A Remark on IVP and TVP Non-Smooth Viscosity Solutions to Hamilton-Jacobi Equations

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***Abstract*— First-order PDEs arise in many applications in physics, control, image processing and other. Some boundary conditions for the PDE need to be formulated to close the problem. In time dependent problems very often the boundary conditions are specified at the initial time instant, and the problem is called the Initial Value Problem (IVP). Respectively, such conditions may be needed at the terminal time instant, in which case one has the Terminal Value Problem (TVP). For time invariant problems these two types of problems still exist, though the difference between them is not that explicit. For computational convenience one can transform from TVP to IVP or vice versa.**

In control problem a TVP for HJBI equation naturally arises, while in mechanics (physics) a IVP originally arises. Which type of the problems should be considered depends also on the fact that the sought for function (value function or action) in these problems is introduced as the function of the left or right endpoint of the cost function in integral form.

As to first-order PDEs arising in image processing there is no straightforward indication what kind of problem must be considered.

This paper discusses the difference between IVP and TVP solutions, the connection between Hamiltonians arising in formulation of IVP and TVP, several illustrative examples are demonstrated, one of them showing a smoothening phenomenon in the consequent solutions to IVP, TVP.

I. INTRODUCTION

For all nonlinear first-order PDEs arising in control, physics and other applications, generally, one has a boundary value problem of the form:

$$F(x, u, \partial u/\partial x) = 0, \quad x \in \Omega \subset R^n \tag{1}$$

$$u(x) = v(x), \quad x \in M \subset \partial\Omega$$

This form embraces also the Hamilton-Jacobi equation

$$\frac{\partial u}{\partial t} + H(x, \frac{\partial u}{\partial x}, t) = 0, \quad (x, t) \in \Omega \tag{2}$$

if we treat the time variable as one of the components of the vector x: $(x, t) \to x$.

Under the condition $u(x), F(x, u, p) \in C^2$ the local construction of the solution to the problem (1) is known to be reduced to the integration of the following system of (regular) characteristics ($p = \partial u/\partial x$):

$$\dot{x} = F_p, \quad \dot{u} = \langle p, F_p \rangle, \quad \dot{p} = -F_x - pF_u \tag{3}$$

In image processing, particularly in Shape-From-Shading, a first order PDE must be solved in order to reconstruct the shape of a 2D surface in 3D space using a 2D flat image of the surface, for instance, a photograph of the surface.

The 2D image is described by the *intensity function* $I(x)$, where $x = (x_1, x_2) \in G$ is a point of the image region G. Under certain assumptions about reflection physics the intensity appears to be a function of the form:

$$I(x) = \langle \gamma, n(x) \rangle$$

where $n(x)$ is the unit normal to the surface at the point $x \in G$, and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ is the fixed direction from which the light is coming. Expressing the normal through the partials of the surface height function $x_3 = u(x_1, x_2)$ leads to the following first-order PDE [6], [14]:

$$I(x) = \frac{-\gamma_1 u_{x_1} - \gamma_2 u_{x_2} + \gamma_3}{\sqrt{1 + u_{x_1}^2 + u_{x_2}^2}} \tag{4}$$

In case of a special geometry for the pictures taken from a satellite the intensity is expressed through the surface function partials in a different way which yields a slightly different PDE of the form [5]:

$$I(x) = \frac{u_{x_2}^2}{\sqrt{1 + u_{x_1}^2 + u_{x_2}^2}} \tag{5}$$

In many problems of Physics and Control one or both of the functions $u(x), F(x, u, p)$ may be nonsmooth and the solution to the problem (1) must be understood in a generalized sense. One of most powerful approaches to the generalization of the solution developed during recent decades is the theory of viscosity solutions [11], [10].

As discussed in [6], the notion of viscosity solution meets also the requirements of the shape-from-shading problem.

One of the attractive construction methods for the generalized (viscosity) solution is still the method of characteristics, appropriately generalized [4].

