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Gain-Scheduled Control under Common Lyapunov Functions: Conservatism Revisited

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Abstract— **In this paper, we revisit the conservatism of gain-scheduled control design under common Lyapunov functions. Although recent research tends to seek parameter-dependent Lyapunov functions to reduce the conservatism, we point out that the conservatism arising from seeking a common Lyapunov function can be reduced in a different manner to the conventional method. If a condition is satisfied, we obtain a set of extreme controllers that achieve the best performance at vertices. Otherwise, an interpolated controller can be constructed via an interesting combined convex structure. An illustrated example demonstrates the applicability of the proposed method.**

I. INTRODUCTION

Since 1990's, much attention has been paid on LMI (Linear Matrix Inequality) - based control design, and a variety of control problems have been solved via LMIs under common Lyapunov functions [1], [3], [7], [8], [10].

Regarding gain-scheduled (GS) control as in [1], which can be considered for some class of nonlinear systems which can be described as LPV (Linear-Parameter Varying), if one select a common Lyapunov function for a whole operating range, the overall stability of the closed-loop system as time varying is guaranteed for any changing rate of the scheduling variable. However, selecting a common Lyapunov function for the whole operating range leads to conservatism of design. When one attempts to design a gain-scheduled controller, it sometimes results in almost the same one as a robust controller. In this case, most of gain-scheduled control gains designed at each vertex are similar to one another. (An illustrated example in such a case shall be given later.)

Many researchers have judged that this conservatism arises from selecting a common Lyapunov function and shifted their research into parameter dependent Lyapunov functions [2], [4], [5], [6], [9], [11], [12], [16], [17] (including research for robust control and multiobjective control).

However, theory of parameter dependent Lyapunov functions are more complicated and sometimes installed additional sufficient conditions or a line search parameter to make the problem convex. In addition, changing rates of scheduling variables are restricted in many cases. As a result, it has not been so useful for practitioners to use so far.

If the reason of the past conservatism of the gain-scheduled control under common Lyapunov functions is only in using common Lyapunov functions, it is natural to proceed to parameter-dependent Lyapunov functions. However, do we have no possibility to reduce the conservatism within using common Lyapunov functions? Shortly speaking, the answer is "YES." We still have some possibility to reduce the past conservatism even if we seek a common Lyapunov function for a whole operating range. In this paper, we shall clarifies theoretically what freedom still remains to reduce the conservatism and propose a new method to reduce the conservatism by utilizing this freedom.

To demonstrate the applicability of the proposed method, in this paper, we use a MDS (Mass-Damper-Spring) system whose damping and spring coefficients vary in proportion to the velocity V and its square V^2, respectively.

The rest of this paper is organized as follows. Section II gives preliminaries. In Sections III and IV, we present the main result. Section V provides an illustrated example, Section VI gives an alternative method if the proposed one cannot be applied straight, and Section VII concludes the paper. The notation is fairly standard. For a matrix M, M' denotes the transpose, and $M > 0$ ($M \geq 0$) means that $M = M'$ and that M is positive (semi-) definite. Tr (M) denotes the trace of M, Let $H_{zw}(s)$ denote the closed-loop transfer function from w to z. Finally, $\|\cdot\|_2$ denotes the H_2 norm.

II. PRELIMINARIES

A. Matrix Polytope

Let us denote the matrix polytope as follows.

$$\mathrm{Co}[\alpha](M_1,\dots,M_p) := \left\{ \sum_{i=1}^{p}\alpha_i M_i \ : \ \alpha_i \geq 0, \quad \sum_{i=1}^{p}\alpha_i = 1 \right\}, \quad \alpha = \begin{bmatrix}\alpha_1\\ \vdots \\ \alpha_p\end{bmatrix} \quad (1)$$

B. Polytopic Operating Range

Let us consider the following polytopic operating range:

$$\forall A(\rho) \in \mathcal{A} \ := \ \mathrm{Co}[\alpha](A_1,\dots,A_p) \quad (2)$$

$$\Omega \ := \ \{\ \rho : A(\rho) \ \in \ \mathcal{A}\ \} \quad (3)$$

Assuming that the scheduling variable $\rho(t)$ and the corresponding $\alpha_i(t)$, $i = 1, 2, \dots, p$ are measurable, we consider the gain-scheduled controller $K(\rho) = \mathrm{Co}[\alpha](K_1,\dots,K_p)$ for the given LPV system. Occasionally, if it is needed, assuming they are not measurable, we consider the robust controller $K = K_1 = \cdots = K_p$ for the given polytopic uncertainty.

