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Extended Applications of Generating Functions to Optimal Feedback Control Problems

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Abstract— **As a natural extension of our recent work on finding optimal feedback control laws based on generating functions of a Hamiltonian system, we consider an optimal control problem with control constraints and a singular optimal control problem. For the problem with control constraints, we consider the time optimal control of the double integrator, and show that our approach can recover the necessary and sufficient conditions of optimal feedback control laws directly. For the singular optimal control problem, we study the linear quadratic problem and show that our method reproduces the conventional solution satisfying the necessary conditions for optimality. The current study is used to more fully understand our approach with the goal of defining a method that is applicable to more general systems.**

Key Words. Minimum-time problem, Singular Optimal Control, Hamiltonian System, Generating Function, Hamilton-Jacobi Equation, Legendre Transformation

I. Introduction

Over the past 40 years, there have been tremendous developments in solving optimal control problems with control bounds and with singular control intervals. However, these problems remain challenging due to the inherent switching structure of the control which is not known a priori, the possible existence of singular control regimes, non-smoothness, and the possible singularity of the cost function. These technical difficulties have been serious barriers for all approaches, including the two main branches of inquiry; necessary conditions for optimality derived from Pontryagin's principle [1], and employment of the sufficient condition characterized by the Hamilton-Jacobi-Bellman (HJB) equation [2].

Recently, Park and Scheeres studied the optimal feedback control problem in the context of Hamiltonian dynamical systems [3][4][5]. They formally proved an underlying link between the optimal cost function and a generating function for a class of canonical transformations. This enabled them to devise a systematic methodology to evaluate the optimal feedback control and cost function satisfying both necessary and sufficient conditions, with the assumption of no constraints on states and control variables and of an analytic cost function. This approach was successfully applied to a nonlinear optimal control problem [6].

The current work is an extension of this approach into problems with control bounds and singular optimal control problems. The whole document is structured as follows. We formulate the optimal control problem as a Hamiltonian system. Then, we present a theorem which relates the optimal cost function and a certain kind of generating function, and discuss its implications. (section II).

We first consider a simple yet representative time optimal problem subject to control constraints. Given the limitation of solving the HJB equation directly due to the inherent singularity at the terminal boundary condition, we alternatively solve the Hamilton-Jacobi (HJ) equation for a certain generating function whose terminal boundary condition is always well-defined and non-singular. Then, this is transformed into the cost function. Unlike conventional methods based on the necessary conditions, our solution satisfies both necessary and sufficient conditions simultaneously. The cost of these favorable properties lies in the increase of dimension; for a system with n states, we must take into account $2n$ variables in the HJ equation for a generating function, whereas the HJB equation is a function of n states (section III).

Next we show that our method applies to the traditionally difficult singular problems encountered in optimal control problems. Studying the linear quadratic singular optimal control problem, we show that our approach reproduces the same solution as the conventional approach based on the necessary conditions, and indicate how our approach can be directly generalized to nonlinear systems (section IV).

II. Generating Functions for Solving Optimal Feedback Control Problems

Consider minimization of the following performance index

$$J = \phi(x(t_f), t_f) + \int_{t_0}^{t_f} L(x(\tau), u(\tau), \tau) d\tau$$

subject to the following system with initial condition

$$\dot{x}(t) = F(x(t), u(t), t) \quad , \quad x(t_0) = x_0$$

Here $x \in \Re^n$, $u \in \Re^m$, $t \in \Re$, $\phi(x(t_f), t_f) : \Re^n \times \Re \rightarrow \Re$, $L(x(\tau), u(\tau), \tau) : \Re^n \times \Re^m \times \Re \rightarrow \Re$, and $F(x(t), u(t), t) : \Re^n \times \Re^m \times \Re \rightarrow \Re^n$. Also t_0 and t_f represent the initial and terminal time index, respectively. We impose the relevant control bounds later. For the terminal boundary conditions, two extreme types are treated: (1) terminal states are completely specified to a fixed point in $\Re^n$ and (2) terminal states are completely unspecified. Given this problem statement, our objective is to evaluate the optimal trajectory which minimizes the cost function and satisfies the imposed boundary conditions and constraints,

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and to find the optimal *feedback* control law for a domain considered in $(x,t) \in \Re^n \times \Re$.