II. VISCOSITY SOLUTION AND BOUNDARY CONDITIONS

To get a unique solution to the PDE (4) or (5) one needs to specify appropriate boundary (initial, terminal) conditions. A comprehensive analysis of possible boundary conditions in shape-from-shading problems is given in [6].

Both, for viscosity and classical solution one must distinguish between two types of problems – the Initial value problem (IVP) and the Terminal value problem (TVP). The difference between IVP and TVP is clear for the case of the Hamilton-Jacobi form (2) of the equation (1), when the boundary conditions are specified at the initial (terminal) time instant $t = t_0$ $(t = t_1)$. The surface M in (1) for the

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IVP and TVP correspondingly has the form:

$$M = M_0 = \{(x,t) \in R^{n+1} : t = t_0\},$$
$$M = M_1 = \{(x,t) \in R^{n+1} : t = t_1\} \quad (6)$$

so that M is a "half" of the boundary $\partial\Omega$.

In time-invariant setting the surface M may coincide with $\partial\Omega$, or needs to be specified specially. The viscosity solution theory suggests general approach to specifying M and boundary condition formulation [10].

In differential game theory M is called the "usable" part of the boundary, and certain necessary conditions for M are given in [1].

We recall the definitions of viscosity solutions for both IVP and TVP in terms of scalar test-functions $\varphi(x) \in C^1(\Omega)$.

A continuous function $u : \Omega + M \to R^1$ is called the viscosity solution of IVP (1) if:

1) $u(x) = v(x), \quad x \in M$;

2) for every test-function $\varphi(x) \in C^1(\Omega)$ such that local minimum (maximum) of $u(x) - \varphi(x)$ is attained at $x^0 \in \Omega$ the following inequality holds:

$$F(x^0, u(x^0), \varphi_x(x^0)) \geq 0$$
$$(F(x^0, u(x^0), \varphi_x(x^0)) \leq 0) \quad (7)$$

For the definition of a viscosity solution of TVP the inequalities in (7) must be changed to opposite ones with, generally, a different terminal surface M.

For the surface reconstruction in shape-from-shading problems one needs to know the shape on some curve, a part of boundary. Such a boundary shape generally is not available. So, one can reconstruct the shape up to an arbitrary boundary function. On the other hand, the shape reconstruction is sufficient up to an additive constant: $u(x)+C$. This constant C can be considered as a boundary data if the boundary surface is degenerate to a point. Such a boundary conditions are known and used in differential games [1] in the frame of the method of characteristics.

In several papers on shape-from-shading the extremal points (maximum or minimum) of the solution are considered as such a degenerate boundary.

The boundary conditions may have a certain type of singularities. As shown in [6], generically three types of edges (parts of boundary) may arise: apparent contours, grazing light edges and shadow edges. The first and second types of boundary edge are shown to have a singularity similar to that of in state constraint optimal control problem [7], [8]. The method of (generalized) characteristics states that a family of regular or singular characteristics is projected on this boundary [9].

III. Initial and Terminal Value Problems in Optimal Control

In control problem the TVP naturally arise, while in mechanics (physics) the IVP ones. Many practical variation calculus or optimal control problems require the consideration of the TVP, though for some constructions the IVP is also requested. A TVP arises when one is interested in the cost functional as a function of the left end of the optimal path, while an IVP arises in case of the interest in the right end of the path. It is clear that generally the IVP and TVP have different solutions.

In this section we use the letter u for control variable, while the scalar solutions to the PDE will be denoted as S or V.

Consider a traditional formulation of the Optimal Control problem which includes the dynamic equations:

$$\dot{x} = f(x,u,t), \quad u \in U \subset R^m, \quad t \in [t_0, t_1] \quad (8)$$

endpoint conditions:

$$x(t_0) = x^0, \quad x(t_1) = x^1, \quad x \in R^n \quad (9)$$

and the cost function (the functional):

$$J[x,u] = \int_{t_0}^{t_1} L(x,u,t)dt + \Phi_0(x(t_0), t_0) + \Phi_1(x(t_1), t_1) \to \min_{u(t)} \quad (10)$$

Here x is the n-dimensional state vector, u is the m-dimensional vector of control variables, U is a convex control constraint set in R^m, and $t_1 > t_0$.

