mathcal{S}^c \triangleq \mathcal{I}/\mathcal{S} = \{j_1, j_2, \cdots, j_\nu\}$ (with $\mu+\nu = n$) such that $E_{i_\alpha j_\beta} = 0$, where $\alpha = 1, \cdots, \mu$ and $\beta = 1, \cdots, \nu$. Matrix

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E is said to be *irreducible* if it is not reducible. A square matrix $E \in \Re^{n\times n}$ can be used to define a directed graph with n nodes $v_1, \cdots, v_n$, and there is a directed arc from v_i to v_j if and only if $E_{ij} \neq 0$. A directed graph represented by E is *strongly connected* if between every pair of distinct nodes v_i, v_j there is a directed path of finite length that begins at v_i and ends at v_j. The fact that a directed graph represented by E is strongly connected is equivalent to that matrix E is irreducible [4].

II. New Results on Products of Row Stochastic Matrices

The classical convergence result of the infinite products of sequences of row stochastic matrices [10] has been recently applied in the study of coordination behavior of groups of agents [5][11]. In this section, we further exploit the convergence properties of the infinite products of sequences of row stochastic matrices and establish some new relaxed conditions which have direct applications to the cooperative control design of mobile robots.

Lemma 2.1: [10][11] Consider a finite or infinite set of row stochastic matrices $\{P_i \in \Re^{R\times R}\}$ satisfying $0 \leq \lambda(P_i) \leq \delta < 1$. Then, for each infinite sequence, $P_{l_1}, P_{l_2}, \cdots$, there exists a row vector $c \in \Re^{1\times R}$ such that

$$\lim_{k\to\infty} \prod_{j=0}^{k-1} P_{l_{k-j}} = \mathbf{1}_R c.$$

Lemma 2.1 presents a powerful result on the convergence of the product of arbitrary infinite sequence generated from set $\{P_i \in \Re^{R\times R}\}$. However, in the applications to cooperative control, we are more interested in the study of the convergence property of a given sequence of row stochastic matrices. Given a sequence of row stochastic matrices $P(k) \in \Re^{R\times R}$, it follows from lemma 2.1 that $\lim_{k\to\infty} \prod_{l=0}^{k-1} P(k-l) = \mathbf{1}_R c$ if $0 \leq \lambda(P(k)) \leq \delta < 1$ for every k. However, this condition on $P(k)$ is relatively strong and is also not necessary. For example, given a matrix

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0.1 & 0.9 & 0 \\ 0 & 0.7 & 0.3 \end{bmatrix},$$

we have $\lambda(P) = 1$, but it can be easily verified that $\lim_{l\to\infty} P^l = \mathbf{1}c$ with $c = [1, 0, 0]$.

In what follows, we will find the relaxed conditions on $P(k)$ such that $\prod_{l=0}^{k-1} P(k-l)$ will converge as $k \to \infty$, which is useful for establishing the less-restrictive conditions on the connectivity requirements for groups of robots in the design of cooperative control. The following lemma can be concluded from lemma 2.1 and provides a relaxed condition on the convergence of a class of sequence $P(k)$.

Lemma 2.2: Given a sequence of row stochastic matrices $P(k) \in \Re^{R\times R}$. If there is a sub-sequence $\{l_v, v = 0, 1, \cdots, \infty, l_0 = 0\}$ of $\{0, 1, 2, \cdots, \infty\}$ such that $\lim_{v\to\infty} l_v = +\infty$ and $0 \leq \lambda(Q(k)) \leq \delta < 1$, where $Q(k) = P(l_k)P(l_k - 1)\cdots P(l_{k-1} + 1)$, $k = 1, 2, \cdots, \infty$, then $\lim_{k\to\infty} \prod_{l=0}^{k-1} P(k-l) = \mathbf{1}_R c$ with c being a constant vector. Furthermore, if $P(k)$ is in the lower-triangular structure with positive diagonal elements, then the condition is both sufficient and necessary.

Proof: Omitted. □

Remark 2.1: For general $P(k)$, the condition in lemma 2.2 is not necessary. For example, given $P = [0.2, 0.5, 0.3; 0.3, 0.5, 0.2; 0.2, 0.5, 0.3]$ and $Q = [0.1, 0, 0.9; 0, 1, 0; 0.2, 0, 0.8]$, the product $P * Q^n = [0.0909, 0.5000, 0.4091; 0.0909, 0.5000, 0.4091; 0.0909, 0.5000, 0.4091]$ as $n \to \infty$. However, since $\lambda(Q^n) = 1$, we cannot find the sub-sequence $\{l_v\}$. ◇

Lemma 2.2 gives a condition on convergence of sequence $P(k)$. However, it is usually difficult to find the sub-sequence $\{l_v\}$ and also not easy to check the value of $\lambda(Q(k))$. It would be desirable to have the direct conditions on $P(k)$. In what follows, we first present such a condition for a sequence of row stochastic matrices in the lower-triangular structure, and then extend it to the more general case. Before proceeding, the following lemma is required which gives a general result on the convergence of a discrete-time system satisfying some properties.