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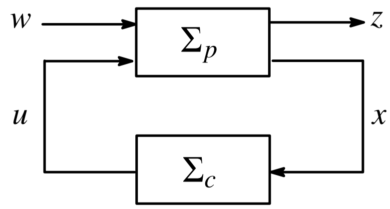

Fig. 1. Generalized plant-controller configuration.

C. Plant and Controller

Let us consider the LPV system:

$$\Sigma_p : \begin{cases} \dot{x} = A(\rho)x + Bu + Ew \\ z = Cx + Du \end{cases} \quad (4)$$

and the state-feedback gain-scheduled controller:

$$\Sigma_c : \quad u = -K(\rho)x, \quad (5)$$

where $x(t) \in \mathbf{R}^n$ is the state vector, $u(t) \in \mathbf{R}^m$ the control input, $w(t) \in \mathbf{R}^{n_w}$ the disturbance input, $z(t) \in \mathbf{R}^{n_z}$ the performance output, and $\rho := [\rho_1, \ldots, \rho_r]$ the scheduling variable. All the matrices in (4) have appropriate dimensions. Defining $Q := C'C$ and $R := D'D$, we assume that

(Assumption 1) $R > 0, \quad C'D = 0$;

(Assumption 2) $(A,\ B)$: controllable, $(C,\ A)$: observable.

D. Closed-loop System

Σ_c is connected to Σ_p by standard negative feedback, the closed-loop system is given as follows (Fig. 1).

$$\Sigma_{cl} : \begin{cases} \dot{x} = A_{cl}(\rho)x + B_{cl}w \\ z = C_{cl}(\rho)x \end{cases} \quad (6)$$

$$A_{cl}(\rho) := A(\rho) - BK(\rho),\ B_{cl} := E,\ C_{cl}(\rho) := C - DK(\rho)$$

E. Lyapunov Function

Using $P > 0$, the Lyapunov function is defined by

$$\Gamma(t) = x'Px > 0. \quad (7)$$

F. $\mathcal{H}_2$ Performance

For an impulse disturbance $w(t) = w_0\delta(t)$, $\|w_0\|_2 = 1$, let us consider the performance index to be minimized:

$$J_{zw} = \int_0^\infty z(t)'z(t)dt, \quad (8)$$

which is equivalent to

$$J_{zw} = \int_0^\infty (x'Qx + u'Ru)dt. \quad (9)$$

When the closed-loop system Σ_{cl} is time-invariant, this is identical to the square of the $\mathcal{H}_2$ norm of the closed-loop transfer function from w to z:

$$J_{zw} = \|H_{zw}(s)\|_2^2 \quad (10)$$

When the Lyapunov variable is time-varying, minimizing J_{zw} is equivalent to

$$\begin{array}{l} \inf[\mathrm{Tr}(B_{cl}'PB_{cl})] \quad \text{subject to} \\ P > 0, \quad \dot{P} + PA_{cl} + A_{cl}'P + C_{cl}'C_{cl} < 0. \end{array} \quad (11)$$

When the Lyapunov variable is time-invariant with $\dot{P} = 0$, minimizing J_{zw} is equivalent to

$$\begin{array}{l} \inf[\mathrm{Tr}(B_{cl}'PB_{cl})] \quad \text{subject to} \\ P > 0, \quad PA_{cl} + A_{cl}'P + C_{cl}'C_{cl} < 0. \end{array} \quad (12)$$

G. Definition of Matrix Functions

For convenience, let us define the following matrix functions related to $\mathcal{H}_2$ performance. Pre- and post-multiply (12) by $X = P^{-1} > 0$ and apply the Schur complement formula, we have

$$\begin{bmatrix} A_{cl}X + XA_{cl}' & XC_{cl}' \\ C_{cl}X & -I_{n_z} \end{bmatrix} < 0. \quad (13)$$

From $A_{cl} = A - BK$, $C_{cl} = C - DK$, (13) can be written as

$$\Phi_{\mathcal{H}_2}(A, K, X) := \begin{bmatrix} (A - BK)X + X(\bullet)' & * \\ (C - DK)X & -I_{n_z} \end{bmatrix} < 0. \quad (14)$$

Using the *change of variables* $W := KX$, (14) is transformed into

$$\Psi_{\mathcal{H}_2}(A, W, X) := \begin{bmatrix} (AX - BW) + (\bullet)' & * \\ CX - DW & -I_{n_z} \end{bmatrix} < 0. \quad (15)$$

Related to the $\mathcal{H}_2$ objective function, introducing a slack variable Z and noticing $B_{cl} = E$, we define

$$\Phi_0(X,\ Z) := \begin{bmatrix} X & * \\ E' & Z \end{bmatrix} > 0. \quad (16)$$

III. ROBUST CONTROL (GUARANTEED $\mathcal{H}_2$ COST)

First of all, let us review the conventional robust control under common Lyapunov functions. Here, we consider not only Lyapunov stability but also so-called guaranteed $\mathcal{H}_2$ cost control.