By minimizing the functional introduce the function of both (left and right) ends of the optimal path:

$$J^*(x^0, t_0; x^1, t_1) = \min_{u(t)} \Big(\int_{t_0}^{t_1} L(x,u,t)dt + \Phi_0(x^0, t_0) + \Phi_1(x^1, t_1) \Big)$$

The dynamic programming approach allows to show that the function (of the left end)

$$S(x,t) = \min_{(x^1,t_1)} J^*(x,t;x^1,t_1), \quad (x^1, t_1) \in M_1$$

is the solution of the (terminal value) problem

$$\frac{\partial S}{\partial t} + H^l(x, \frac{\partial S}{\partial x}, t) = 0, \quad (x,t) \in \Omega \quad (11)$$
$$S(x,t) = \Phi_1(x,t), \quad (x,t) \in M_1 \quad (M_1 \subset \partial\Omega)$$

while the right end function

$$V(x,t) = \min_{(x^0,t_0)} J^*(x^0, t_0; x, t), \quad (x^0, t_0) \in M_0$$

is the solution of the (initial value) problem

$$\frac{\partial V}{\partial t} + H^r(x, \frac{\partial V}{\partial x}, t) = 0, \quad (x,t) \in \Omega \quad (12)$$
$$V(x,t) = \Phi_0(x,t), \quad (x,t) \in M_0 \quad (M_0 \subset \partial\Omega)$$

where M_1 and M_0 are given terminal and initial surface (manifold).

For the left (H^l) and right (H^r) Hamiltonians the dynamic programming technique gives the following relations:

$$\begin{aligned} H^l(x,p,t) &= \min_{u\in U}\left(\langle p, f(x,u,t)\rangle + L(x,u,t)\right)\\ H^r(x,p,t) &= \max_{u\in U}\left(\langle p, f(x,u,t)\rangle - L(x,u,t)\right)\end{aligned} \tag{13}$$

so that one has

$$H^r(x,p,t) = -H^l(x,-p,t) \tag{14}$$

Similar relation holds for the extended Hamiltonians, i.e. including the time partials $\partial S/\partial t$, $\partial V/\partial t$. In a typical Optimal Control problem the left Hamiltonian (13) is used.

For the Variation Calculus problem, where $f \equiv u$, $U = R^n$, the Hamiltonians H^l and H^r are Legandre transforms of $L(x,u,t)$:

$$\begin{aligned} H^l(x,p,t) &= \min_{u}\left(\langle p, u\rangle + L(x,u,t)\right)\\ H^r(x,p,t) &= \max_{u}\left(\langle p, u\rangle - L(x,u,t)\right)\end{aligned} \tag{15}$$

and the Hamilton-Jacobi equations have the same form (11), (12).

Note that the time-optimal control problem and the variation calculus problem with homogeneous Lagrangian, $L(x,\lambda\dot{x}) = \lambda L(x,\dot{x})$, $\lambda > 0$, possess the Hamilton-Jacobi-Bellman equation of the form

$$H(x,p) = 1$$

For an optimal control problem such Hamiltonian one can get from (15) by setting $L \equiv 1$; for variation calculus one has

$$H^r(x,p) = L(x,\omega(x,p)) \tag{16}$$

where the function $\dot{x} = \omega(x,p)$ is given implicitly by $p = (L^2)_{\dot{x}}/2$.

The terms "initial" or "terminal" come from the fact that optimal trajectories in IVP move from the surface M_0, while for a TVP they move towards M_1. In general problem (1) this terminology corresponds to the direction of flow of the auxiliary "time"-variable introduced naturally by the equations of regular characteristics (3): $\dot{x} = dx/dt = F_p$.