Lemma 2.3: Consider the following discrete-time equation

$$Q_i(k) = P_{ij}(k)Q_j(k-1) + P_{ii}(k)Q_i(k-1), \quad i \neq j \quad (1)$$

where $Q_i(k) \in \Re^{r_i\times R}$ and $Q_j(k) \in \Re^{r_j\times R}$ satisfying $Q_i(k)\mathbf{1}_R = \mathbf{1}_{r_i}$ and $Q_j(k)\mathbf{1}_R = \mathbf{1}_{r_j}$, and $P_{ij}(k) \in \Re^{r_i\times r_j}$ and $P_{ii}(k) \in \Re^{r_i\times r_i}$ satisfying $P_{ij}(k)\mathbf{1}_{r_j} + P_{ii}(k)\mathbf{1}_{r_i} = \mathbf{1}_{r_i}$, and $Q_i(k) \geq 0$, $Q_j(k) \geq 0$, $P_{ij}(k) > 0$ and $P_{ii}(k) > 0$. If $\lim_{k\to\infty} Q_j(k) = \mathbf{1}_{r_j} c_j$ with $c_j \in \Re^{1\times R}$ being a constant vector, and there exists a constant $\epsilon < 1$ such that

$$|\tilde{Q}_j(k)| = |Q_j(k) - \mathbf{1}_{r_j} c_j| \leq \epsilon^k \mathbf{J}_{r_j\times R}, \quad (2)$$

then,

$$\lim_{k\to\infty} Q_i(k) = \mathbf{1}_{r_i} c_j. \quad (3)$$

Proof: omitted. □

Theorem 2.1: Consider a sequence of nonnegative, row stochastic matrices in the lower-triangular structure

$$P(k) = \begin{bmatrix} P_{11}(k) & & & \\ P_{21}(k) & P_{22}(k) & & \\ \vdots & \vdots & \ddots & \\ P_{m1}(k) & P_{m2}(k) & \cdots & P_{mm}(k) \end{bmatrix} \in \Re^{R\times R},$$

where $R = \sum_{i=1}^{m} r_i$, sub-blocks $P_{ii}(k)$ on the diagonal are square and of dimension $\Re^{r_i\times r_i}$, sub-blocks $P_{ij}(k)$ off diagonal are of appropriate dimensions. Suppose that $P_{ii}(k) \geq \epsilon_i \mathbf{J}_{r_i\times r_i}$ for some constant $\epsilon_i > 0$ and for all $(i = 1, \cdots, m)$, and in the ith row of $P(k)$ $(i > 1)$, there is at least one j $(j < i)$ such that $P_{ij} \succ 0$. Then,

$$\lim_{k\to\infty} \prod_{l=0}^{k-1} P(k-l) = \mathbf{1}_R c,$$

where constant vector $c = [c_1, 0, \cdots, 0] \in \Re^{1\times R}$ with $c_1 \in \Re^{1\times r_1}$.

Proof: The proof can be done by induction using lemma 2.3. Define $Q(k) = \prod_{l=0}^{k-1} P(k-l)$ and the ith row of $Q(k)$ being $Q_i(k) = [Q_{i1}(k), Q_{i2}(k), \cdots, Q_{ii}(k), 0, \cdots, 0]$.
(i) Since sub-block $P_{11}(k) \geq \epsilon_1 \mathbf{J}_{r_1\times r_1}$ is row stochastic, and $0 \leq \lambda(P_{11}(k)) \leq \sigma_1 < 1$, where $\sigma_1 = 1 - r_1\epsilon_1$, thus, it follows from lemma 2.2 that

$$\lim_{k\to\infty} \prod_{l=0}^{k-1} P_{11}(k-l) = \mathbf{1}_{r_1} c_1 \triangleq P_{11}^*.$$