Problem 1 (Synthesis): Using a common Lyapunov solution X for all the plants $(A_i,\ B)$, $1 \le i \le p$ at each vertex of the convex hull $\mathcal{A} = \mathrm{Co}[\alpha](A_1, \ldots, A_p)$, we solve the following problem:

$$\begin{array}{l} J_{\mathrm{syn}} := \inf\limits_{W,\ X,\ Z}[\mathrm{Tr}(Z)] \quad \text{subject to} \\ \Phi_0(X,\ Z) > 0, \\ \Psi_{\mathcal{H}_2}(A_i, W, X) < 0, \quad 1 \le i \le p. \end{array} \quad (17)$$

Using the optimal solution set $(W,\ X)$, the control gain is given by $K = WX^{-1}$.

Theorem 1 (Analysis): Using a common Lyapunov solution X for all the plants $(A_i,\ B)$ at each vertex $1 \le i \le p$ of the convex hull $\mathcal{A} = \mathrm{Co}[\alpha](A_1, \ldots, A_p)$ and a given control gain K, if the following problem:

Problem 2 (Analysis at Vertices):

$$\begin{array}{l} J_{\mathrm{ana}} := \inf\limits_{X,\ Z}[\mathrm{Tr}(Z)] \quad \text{subject to} \\ \Phi_0(X,\ Z) > 0, \\ \Phi_{\mathcal{H}_2}(A_i, K, X) < 0, \quad 1 \le i \le p. \end{array} \quad (18)$$

is feasible, the K robustly stabilizes all the plants $(A(\rho),\ B)$, $\rho \in \Omega$ and the following problem:

Problem 3 (Analysis at Any Operating Point):

$$\begin{aligned} & J(\rho) := \inf_{X,\ Z}[\mathrm{Tr}(Z)] \quad \text{subject to} \\ & \Phi_0(X,\ Z) > 0, \\ & \Phi_{\mathcal{H}_2}(A(\rho), K, X) < 0, \quad \rho \in \Omega. \end{aligned} \tag{19}$$

is always feasible as well as we have $J(\rho) \le J_{\mathrm{ana}}$.

Proof: It is easy to prove both Theorem 1 and Corollary 1. Due to lack of space, it is omitted here. ■

Corollary 1: If we solve Problem 2 using K determined by Problem 1, then Problems 2 and 3 are feasible and we have $J(\rho) \le J_{\mathrm{ana}} = J_{\mathrm{syn}}$.

IV. GAIN-SCHEDULED CONTROL (GUARANTEED $\mathcal{H}_2$ COST)

A. Synthesis I (Conventional)

Problem 4 (Synthesis I): Using a common Lyapunov solution X for all the plants $(A_i,\ B)$ at each vertex $1 \le i \le p$ of the convex hull $\mathcal{A} = \mathrm{Co}[\alpha](A_1, \ldots, A_p)$, we solve the following problem:

$$\begin{aligned} & J_{\mathrm{syn}}^{\mathrm{I}} := \inf_{W_i, X, Z}[\mathrm{Tr}(Z)] \quad \text{subject to} \\ & \Phi_0(X,\ Z) > 0, \\ & \Psi_{\mathcal{H}_2}(A_i, W_i, X) < 0, \quad 1 \le i \le p. \end{aligned} \tag{20}$$

Using the optimal solution sets $(W_i,\ X)$, the control gains are given by $K_i = W_i X^{-1},\ 1 \le i \le p$.

B. Synthesis II (Proposed)

Problem 5 (Synthesis II): Using distinct Lyapunov solutions X_i for each plant $(A_i,\ B)$ at each vertex $1 \le i \le p$ of the convex hull $\mathcal{A} = \mathrm{Co}[\alpha](A_1, \ldots, A_p)$, we solve the following p problems:

$$\begin{aligned} & J_{\mathrm{opt\_}i} := \inf_{W_i, X_i, Z_i}[\mathrm{Tr}(Z_i)] \quad \text{subject to} \\ & \Phi_0(X_i,\ Z_i) > 0, \\ & \Psi_{\mathcal{H}_2}(A_i, W_i, X_i) < 0; \quad 1 \le i \le p. \end{aligned} \tag{21}$$

Using the optimal solution sets $(W_i,\ X_i)$, the control gains are given by $K_i = W_i X_i^{-1},\ 1 \le i \le p$.
In the sequel, we sometimes use the suffix opt_i instead of just i for K_i, W_i, X_i and Z_i in Problem 5 such as $K_{\mathrm{opt\_}i}$ in order to distinguish them from those in Problem 4.