Depending upon the signs of the inequalities in (7), the problem (1) produces two viscosity solutions, each of them having its specific boundary surface $M_0, M_1 \subset \partial\Omega$. For the problems with time variables, like (11), (12), one typically has:

$$\begin{aligned} M_0 &= \{(x,t) \in R^{n+1} : t = t_0\},\\ M_1 &= \{(x,t) \in R^{n+1} : t = t_1\}\end{aligned} \tag{17}$$

Thus the pairs (F, M_0) and (F, M_1), consisting of the Hamiltonian and the boundary surface, produce two generally different viscosity solutions. One of these solutions uses the same inequalities in (7) as the initial value problem with M_0 as in (17), the other one has the inequalities corresponding to the terminal value problem with M_1 as in (17). This allows to call one of these functions the solution of IVP (1), and the other one — the solution of TVP (1).

In some problems the boundary surfaces for the IVP and TVP may coincide, $M_0 = M_1 = M$, say, $M = \partial\Omega$. This is not the case for the problems with boundary surfaces of the form (17). We will denote the viscosity solution $u(x)$ of the IVP (of the TVP) produced by the pair (F, M) by $I(F, M)$ (by $T(F, M)$). Generally, $I(F, M) \neq T(F, M)$. One can verify that $I(F, M) = T(-F, M)$. Thus,

$$I(F,M) \neq T(F,M) = I(-F,M)$$

In that sense, one can state that the equations $F = 0$ and $-F = 0$ have generally different viscosity solutions, which is not the case for classical solution.

If one prefers to solve IVP rather than TVP then the inverse time must be used, which is equivalent to changing sign in front of F (or H).

Generally, with a given Hamiltonian $H(x,p)$ one can associate eight boundary value problems since for each of four Hamiltonians

$$H(x,p), \quad -H(x,p), \quad -H(x,-p), \quad H(x,-p)$$

one can formulate a TVP or IVP. This is a general mathematical possibility, and only half of them have a physical meaning, see examples.

Thus, summarizing the considerations of IVP and TVP in terms of Bellman functions as the functions of the left and right ends of an optimal path, one can state that the following Hamiltonians may arise in considerations:

$$\pm H(x,\pm p)$$

The sign in front of H switches between the IVP and TVP, while the sign in front of p switches between the left and right ends.

The relation of the above considerations to general Two Point Boundary Value Problem (TPBVP) is the following. For the solution of the TPBVP one uses the conventional Hamiltonian of the form $H = H^l$ in (15), and some values of $x(t)$, $p(t)$ are (numerically) obtained. Changing the sign at H one can reverse the time, switching the roles of t_0 and t_1. Sensitivity at the points (x^0, t_0), (x^1, t_1) is given by the vectors $-p$ and p, respectively (Newton-Leibnitz formula), in accordance with the definition of IVP and TVP for the problem (13)-(15).

We consider here three examples demonstrating the diversity of the IVP and TVP solutions of the left and right HJB-equations.

A. Control of a car.

This time-optimal control problem is a particular case of the game of two cars by R.Isaacs. The dynamic equations are given by

$$\dot{x} = -uy, \quad \dot{y} = ux - 1, \quad |u| \leq 1$$

The goal of control is to bring the state vector (x, y) to the terminal circle

$$M: \quad x^2 + y^2 = l^2 \quad (0 \le l < 1)$$

in a minimum time.

The Hamiltonian for the time-optimal bringing to the terminal surface M has the form:

$$\begin{aligned} H(x, y, p, q) &= \min_u(-uyp + uxq - q) + 1 \\ &= -|qx - py| - q + 1 \end{aligned}$$

The optimal time $V(x, y)$ for this problem is the TVP solution to

$$\begin{aligned} &H(x, y, \partial V/\partial x, \partial V/\partial y) = 0, \\ &V(x, y) = 0,\ (x, y) \in M \end{aligned}$$

The solution of this problem one can reconstruct from the considerations in R.Isaacs' book. The so called usable part of the boundary, the set M_1 happens to be the upper part of the circle M.