On the other hand, since

$$\left| \prod_{l=0}^{k-1} P_{11}(k-l) - P_{11}^* \right| \leq \delta \left(\prod_{l=0}^{k-1} P_{11}(k-l) \right) \mathbf{J}_{r_1\times r_1},$$

it then follows from $\delta\left(\prod_{l=0}^{k-1} P_{11}(k-l)\right) \leq \prod_{l=0}^{k-1} \lambda(P_{11}(k-l))$ [10] that, $\left|\prod_{l=0}^{k-1} P_{11}(k-l) - P_{11}^*\right| \leq \sigma_1^k \mathbf{J}_{r_1\times r_1}$, and $|Q_1(k) - \mathbf{1}_{r_1} c_1| \leq \sigma_1^k \mathbf{J}_{r_j\times R}$.
(ii) Consider $i = 2$. Since for $P_{21}(k) \succ 0$, we can always construct a new sequence $S(k)$ such that $S_{21}(k) > 0$ for all k [7]. Without loss of generality, we assume that $P_{21}(k) > 0$. It follows that

$$Q_2(k) = P_{21}(k) Q_1(k-1) + P_{22}(k) Q_2(k-1).$$

Noting that $P_{21}(k)\mathbf{1}_{r_1} + P_{22}(k)\mathbf{1}_{r_2} = \mathbf{1}_{r_2}$, and $P_{21}(k) > 0$ and $P_{22}(k) > 0$, it then follows from lemma 2.3 that $\lim_{k\to\infty} Q_2(k) = \mathbf{1}_{r_2} c_2$.
(iii) For general $i \geq 2$, without loss of generality, suppose that $P_{ij_1} \succ 0$ and $P_{ij_2} \succ 0$, and

$$\begin{aligned} \lim_{k\to\infty} Q_{j_1}(k-1) &= \mathbf{1}_{r_{j_1}} c_{j_1} = \mathbf{1}_{r_{j_1}} c_j, \\ \lim_{k\to\infty} Q_{j_2}(k-1) &= \mathbf{1}_{r_{j_2}} c_{j_2} = \mathbf{1}_{r_{j_2}} c_j. \end{aligned}$$

Note that $Q_i(k) = P_{ij_1}(k) Q_{j_1}(k-1) + P_{ij_2}(k) Q_{j_2}(k-1) + P_{ii}(k) Q_i(k-1)$. Similarly, it follows from $P_{ij_1}(k)\mathbf{1}_{r_{j_1}} c_j + P_{ij_2}(k)\mathbf{1}_{r_{j_2}} c_j + P_{ii}(k)\mathbf{1}_{r_i} c_j = \mathbf{1}_{r_i} c_j$ and using lemma 2.3, we have $\lim_{k\to\infty} Q_i(k) = \mathbf{1}_{r_i} c_j$. □

Theorem 2.1 provides an easy-to-check condition for the convergence of a sequence of row stochastic matrices $P(k)$ in the lower-triangular form, which can find the application to the design and analysis of leader-follower cooperative control [7]. However, for the design and analysis of cooperative controls in the case of general connectivity topologies (leaderless) within the network, we need to study the convergence properties of sequences of lower-triangular row stochastic matrices with mixed permutation matrices. That is, we need further study the convergence of the product of a sequence of row stochastic matrices $T(k)P(k)T^T(k)$ where $P(k)$ is in the triangular form and $T(k)$ is the permutation matrix.

In general, given $P(k)$ satisfying the conditions in theorem 2.1, the convergence cannot be guaranteed any more when permutation matrices are incorporated into the sequence. New conditions have to be exploited. For example, consider

$$P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0.1 & 0 & 0.9 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.2 & 0.8 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix}$$

It is easy to verify that $\lim_{n\to\infty}(P_1P_2)^n = \mathbf{1}\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$. However, given permutation matrix

$$T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

the product $(P_1TP_2T^T)^n$ will not converge as

$$\lim_{n\to\infty}(P_1TP_2T^T)^n = \begin{bmatrix} 0.2857 & 0 & 0.7143 & 0 \\ 0 & 1 & 0 & 0 \\ 0.2857 & 0 & 0.7143 & 0 \\ 0.2857 & 0 & 0.7143 & 0 \end{bmatrix}.$$

An important convergence result on the combination of two sequences of row stochastic matrices is stated in the following theorem. Before presenting this theorem, we first give the following lemma which shows how the property of $\lambda(\cdot) < 1$ is preserved when additional row stochastic matrices are inserted into the product of row stochastic matrices.