C. Analysis

Theorem 2 (Analysis): Using a common Lyapunov solution X for all the plants $(A_i,\ B)$ with the control gains K_i at each vertex $1 \le i \le p$ of the convex hull $\mathcal{A} = \mathrm{Co}[\alpha](A_1, \ldots, A_p)$, if the following problem:

Problem 6 (Analysis at Vertices):

$$\begin{aligned} & J_{\mathrm{ana}} := \inf_{X,\ Z}[\mathrm{Tr}(Z)] \quad \text{subject to} \\ & \Phi_0(X,\ Z) > 0, \\ & \Phi_{\mathcal{H}_2}(A_i, K_i, X) < 0, \quad 1 \le i \le p. \end{aligned} \tag{22}$$

is feasible, the gain-scheduled controller:

$$K(\rho) = \mathrm{Co}[\alpha](K_1, \ldots, K_p) \tag{23}$$

(α is determined from the operating point ρ) robustly stabilizes the time-varying plant $(A(\rho),\ B)$, $\rho \in \Omega$ and the following problem:

Problem 7 (Analysis at Any Operating Point):

$$\begin{aligned} & J(\rho) := \inf_{X,\ Z}[\mathrm{Tr}(Z)] \quad \text{subject to} \\ & \Phi_0(X,\ Z) > 0, \\ & \Phi_{\mathcal{H}_2}(A(\rho), K(\rho), X) < 0, \quad \rho \in \Omega. \end{aligned} \tag{24}$$

is always feasible as well as we have $J(\rho) \le J_{\mathrm{ana}}$.

Proof: It is easy to prove Theorem 2 with Corollaries 2 and 3. Due to lack of space, it is omitted here. ■

For convenience, let K_i^{I} and K_i^{II} denote $K_i,\ 1 \le i \le p$ determined by Problems 4 and 5, respectively. Let $J_{\mathrm{ana}}^{\mathrm{I}}$ and $J_{\mathrm{ana}}^{\mathrm{II}}$ denote J_{ana} when Problem 6 is solved with K_i^{I} and K_i^{II}, respectively. Let K^{I} and K^{II} denote the corresponding gain-scheduled controllers. Similarly, let $J^{\mathrm{I}}(\rho)$ and $J^{\mathrm{II}}(\rho)$ denote $J(\rho)$ of Problem 7 solved with K_i^{I} and K_i^{II}.

Corollary 2 (conventional): When we solve Problem 6 using $K_i^{\mathrm{I}},\ 1 \le i \le p$ determined by Problem 4, then Problem 6 is feasible and the gain-scheduled controller K^{I} stabilizes the time-varying plant $(A(\rho),\ B)$, $\rho \in \Omega$. Furthermore, Problem 7 is feasible and we have $J(\rho) \le J_{\mathrm{ana}}^{\mathrm{I}} = J_{\mathrm{syn}}^{\mathrm{I}}$.

Corollary 3 (proposed): When we solve Problem 6 using $K_i^{\mathrm{II}},\ 1 \le i \le p$ determined by Problem 5, if Problem 6 is feasible, the gain-scheduled controller K^{II} stabilizes the time-varying plant $(A(\rho),\ B)$, $\rho \in \Omega$. Furthermore, Problem 7 is feasible and we have $J(\rho) \le J_{\mathrm{ana}}^{\mathrm{II}}$.

Theorem 3: $J_{\mathrm{ana}}^{\mathrm{I}}$ of Problem 6 with $K_i^{\mathrm{I}},\ 1 \le i \le p$ determined by Problem 4, is minimum among all J_{ana} of Problem 6 with any $K_i,\ 1 \le i \le p$. That is, we have $J_{\mathrm{ana}}^{\mathrm{I}} \le J_{\mathrm{ana}}$.

Proof: It is easy to prove both Theorem 3 and Corollary 4. Due to lack of space, it is omitted here. ■

Corollary 4: Between $J_{\mathrm{ana}}^{\mathrm{I}}$ and $J_{\mathrm{ana}}^{\mathrm{II}}$ of Problem 6 with K^{I} and K^{II}, respectively, we have the relation: $J_{\mathrm{ana}}^{\mathrm{I}} \le J_{\mathrm{ana}}^{\mathrm{II}}$.