Usually this problem is solved by switching to the IVP for the Hamiltonian with the opposite sign $-H(x, y, p, q)$ with the same usable boundary M_1.

The formally considered IVP for the problem

$$\begin{aligned} &H(x, y, \partial V/\partial x, \partial V/\partial y) = 0, \\ &V(x, y) = 0,\ (x, y) \in M_0 \end{aligned}$$

has the usable part M_0 (lower part of M) and a negative solution.

The physical interpretation have the IVPs for:

$$\begin{aligned} \text{problem2}: &\quad -H(x, y, p, q) = 0 \\ \text{problem3}: &\quad -H(x, y, -p, -q) = 0 \end{aligned}$$

and TVPs for:

$$\begin{aligned} \text{problem1}: &\quad H(x, y, p, q) = 0 \\ \text{problem4}: &\quad H(x, y, -p, -q) = 0 \end{aligned}$$

The solutions $V_i(x, y)$ of the i-th problem are related as follows:

$$V_1(x, y) = V_2(x, y), \quad V_3(x, y) = V_4(x, y) = V_1(x, -y)$$

Thus, the solutions $V_1(x, y)$, $V_2(x, y)$ and $V_3(x, y)$, $V_4(x, y)$ coincide, but $V_1(x, y)$, $V_3(x, y)$ are different.

B. A 2D differential game.

Consider an IVP with non-smooth initial data given by [4]:

$$F(x, y, p, q) = p + \sqrt{a^2 + q^2} - x\sqrt{b^2 + q^2} = 0,$$

$$x > 0, \quad u(0, y) = -|y| + cy, \tag{18}$$

$$(p = \partial u/\partial x, \quad q = \partial u/\partial y, \quad a, b, c = \text{const})$$

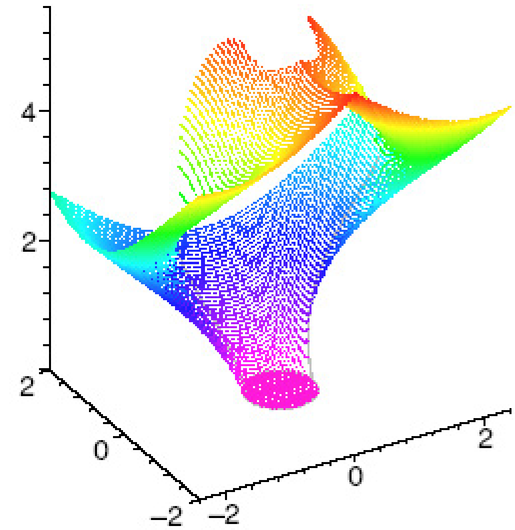

Fig. 1. Value function.

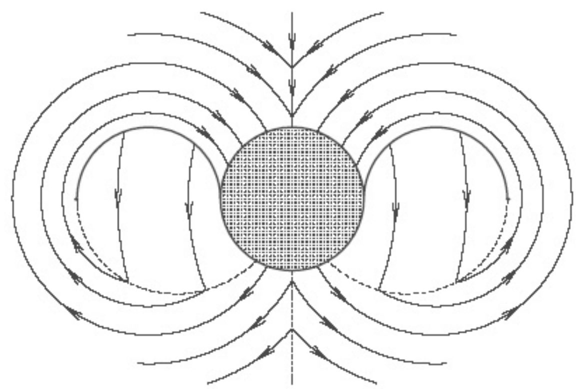

Fig. 2. Optimal characteristics.

One can show that the equation (18) is the HJBI equation for a fixed-time differential game with one spatial variable and a non-smooth terminal cost-function.