Lemma 2.4: Given row stochastic matrices $L \in \Re^{R\times R}$, $M \in \Re^{R\times R}$, and $Q \in \Re^{R\times R}$. Suppose that the values of the positive elements of L and M are greater than ϵ_l and ϵ_m, respectively. Let $S = LM$ and $P = LQM$. If $0 \leq \lambda(S) \leq \delta_s < 1$, and $Q_{ii} \geq \epsilon_q > 0$, then $\lambda(P) \leq \delta_p < 1$ with $\delta_p = 1 - (1-\delta_s)\epsilon_q$.

Proof: Consider any pair of rows of S, say rows 1 and 2. Since S satisfies $\lambda(S) \leq \delta_s < 1$, there will exist γ such that both $S_{1\gamma} > 0$ and $S_{2\gamma} > 0$. It follows from

$$S_{1\gamma} = \sum_{\beta=1}^{R} L_{1\beta} M_{\beta\gamma}, \quad S_{2\gamma} = \sum_{\beta=1}^{R} L_{2\beta} M_{\beta\gamma},$$

that there are some values of β for which $L_{1\beta} > 0$ and $M_{\beta\gamma} > 0$. Let β_1 be such a value. Similarly, let β_2 be such that $L_{1\beta_2} > 0$ and $M_{\beta_2\gamma} > 0$. It follows from

$$[LQ]_{1\beta_1} = \sum_{\alpha=1}^{R} L_{1\alpha} Q_{\alpha\beta_1} \geq L_{1\beta_1} Q_{\beta_1\beta_1} > 0,$$

that

$$P_{1\gamma} = \sum_{\alpha=1}^{R} [LQ]_{1\alpha} M_{\alpha\gamma} \geq [LQ]_{1\beta_1} M_{\beta_1\gamma} > 0.$$

Similarly, we have $P_{2\gamma} > 0$. To this end, we can find a positive constant δ_p such that $\lambda(P) \leq \delta_p < 1$. It follows from the fact that the values of the positive elements of L and M are greater than ϵ_l and ϵ_m, that $\delta_s = 1 - \epsilon_l\epsilon_m$. On the other hand, it is easy to see that $\delta_p = 1-\epsilon_l\epsilon_q\epsilon_m$. Thus, we have $\delta_p = 1-(1-\delta_s)\epsilon_q$. □

Theorem 2.2: Consider sequences of row stochastic matrices $L(k) \in \Re^{R\times R}$ and $M(k) \in \Re^{R\times R}$. If there exists a subsequence $\{k_v, v = 1, \cdots, \infty,\ k_1 = 1\}$ of $\{1, 2, \cdots, \infty\}$ such that $\lim_{v\to\infty} l_v = +\infty$ and

$$0 \le \lambda(L(k_{v+1}-1)\cdots L(k_v+1)L(k_v)) \le \delta_{lk_v} < 1, \quad (4)$$

and $M(k)$ satisfies $M_{ii}(k) \ge \epsilon_m > 0$, then

$$\lim_{k\to\infty} \prod_{l=0}^{k-1} L(k-l)M(k-l) = \mathbf{1}_R c_1, \quad (5)$$

and

$$\lim_{k\to\infty} \prod_{l=0}^{k-1} M(k-l)L(k-l) = \mathbf{1}_R c_2, \quad (6)$$

where c_1 and c_2 are constant vectors.

Proof: To prove (5), we only need show that

$$0 \le \lambda\left(\prod_{l=1}^{k_{v+1}-k_v} L(k_{v+1}-l)M(k_{v+1}-l)\right) \le \delta_{lmk_v}, \quad (7)$$

where $\delta_{lmk_v} < 1$. It follows from (4) and lemma 2.4 and $M_{ii}(k_{v+1}-1) \ge \epsilon_m > 0$ that $\lambda(L(k_{v+1}-1)M(k_{v+1}-1)L(k_{v+1}-2)\cdots L(k_v+1)L(k_v)) \le 1-(1-\delta_{lk_v})\epsilon_m < 1$. Repeatedly using lemma 2.4 leads to (7) with $\delta_{lmk_v} = 1-(1-\delta_{lk_v})\epsilon_m^{k_{v+1}-k_v}$. To this end, by using lemma 2.2, we have (5). (6) can proved similarly. □

By using theorem 2.2, one can obtain the following corollary, which provides a general guideline for the choice of cooperative control while guaranteeing the convergence of overall system. In next section, we will show a direct application of the following corollary, and establish the less-restrictive conditions on the connectivity of robots and the design of cooperative control.