D. Conservatism Revisited

The reason why the conventional gain-scheduled control design is carried out by Problem 4 (Synthesis I), is in Corollary 2 and Theorem 3. In Problem 4, even if the Lyapunov solution X is constrained to be common for all the vertices, the variables W_i can be selected independently. Therefore, it seems that the controller gains K_i can be determined independently for each vertex. However, the variables W_i sometimes become similar to one another, affected by the above constraint of the Lyapunov variable X. As a result, the controller gains K_i sometimes become similar to one another even though they should be determined independently. Such an example shall be given in the sequel. In this case, the gain-scheduled control will become almost the same as robust control.
Faced with this situation, many researchers have judged this conservatism arises from using common Lyapunov functions and shifted their research into parameter dependent Lyapunov functions. However, from the difficulties as described in INTRODUCTION, theory of parameter dependent Lyapunov functions has not been so useful for practitioners to use so far. If the reason of conservatism

arising from seeking a common Lyapunov function of gain-scheduled control is only in using common Lyapunov functions, it is natural to proceed to parameter-dependent Lyapunov functions. However, as described in INTRODUCTION, we still have some possibility to reduce the past conservatism even if we seek common Lyapunov functions. Since $J_{\rm ana}$ gives an upper bound on the actual cost $J(\rho)$ for any operating point $\rho \in \Omega$ and it is said to be the "*worst case guaranteed cost*," it seems that minimization of $J_{\rm ana}$ leads to minimization of $J(\rho)$. However, $J_{\rm ana}$ is just an upper bound and there may be some amount of (sometimes very large) gap between $J_{\rm ana}$ and $J(\rho)$. Even if one minimizes the upper bound cost $J_{\rm ana}$, the actual cost $J(\rho)$ is not always minimized. The conventional gain-scheduled control design based on Problem 4 (Synthesis I), carries out this "*upper bound cost minimization*." It is very important to point out that this is the TRUE CAUSE of the past conservatism. Noticing this fact, one can consider a new way to reduce the past conservatism. Instead of Problem 4 (Synthesis I), the problem to be actually solved here is given as follows.

Problem 8: Among sets of controllers K_i, $1 \le i \le p$ with which Problem 6 is feasible, find a set of controllers such that the *actual worst case cost*:

$$\bar{J} := \max_{\rho}\{\ J(\rho) : \rho \in \Omega\ \} \tag{25}$$

is minimized.

However, it is very difficult to solve this problem directly. Therefore, we focus on the actual costs only at the vertices $J(\rho_i)$, $1 \le i \le p$. The problem is described as follows.

Problem 9: Among sets of controllers K_i, $1 \le i \le p$ with which Problem 6 is feasible, find a set of controllers such that the *actual costs at vertices* $J(\rho_i)$, $1 \le i \le p$ are minimized.

Minimization of $J(\rho_i)$, $1 \le i \le p$ may not lead to minimization of $\bar{J}$. However, if we minimize $J(\rho)$ directly at least at the vertices, it may be minimized over a whole operating range. Furthermore, one of $J(\rho_i)$, $1 \le i \le p$ sometimes coincides with the *actual worst case cost* $\bar{J}$. In this case, Problem 8 reduces to Problem 9. We shall show such an example in the sequel.

Theorem 4: The theoretical lower bound of $J(\rho)$ of Problem 7 is given by $J_{{\rm opt}\_i}$ of Problem 5 at the vertices.

$$J_{{\rm opt}\_i} \le J(\rho_i), \qquad 1 \le i \le p. \tag{26}$$

Proof: It is easy to prove Theorem 4 with Corollaries 5 and 6. Due to lack of space, it is omitted here. ■

Corollary 5 (proposed): When we solve Problem 6 using $K_i^{\rm II}$, $1 \le i \le p$ determined by Problem 5, if Problem 6 is feasible, the gain-scheduled controller $K^{\rm II}$ stabilizes the time-varying plant $(A(\rho), B)$, $\rho \in \Omega$. Furthermore, Problem 7 is feasible and we have $J(\rho) \le J_{\rm ana}^{\rm II}$. Especially, $J(\rho)$ achieves its theoretical lower bound $J_{{\rm opt}\_i}$ at the vertices.

$$J(\rho_i) = J_{{\rm opt}\_i}, \qquad 1 \le i \le p \tag{27}$$

Corollary 6: Between $J_{\rm ana}^{\rm I}$ and $J_{\rm ana}^{\rm II}$ determined through Problem 6 by the gain-scheduled controllers $K^{\rm I}$ and $K^{\rm II}$, respectively, we have the relation $J_{\rm ana}^{\rm I} \le J_{\rm ana}^{\rm II}$. Especially, at the vertices, we have

$$J^{\rm II}(\rho_i) \le J^{\rm I}(\rho_i) \le J_{\rm ana}^{\rm I} \le J_{\rm ana}^{\rm II}, \quad 1 \le i \le p. \tag{28}$$

Thus, regarding the upper bounds, $J_{\rm ana}^{\rm I}$ is smaller than (or equal to) $J_{\rm ana}^{\rm II}$, while regarding the actual cost $J(\rho)$, the relation between the values corresponding to $K^{\rm I}$ and $K^{\rm II}$ is reversed at least at the vertices. Moreover, $J^{\rm II}(\rho_i)$ achieves the theoretical lower bound $J_{{\rm opt}\_i}$ at the vertices.