Set $a = b$ and $c = 0$. The solution of the IVP is given by the formula

$$\begin{aligned} u(x, y) &= \min[u^+(x, y), u^-(x, y)] \\ &= -|y| + \sqrt{a^2 + 1}(x^2/2 - x) \end{aligned} \tag{19}$$

$$u^{\pm}(x, y) = \mp y + \sqrt{a^2 + 1}(x^2/2 - x)$$

everywhere in the half-plane $y \ge 0$ except for the region:

$$x \ge 1, \quad |y| \le \frac{(x - 1)^2}{2\sqrt{a^2 + 1}}$$

where the solution equals to the following smooth function $v(x, y)$:

$$v(x, y) = -\sqrt{a^2 + 1}/2 + a\sqrt{(x - 1)^4/4 - y^2}$$

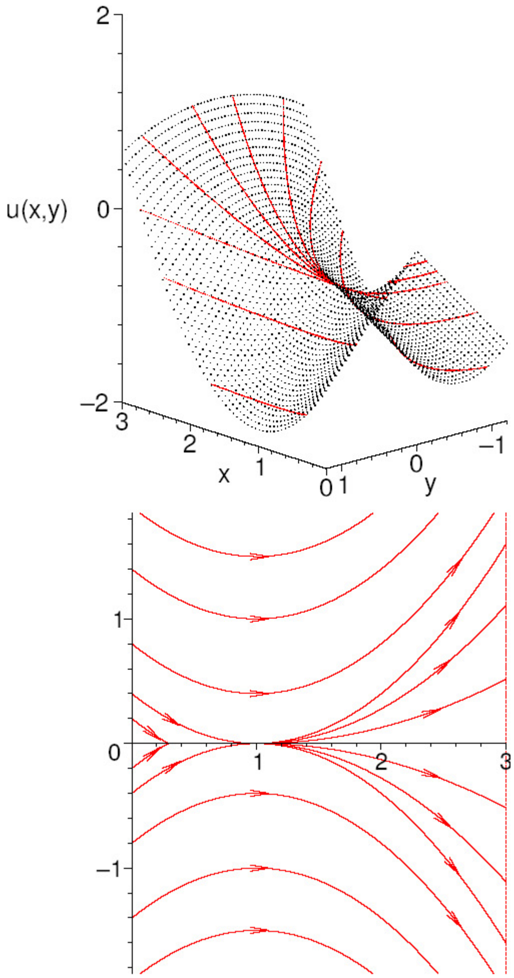

Fig. 3.

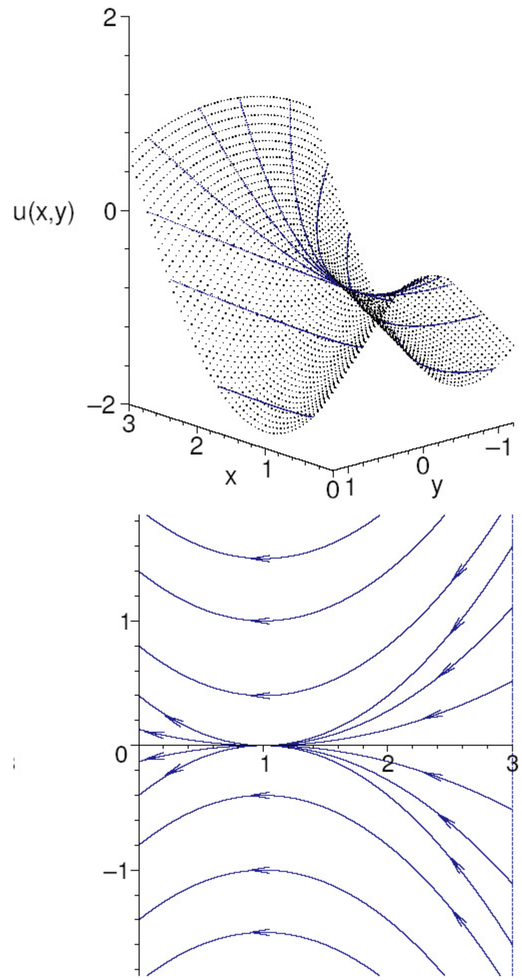

Fig. 4.