Corollary 2.1: Given sequences of row stochastic matrices $P(k) \in \Re^{R\times R}$ and $P'(k) \in \Re^{R\times R}$, where $P(k)$ is in the lower-triangular structure and $P'(k)$ satisfying $P'_{ii}(k) \ge \epsilon_p > 0$. Then,

$$\lim_{k\to\infty} \prod_{l=0}^{k-1} P(k-l)P'(k-l) = \mathbf{1}_R c_1, \quad (8)$$

if and only if $\lim_{k\to\infty} \prod_{l=0}^{k-1} P(k-l) = \mathbf{1}_R c_2$, where c_1 and c_2 are constant vectors.

Proof: Omitted. □

III. Applications to Cooperative Control

Consider a group of autonomous robots whose dynamics can be transformed into the following canonical form

$$\dot{x}_i = A_i x_i + B_i u_i, \quad y_i = C_i x_i, \quad \dot{\eta}_i = g_i(\eta_i, x_i), \quad (9)$$

where $i = 1, \cdots, q$, $l_i \ge 1$ is an integer, $x_i \in \Re^{l_i m}$, $\eta_i \in \Re^{n_i - l_i m}$, $I_{m\times m}$ is the m-dimensional identity matrix, $\otimes$ denotes the Kronecker product, J_k is the kth order Jordan canonical form given by

$$J_k = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & -1 \end{bmatrix} \in \Re^{k\times k},$$

$A_i = J_{l_i} \otimes I_{m\times m} \in \Re^{(l_i m)\times(l_i m)}$, $B_i = \begin{bmatrix} 0 \\ I_{m\times m} \end{bmatrix} \in \Re^{(l_i m)\times m}$, $C_i = \begin{bmatrix} I_{m\times m} & 0 \end{bmatrix} \in \Re^{m\times(l_i m)}$, $y_i \in \Re^m$ is the output, $u_i \in \Re^m$ is the cooperative control law to be designed, and subsystem $\dot{\eta}_i = g_i(\eta_i, x_i)$ is input-to-state stable. Without loss of any generality, in this paper we consider the case that $l_1 = l_2 = \cdots = l_q = l$. The objective is to synthesis a general class of cooperative control and establish the conditions on network connectivity requirements such that the all states of the overall system converge to the same steady state.

A. Leaderless Cooperative Control

Let the cooperative control be given by: for $i = 1, \cdots, q$,

$$u_i = G_i(t)y, \quad (10)$$

where $G_i = \begin{bmatrix} G_{i1} & \cdots & G_{iq} \end{bmatrix}$ with $G_{ij} \in \Re^{m\times m}$ being the interconnection matrix satisfying the properties that $G_i \mathbf{1}_{mq} = \mathbf{1}_m$, and $y = [y_1^T \ \cdots \ y_q^T]^T$. Assume that $G_i(t)$ be piecewise constant for all i. It is obvious that the value of G_{ij} depends on the connection between the ith robot and the jth robot. In practise, the feedback matrix $G(t)$ must change over time according to physical surroundings. In this paper, we consider the situation that robot sensors have a limited field of view, such as a cone-like field of view.

It follows from (9) and (10) that the closed-loop overall system is given by

$$\dot{x} = (A + D(t))x = (-I_{N_q\times N_q} + \bar{D}(t))x, \quad (11)$$

where $x = [x_1^T, \ \cdots, \ x_q^T]^T \in \Re^{N_q}$, $N_q = mql$, $x_i = [x_{i1}^T, x_{i2}^T, \cdots, x_{il}^T]^T \in \Re^{ml}$, $x_{ij} = [x_{ij1}, x_{ij2}, \cdots, x_{ijm}]^T \in \Re^m$ with $i = 1, \cdots, q$, and $j = 1, \cdots, l$, $A = diag\{A_1, \ \cdots, \ A_q\} \in \Re^{N_q\times N_q}$, $C = diag\{C_1, \ \cdots, \ C_q\} \in \Re^{(mq)\times N_q}$, $B = diag\{B_1, \ \cdots, \ B_q\} \in \Re^{N_q\times(mq)}$, $G = \begin{bmatrix} G_1^T & \cdots & G_q^T \end{bmatrix}^T \in \Re^{(mq)\times(mq)}$, $D = BGC$, and $\bar{D}(t) = [\bar{D}_{ij}]$ with $(i = 1, \cdots, q)$

$$\bar{D}_{ii} = \begin{bmatrix} 0 & I_{(l-1)\times(l-1)} \otimes I_{m\times m} \\ G_{ii} & 0 \end{bmatrix} \in \Re^{lm\times lm},$$

and

$$\bar{D}_{ij} = \begin{bmatrix} 0 & 0 \\ G_{ij} & 0 \end{bmatrix} \in \Re^{lm\times lm}, \quad i, j = 1, \cdots, q, \quad i \ne j.$$

B. Conditions on Cooperative Control Design

To achieve the cooperative control objective, most recent results require that the sensor graphs (undirected or directed) are strongly connected at lease once in each time interval of some length [5][11]. In this paper, we extend the result to the case of that the directed sensor graph is not strongly connected.