V. NUMERICAL EXAMPLE

A. LPV Model

As an illustrated example of LPV systems, we consider a MDS system, in which the spring and damping coefficients vary in proportion to the velocity V and its square V^2 as follows ($k = 0.1, \quad d = 0.01$).

$$\Sigma_p : \left\{ \begin{array}{l} \dot{x} = \begin{bmatrix} 0 & 1 \\ -kV & -dV^2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 5 \end{bmatrix} w + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ z = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \end{array} \right. \tag{29}$$

B. Construction of Convex Hulls

Let us define the numbers of the vertices depicted as in Fig. 2 and consider the case where the MDS system is accelerated from $V = 100$ (m/s) to $V = 300$ (m/s). It is noted that this LPV system linearly depends on the scheduling variables V and V^2 which are not independent of each other. To obtain less conservative design result, dependency of them should be taken into account.

When we consider as if the variables V and V^2 were independent of each other, the operating range would be the rectangular 1-9-7-10 as depicted in Fig. 2. However, the actual path of the operating points goes only through the curve $\eta = V^2$. Any other areas are not used at all. When we consider tighter convex hulls, the tightest (area-minimized) one is △1-8-7 in the case of three vertices, whereas it

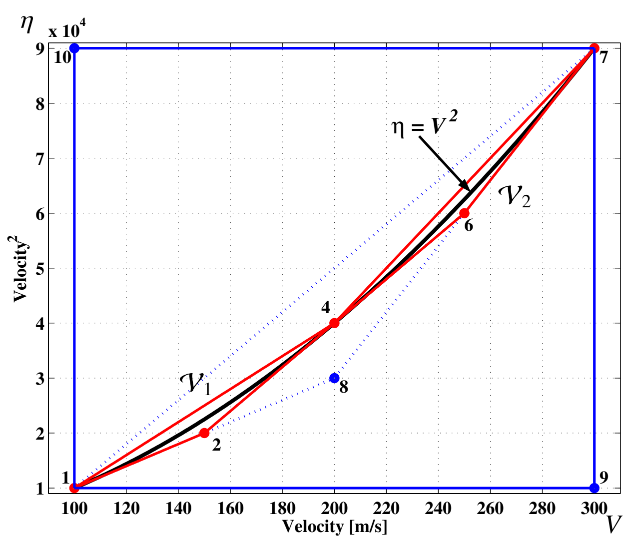

Fig. 2. Division of a whole operating range into two subregions.

TABLE I

ROBUST CONTROL

Region	K_{rob}
$\mathcal{V}_1$	[1.8480 5.4244]
$\mathcal{V}_2$	[1.0395 1.9952]

TABLE II

GAIN-SCHEDULED CONTROL

Region	i	$K_i^{I} = K_i$ (conventional)	$K_i^{II} = K_{opt\_i}$ (proposed)
$\mathcal{V}_1$	1	[1.8630 5.2969]	[1.1803 0.0118]
	2	[1.8481 5.4243]	[0.8114 0.0041]
	4	[1.8375 5.3282]	[0.6155 0.0015]
$\mathcal{V}_2$	4	[1.0395 1.9950]	[0.6155 0.0015]
	6	[1.0397 1.9954]	[0.4951 0.0008]
	7	[1.0394 1.9948]	[0.4138 0.0005]

becomes the trapezoid 1-2-6-7 in the case of four vertices. Since the gap in the concave side of the curve $\eta = V^2$ cannot be removed by increasing the number of vertices, we divide the whole operating range into two convex hulls of $\triangle$1-2-4 and $\triangle$4-6-7 in this paper.

Using a common Lyapunov function within each convex hull while allowing for distinct ones over them, we design two gain-scheduled controllers and switch them from one to another at the intersection ($i = 4$) of both convex hulls.

C. Relation between ρ and the combination factor α

In the current numerical example, the operating point ρ is given by $\rho = [V \quad V^2]'$. In application of gain-scheduled control, we should obtain an explicit relation $\alpha = f(\rho)$ between the operating point ρ and the combination factor α from Fig. 2. The result is given as follows.

$$\alpha_1 = \frac{(V_j - V)^2}{(V_j - V_i)^2}, \quad \alpha_2 = \frac{2(V_j - V)(V - V_i)}{(V_j - V_i)^2}, \quad \alpha_3 = \frac{(V - V_i)^2}{(V_j - V_i)^2}, \tag{30}$$

where V_i and V_j is set to the lowest and the highest velocities of the convex hull under consideration, respectively.