Fix now some positive value of x, say $x_T = 3$. For this fixed x_T the above IVP solution takes the values:

$$u(3,y) = -|y| + \frac{3}{2}\sqrt{a^2+1}, \quad |y| \geq \frac{2}{\sqrt{a^2+1}} \tag{20}$$

$$u(3,y) = -\sqrt{a^2+1}/2 + a\sqrt{4-y^2}, \quad |y| \leq \frac{2}{\sqrt{a^2+1}}$$

Now consider on the left half-plane $x \leq 3$ the TVP for the equation (18) with the terminal conditions (20). One can show that the solution of this TVP coincides with the previous IVP solution everywhere except for the region

$$x \leq 1, \quad |y| \leq \frac{(x-1)^2}{2\sqrt{a^2+1}}$$

where the TVP solution equals to $v(x,y)$, i.e. the condition $u(0,y) = -|y|$ is not fulfilled, and now the solution is smooth. So, we started with a non-smooth initial data, then "reflected" the solution at $x = 3$ and we get certain smoothening of the solution. If we continue "reflect" the solutions, solving consequently IVP and TVP, the solution will remain the same smooth function.

C. A scalar eikonal equation.

An example of this type presents the following problem with scalar x:

$$F \equiv u_x^2 - 1 = 0, \quad x \in (0,1); \quad u(0) = 0, \quad u(1) = 0$$

where $\Omega = (0,1)$ and $M = \partial\Omega$ consists of two points $\{0,1\}$. One can show that the $I(F,M)$ is the function $u = h(x)$:

$$h(x) = x, \quad x \in [0,1/2];$$
$$h(x) = 1-x, \quad x \in [1/2,1]$$

and the $T(F,M)$ is the function $u = -h(x)$

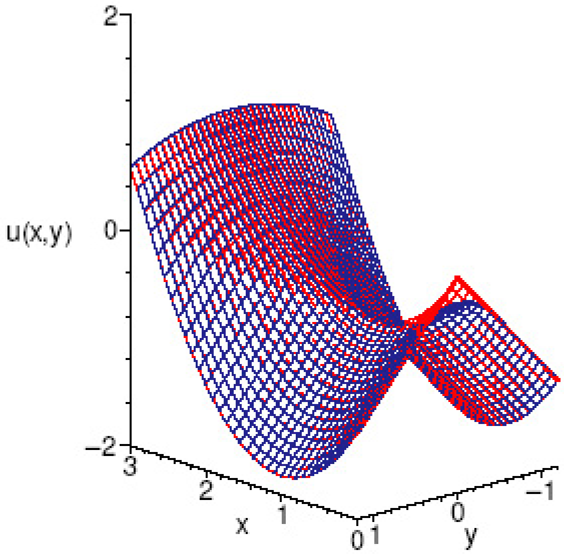

Fig. 5.

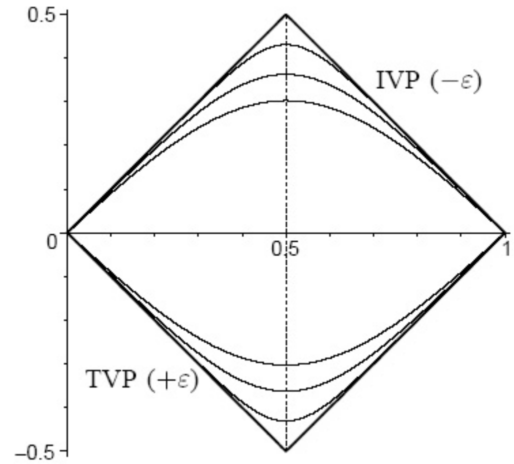

Fig. 6.

For this example one can calculate the functions $\pm h(x)$ as the limits of the solutions to the following second-order ODE

$$\pm\varepsilon u_{xx} + u_x^2 - 1 = 0, \quad u(0) = 0, \quad u(1) = 0$$

as the positive small parameter ε tends to zero. The sign "$-$" generates the IVP, while the sign "$+$" generates the TVP.

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