Define the robot sensor matrix as $S(t) = [S_{ij}(t)] \in \Re^{q\times q}$, where $S_{ii} = 1$ which means that robot always has sensor information itself; $S_{ij} = 1$ for $i \neq j$ if the ith robot can sense the jth robot, otherwise $S_{ij} = 0$. The feedback gain matrix $G(t)$ will be designed according to connectivity topologies of robot sensor matrix. Let $\{t_k^G : k = 0, 1, \cdots\}$ with $t_0^G = t_0$ be the sequence of time instants at which $G(t)$ changes (i.e., $S(t)$ changes). That is, $G(t) = G(t_k^G)$ over the time interval $t \in [t_k^G, t_{k+1}^G)$. If there are only finite changes for $G(t)$, that is, for $t > t_i^G$, $G(t) = G(t_i^G)$, we can always partition the remaining time to generate an infinite time interval $[t_k^G, t_{k+1}^G)$. Suppose that $0 < t_{k+1}^G - t_k^G \le t_{max}$.

If the robot sensor matrix $S(t)$ is irreducible, then the corresponding feedback matrix $G(t)$ can also be made irreducible. In such a situation, the convergence result has been obtained for simple linear system with relative degree one [11], and it is further extended to a general class of MIMO systems of finite but arbitrary relative degree as given by (9) [7]. However, the condition that the robot sensor matrix $S(t)$ is irreducible, is still strong. In what follows, we study the coordination of group of autonomous robots in the case that robot sensor matrix $S(t)$ is always reducible, and propose less-restrictive conditions on the design of cooperative control feedback matrix.

It is shown in [1][3] that, if $S(t)$ is reducible, then there is a permutation matrix $T_1 \in \Re^{q\times q}$ such that $S_{T_1}(t) = T_1^T S(t) T_1$ is in the lower-triangular structure, that is

$$S_{T_1}(t) = \begin{bmatrix} S_{T_1,11}(t) & 0 & \cdots & 0 \\ S_{T_1,21}(t) & S_{T_1,22}(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ S_{T_1,k1}(t) & S_{T_1,k2}(t) & \cdots & S_{T_1,kk}(t) \end{bmatrix},$$

where $S_{T_1,ii} \in \Re^{q_i\times q_i}$, $\sum_{i=1}^k q_i = q$, and $S_{T_1,ii}(t)$ are irreducible. Correspondingly, we have augmented permutation matrices $T_2 = T_1 \otimes I_{m\times m} \in \Re^{qm\times qm}$ and $T_3 = T_1 \otimes I_{lm\times lm} \in \Re^{qlm\times qlm}$, such that

$$\begin{aligned} G_{T_2}(t) &= T_2^T G(t) T_2 \\ &= \begin{bmatrix} G_{T_2,11}(t) & 0 & \cdots & 0 \\ G_{T_2,21}(t) & G_{T_2,22}(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ G_{T_2,k1}(t) & G_{T_2,k2}(t) & \cdots & G_{T_2,kk}(t) \end{bmatrix}, \end{aligned} \quad (12)$$

where $G_{T_2,ii}(t)$ is irreducible, and the state transformation is $x = T_3 z$. To this end, the system dynamic (11) becomes

$$\dot{z} = -(I - T_3^T \bar{D} T_3) z. \quad (13)$$

Remark 3.1: Generally, the case of $q_i \neq 1$ means that the q robots reformulate k subgroups with $k < q$. ◇

The following theorem states the main result of the paper and gives the less-restrictive conditions on the design of cooperative control feedback matrix.