D. Design Result

Let us define the subregions as $\mathcal{V}_1 : 100 \le V \le 200$ (m/s) and $\mathcal{V}_2 : 200 \le V \le 300$ (m/s) for convenience. Tables I and II show the design result. There is little difference between the robust controller (K_{rob}) and the conventional gain-scheduled controller ($K_i^{I} = K_i$). In contrast, the gain of the proposed gain-scheduled controller ($K_i^{II} = K_{opt\_i}$) smoothly transfers as the operating point transferring.

The upper bounds $J_{ana}^{rob} \le J_{ana}^{I} \le J_{ana}^{II}$ and the actual costs $J^{opt}(\rho) \le J^{II}(\rho) \le J^{I}(\rho) \le J^{rob}(\rho)$ calculated by these controllers for the region of $\mathcal{V}_1$ is given in Fig. 3. The cost $J^{I}(\rho)$ given by the conventional gain-scheduled controller K^{I} is almost the same as the cost $J^{rob}(\rho)$ by the robust controller K_{rob} except for the neighbor around $V = 100$ (m/s). In contrast, the cost $J^{II}(\rho)$ given by the proposed controller K^{II} almost achieves the theoretical lower bound $J^{opt}(\rho)$, which is the optimal cost of the $\mathcal{H}_2$ optimal controller $K_{opt}(\rho)$ calculated at any grid of the operating points ($V = 100, 110, 120, \ldots, 200$).

Note that the gap between the worst case upper bounds $J_{ana}^{rob} \le J_{ana}^{I} \le J_{ana}^{II}$ and the actual costs $J^{opt}(\rho) \le J^{II}(\rho) \le J^{I}(\rho) \le J^{rob}(\rho)$ is extremely large. In addition, $J(\rho_i)$, $i = 1$ coincides with the actual worst case cost $\bar{J}$ in this example.

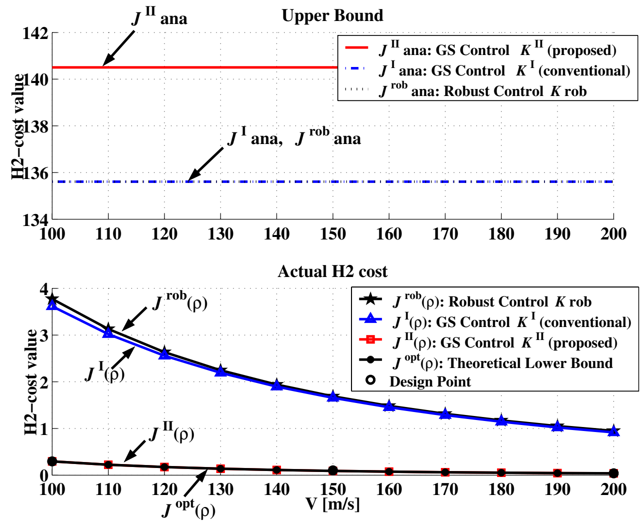

Fig. 3. Comparison of the individual $\mathcal{H}_2$ performances.

E. Time-Response Simulation

Let us consider the case where the MDS system is accelerated from $V = 100$ to $V = 300$ (m/s) by the rate of $\dot{V} = 1$ (m/s^2) and the controllers are switched at the mid point $V = 200$ (m/s) when $t = 100$ (sec). The simulation result of the time response is given in Fig. 4. In both the robust control (K_{rob}) and the conventional gain-scheduled control ($K_i^{I} = K_i$), undesired non-smooth changes of the control input u occur at $V = 100$ (m/s) where the controllers are switched. In contrast, the proposed controller ($K_i^{II} = K_{opt\_i}$) demonstrates very smooth switching. See also Table II for checking discontinuity or continuity between two gains $K_4^{I} = K_4$ of $\mathcal{V}_1$ and $\mathcal{V}_2$ (conventional) or $K_4^{II} = K_{opt\ 4}$ of them (proposed). Another interesting method to avoid discontinuity on switching is included in [13].

VI. ALTERNATIVE METHOD

Even if the proposed method (Design II) cannot be applicable, in other words, Problem 6 is infeasible, we can take at least two alternative methods. The simplest one is to reduce the volume of each convex hull while increasing the number of convex hulls until Problem 6 becomes feasible. Another alternative method is to seek a middle controller K_{mid_i} given by some indirect interpolation of the conventional one (K_i^{I}) and the proposed one (K_i^{II}). Recall that the conventional controller K_i^{I} and the proposed one K_i^{II} are given as follows.

$$\begin{aligned} K_i^{I} &= K_i \ = W_i X^{-1}, \\ K_i^{II} &= K_{opt\_i} = W_{opt\_i} X_{opt\_i}^{-1}, \quad 1 \le i \le p, \end{aligned} \tag{31}$$

where we use the suffix "opt_i" for the proposed controller K_i^{II} because it is an optimal controller at each vertex. The middle controller K_{mid_i} is given by the following indirect