Theorem 3.1: Consider the cooperative control of system (9) under (10). Suppose that the signal transmission matrix $S(t_k^G)$ is reducible for almost all k. If there exists a subsequence $\{s_v, v = 0, 1, \cdots, \infty\}$ of $\{0, 1, 2, \cdots, \infty\}$, such that $S_{T_1}(t_{s_v}^G)$ is in the same lower-triangular structure (that is, $T_1(t_{s_v}^G) = T_{1c}$ for all v, where T_{1c} is a fixed permutation matrices), and satisfies the conditions that (i) $S_{T_1,ii}(t_{s_v}^G)$ is irreducible and (ii) for every $i > 1$, there is at least one j such that $S_{T_1,ij}(t_{s_v}^G) \succ 0$, $j < i$. Then, while the corresponding feedback matrix $G(t_k^G)$ is according to connectivity topology $S(t_k^G)$, in particular, $G(t_{s_v}^G)$ can be chosen to satisfy the following properties:

(a) $G_{T_2,ii}(t_{s_v}^G)$ becomes irreducible;
(b) for every $i > 1$, there is at least one j such that $G_{T_2,ij}(t_{s_v}^G) \succ 0$, $j < i$.

Under such a choice of $G(t)$, the stability of the overall closed-loop system can be guaranteed with

$$\lim_{t\to\infty} x(t) = \mathbf{1}_{N_q} c x(t_0^G), \quad (14)$$

where constant vector $c \in \Re^{1\times N_q}$.

Proof: For briefness, let $T_3(k) = T_3(t_k^G)$ and $P(k) = P(t_k^G)$. It follows that the solution of (13) is given by

$$x(t_{k+1}^G) = \prod_{l=0}^{k} T_3(k-l) P(k-l) T_3(k-l)^T x(t_0^G), \quad (15)$$

where

$$P(i) = e^{-(I - T_3^T(i)\bar{D}(t_i^G)T_3(i))(t_{i+1}^G - t_i^G)}, \; i = 0, \cdots, k. \quad (16)$$

It follows from $G_{T_2}(t_k^G)$ is in the lower-triangular structure that $\bar{D}_{T_3}(t_k^G) = T_3^T(k)\bar{D}(t_k^G)T_3(k)$ and $P(k)$ are also in the lower-triangular structure. Moreover, $P(k)$ is row-stochastic matrix and its diagonal elements are lower-bounded by a positive value [2]. To prove (14), it suffices to prove that

$$\lim_{k\to\infty} \prod_{l=0}^{k} T_3(k-l) P(k-l) T_3(k-l)^T = \mathbf{1}_{N_q} c. \quad (17)$$

Note that if $G_{T_2,ii}(t_{s_v}^G)$ is irreducible, then $\bar{D}_{T_3,ii}(t_{s_v}^G)$ is irreducible and $P_{ii}(s_v) > 0$ [7]. On the other hand, $G_{T_2,ij}(t_{s_v}^G) \succ 0$ leads to $\bar{D}_{T_3,ij}(t_{s_v}^G) \succ 0$ and $P_{ij}(s_v) \succ 0$. It then follows from assumption **??** and theorem 2.1 that

$$\lim_{v\to\infty} P(s_v) P(s_{v-1}) \cdots P(s_0) = \mathbf{1}_{N_q} c_s, \quad (18)$$

where c_s is a constant vector. Define $P'(s_v) = T_3(s_v)^T T_3(s_v - 1) P(s_v - 1) T_3^T(s_v - 1) \cdots T_3(s_{v-1} + 1) P(s_{v-1} + 1) T_3^T(s_{v-1} + 1) T_3(s_{v-1})$. Since $P_{ii}(\cdot) > 0$, then $P'_{ii}(\cdot) > 0$. To this end, the theorem follows from corollary 2.1. □

Remark 3.2: The proposed conditions for cooperative control design in theorem 3.1 is flexible in the sense that we do not require the robot sensor graph to be strongly connected. There is also no need to have the fixed leader in the group. When $T_1(t_k^G)$ are same for all k, the cooperative control strategy in theorem 3.1 becomes the leader-follower strategy with the fixed leader as discussed in [7]. When $T_1(t_k^G) = I_{m\times m}$, $\forall k$, the result in theorem 3.1 recover the cases discussed in [11][5]. $\diamond$

IV. Illustrative Example

In this section, an example is given to illustrate the cooperative control method studied in this paper. Consider a group of three nonholonomic 4-wheel differential driven mobile robots. By taking the robot "hand" position as the guide point, whose model can be feedback linearized to a double integrator with a stable internal dynamics [7]:

$$\begin{aligned} \dot{z}_{i1} &= z_{i2} \\ \dot{z}_{i2} &= v_{i2} \end{aligned} \tag{19}$$

where $i = 1, 2, 3$, $z_{i1} = [z_{i11}, z_{i12}]^T \in \Re^2$ is the position of particle i, $z_{i2} = [z_{i21}, z_{i22}]^T \in \Re^2$ its velocity, and $v_i = [v_{i1}, v_{i2}]^T \in \Re^2$ its acceleration inputs. The cooperative control objective is that all particles move to the same position, this is called an agreement problem.