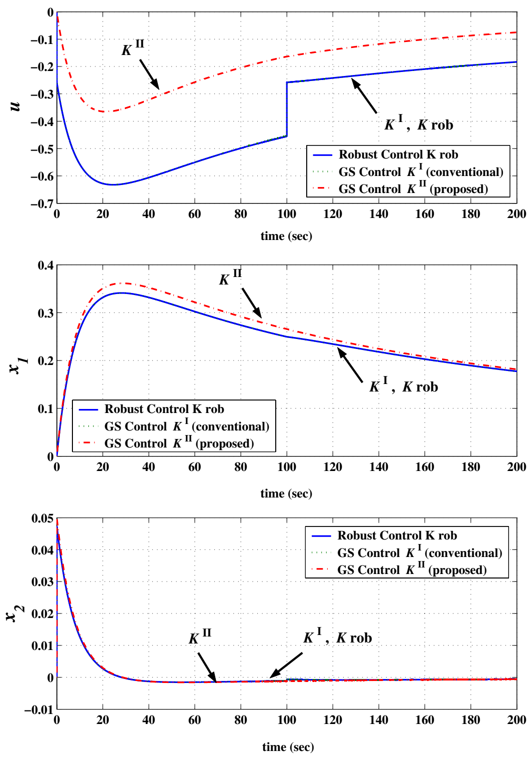

Fig. 4. Time-response simulation encountering switching action.

interpolation of $K_i^{\rm I}$ and $K_i^{\rm II}$. ($\beta := [\beta_1 \ \ \beta_2]'$)

$$K_{{\rm mid}\_i} = W_{{\rm mid}\_i} X_{{\rm mid}\_i}^{-1}, \qquad 1 \le i \le p, \tag{32}$$
$$W_{{\rm mid}\_i} = {\rm Co}[\beta](W_i,\ W_{{\rm opt}\_i}), \quad X_{{\rm mid}\_i} = {\rm Co}[\beta](X_i,\ X_{{\rm opt}\_i}).$$

This interpolation technique is somehow related to [15]. Even if Problem 6 is infeasible for $K_i^{\rm II}$, if Problem 4 is feasible for $K_i^{\rm I}$, Problem 6 becomes feasible for $K_{{\rm mid}\_i}$ as β_1 increases from zero to unity. Using this property, we can seek a middle controller which may improve the past conservatism even if the proposed method (Design II) cannot be applicable. The actual cost $J^{\rm mid}(\rho_i)$ determined by $K_{{\rm mid}\_i}$ lies between the costs of the conventional method and the proposed one as follows [14].

$$J^{\rm II}(\rho_i) \le J^{\rm mid}(\rho_i) \le J^{\rm I}(\rho_i), \quad 1 \le i \le p. \tag{33}$$

One example is calculated as in Fig. 5.

VII. Conclusion

In this paper, we have revisited the conservatism of gain-scheduled control design under common Lyapunov functions. First, it has been clarified what freedom still remains to reduce the past conservatism even if we seek a common Lyapunov function for a whole operating range. Second, utilizing this freedom, we have proposed a simple new method to reduce the past conservatism under common Lyapunov functions. This new method is available if a condition is satisfied. Otherwise, we have proposed an alternative one based on an interesting combined convex structure. Through an illustrated example, the applicability of the proposed method has been demonstrated.

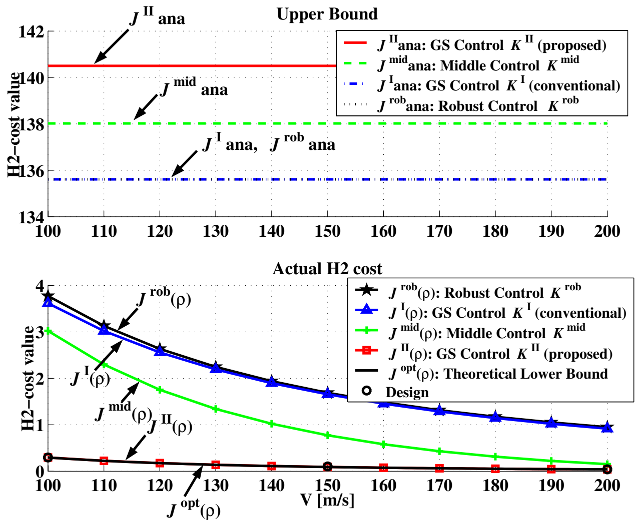

Fig. 5. Comparison of the individual $\mathcal{H}_2$ performances.

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