Define the state and input transformations as follows:

$$x_{i1} = z_{i1}, \;\; x_{i2} = x_{i1} + z_{i2}, \;\; v_i = -2x_{i2} + x_{i1} + u_i.$$

Then system model can be transformed into

$$\dot{x}_{i1} = -x_{i1} + x_{i2}, \;\; \dot{x}_{i2} = -x_{i2} + u_i,$$

where $u = Gy$ with $y = [x_{11}^T, x_{21}^T, x_{31}^T]^T$. To this end, the cooperative control method in this paper can be used for the design of G. For illustration purpose, assume that two kinds of situations run alternatively during the robot motion process: (i) robot 1 as the leader, robot 2 follows robot 1 and robot 3 follows robot 1; (ii) each robot runs by itself. The corresponding robot sensor matrices are

$$S(1) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \;\; S(2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let the corresponding $G(t)$ be

$$G(1) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$G(2) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

It is easy to verify that $G(1)$ satisfies the conditions in theorem 3.1. In the simulation, the initial positions are $[6, 3]^T$, $[2, 5]^T$ and $[4, 2]^T$, respectively. Figure 1 shows the convergence of robots' position, which verifies the proposed design in this paper.

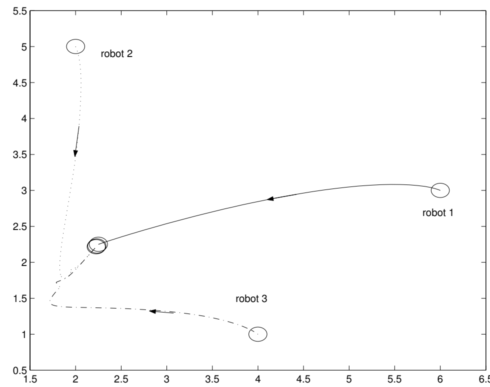

Fig. 1. Convergence of positions under cooperative control

V. Conclusion

In this paper, we studied the cooperative control strategy for a general class of MIMO dynamic systems of finite but arbitrary relative degree. A general guideline for the design of cooperative control is proposed, and the stability of overall closed-loop system is rigorously analyzed. The proposed method is less-restrictive in the sense that neither sensor graph's strong connectivity nor fixed leader are required. The approach can be easily applied to the formation stabilization and formation tracking control of robots due to the general framework established in the paper.

References

[1] R. Bapat and T. Raghavan, *Nonnegative Matrices and Applications*. Cambridge: Cambridge University Press, 1997.
[2] D. Freedman, *Markov Chains*. New York: Springer-Verlag, 1983.
[3] H.Minc, *Nonnegative Matrices*. New York, NY: John Wiley & Sons, 1988.
[4] R. Horn and C. Johnson, *Matrix Analysis*. Cambridge: Cambridge University Press, 1985.
[5] A. Jadbabaie, J. Lin, and A. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Trans. on Automatic Control*, vol. 48, pp. 988–1001, 2003.
[6] L. Moreau, "Leaderless coordination via bidirectional and unidirectional time-dependent communication," in *Proceedings of the 42nd IEEE Conference on Decision and Control*, Maui, Hawaii, Dec 2003.
[7] Z. Qu, J. Wang, and R. A. Hull, "Cooperative control of dynamical systems with application to mobile robot formation," in *The 10th IFAC/IFORS/IMACS/IFIP Symposium on Large Scale Systems: Theory and Applications*, Japan, July 2004.
[8] C. Reynolds, "Flocks, herds, and schools: a distributed behavioral model," *Computer Graphics (ACM SIGGRAPH 87 Conference Proceedings*, vol. 21(4), pp. 25–34, 1987.
[9] T. Vicsek, A. Czirok, E. B. Jacob, I. Cohen, and O. Shochet, "Novel type of phase transition in a system of self-driven particles," *Physical Review Letters*, vol. 75, pp. 1226–1229, 1995.
[10] J. Wolfowitz, "Products of indecoposable, aperiodic, stochastic matrices," *Proc. Amer. Mathematical Soc.*, vol. 14, pp. 733–737, 1963.
[11] Z.Lin, M. Brouchke, and B. Francis, "Local control strategies for groups of mobile autonomous agents," *IEEE Trans. on Automatic Control*, vol. 49, pp. 622–629, 2004.