First define the pre-Hamiltonian $\bar{H}$ as

$$\bar{H}(x,\lambda,u,t) = L(x,u,t) + \lambda^T F(x,u,t) \tag{1}$$

where λ represents the costates. Then, Pontryagin's principle provides the necessary conditions for optimality [7]:

$$\begin{aligned} \dot{x} &= \bar{H}_\lambda(x,\lambda,u,t) && (2)\\ \dot{\lambda} &= -\bar{H}_x(x,\lambda,u,t) && (3)\\ u^*(x,\lambda,t) &= arg \min_{\bar{u}} \bar{H}(x,\lambda,\bar{u},t) && (4) \end{aligned}$$

Substituting (4) into (1), (2), and (3) defines a Hamiltonian canonical system for states and costates only:

$$\begin{aligned} H(x,\lambda,t) &= \bar{H}(x,\lambda,u^*(x,\lambda,t),t) && (5)\\ \dot{x} &= H_\lambda(x,\lambda,t) && (6)\\ \dot{\lambda} &= -H_x(x,\lambda,t) && (7) \end{aligned}$$

Evaluating the (candidate) optimal trajectory corresponds to solving this system of ordinary differential equations (ODEs) satisfying the relevant boundary conditions. If the terminal states are not specified, then the transversality condition provides n additional boundary conditions at the terminal time [7]:

$$\lambda(t_f) = \frac{\partial \phi(x(t_f),t_f)}{\partial x(t_f)}, \tag{8}$$

Since the total $2n$ boundary conditions are split, the optimal control problem is reduced to a two point boundary value problem (TPBVP).

Finally the candidate optimal trajectory found from this TPBVP should satisfy the sufficient condition for optimality, which is given by the following theorem [8]:

Theorem 2.1 (Sufficient Condition for Optimality): If $J(x,t)$ is sufficiently smooth and satisfies the Hamilton-Jacobi-Bellman (HJB) equation with the given boundary condition

$$\frac{\partial J}{\partial t}(x,t) + \min_{\bar{u}} \bar{H}\left(x, \frac{\partial J}{\partial x}, \bar{u}, t\right) = 0 \tag{9}$$
$$J(x_f,t_f) = \phi(x_f,t_f),$$

then it is the optimal cost function. Furthermore, the optimal control law is determined from

$$u = arg \min_{\bar{u}} \bar{H}\left(x, \frac{\partial J}{\partial x}, \bar{u}, t\right).$$

As is seen, this traditional procedure consists of two steps; first the candidate optimal trajectory is evaluated from the necessary condition, and then it is checked to see if it satisfies the HJB equation. This procedure is caused by the extreme difficulty of solving the HJB equation directly; except for very simple formulations the HJB equation does not have closed form solutions. Furthermore, for some types of boundary conditions, the HJB equation becomes singular at the terminal time [3], which adds to the problem difficulty.

Instead of this traditional two-step procedure, we employ our recently developed technique. We treat the trajectory $(x(t),\lambda(t))$ as a transformation between terminal coordinates (x,λ,t) and initial coordinates (x_0,λ_0,t_0), which is by definition a canonical transformation[1]. Then, using generating functions of the given canonical transformation, we evaluate the optimal trajectory, which satisfies both necessary and sufficient conditions simultaneously. The following theorem justifies our single-step approach:

Theorem 2.2 (Optimal Cost and Control Law from F_1): Let x_f be the (fixed) terminal state at t_f and x be the (moving) initial state at t. Also let $F_1(x_f,x,t_f,t)$ be a generating function for the given phase flow. Then, F_1 satisfies the necessary conditions of the TPBVP by definition. Also, the function

$$V(x,t) = -F_1(x_f,x,t_f,t) + \phi(x_f,t_f)$$

is the optimal cost function and satisfies the HJB equation and the sufficient conditions. Furthermore, the optimal feedback control can be expressed as

$$u = arg \min_{\bar{u}} \bar{H}\left(x, \frac{\partial V(x,t)}{\partial x}, \bar{u}, t\right)$$

Proof Refer to Park and Scheeres [5].

III. Application to Optimal Control Problem with Control Constraints

In the above theorem, we make no assumption on whether the control is bounded, assumptions made in our previous investigations in [3] and [4]. Thus, to show this generality in our approach we first consider a system with discontinuity and constraints on its controls.

A. Problem Formulation

We consider a simple yet representative time optimal control of a double integrator system: minimize

$$J = \int_{t_0}^{t_f} dt$$

subject to the double-integrator system with control constraints:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ u \end{bmatrix} \quad , \quad |u| \le 1 \tag{10}$$

The initial and terminal boundary conditions are given by

$$\begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \quad , \quad \begin{bmatrix} x_1(t_f) \\ x_2(t_f) \end{bmatrix} = \begin{bmatrix} x_{1f} \\ x_{2f} \end{bmatrix}$$

Define the pre-Hamiltonian as

$$\bar{H}(x,\lambda,u,t) = 1 + \lambda_1 x_2 + \lambda_2 u$$

Then, the costate equations are

$$\begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\lambda_1 \end{bmatrix} \Rightarrow \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ -c_1 t + c_2 \end{bmatrix} \tag{11}$$

[1]Refer to Greenwood [9], Goldstein [10], and Guibout and Scheeres [11] for a review of canonical transformations and generating functions.

where c_1 and c_2 are constants compatible with the given boundary condition. Pontryagin's principle yields the optimal control logic:

$$u = \arg\min_{\bar{u}} \bar{H}(x, \lambda, \bar{u}, t) = -\text{sgn}(\lambda_2),$$

which enables us to evaluate the Hamiltonian as a function of states and costates:

$$H(x, \lambda, t) = 1 + \lambda_1 x_2 - |\lambda_2|$$

Also the transversality condition for free final time provides [7]

$$H(t_f) = 1 + \lambda_1(t_f)x_2(t_f) - |\lambda_2(t_f)| = 0 \quad (12)$$

As is seen, the optimal control is determined by the sign of λ_2, which is a linear function in time t. The singular interval exists only if $c_1 = c_2 \equiv 0$. However, then the Hamiltonian $H(t) \equiv 1$, which violates (12). These arguments imply that there do not exist any singular intervals and we have only 4 possibilities for the optimal control logic:

$$u = \mp 1 \quad , \quad \mp 1 \to \pm 1$$

B. The Traditional Approach and its Drawbacks

Integrating backward with this control logic and enforcing the corner condition for each switching case, we have the following candidate solutions case-by-case: [8]

- If $u = \mp 1$, then

$$(x_{10}, x_{20}) \in \{(x_1, x_2) | (x_1 - x_{1f}) \pm \frac{1}{2}(x_2^2 - x_{2f}^2) = 0\} \quad (13)$$

$$J = \pm(x_{20} - x_{2f}) \quad (14)$$

- If $u = \mp 1 \to \pm 1$, then

$$\{(x_1, x_2) | (x_1 - x_{10}) \pm \frac{1}{2}(x_2^2 - x_{20}^2) = 0, t \in [t_0, t_s]\} \quad (15)$$

$$\{(x_1, x_2) | (x_1 - x_{1f}) \mp \frac{1}{2}(x_2^2 - x_{2f}^2) = 0, t \in [t_s, t_f]\} \quad (16)$$

$$t_s = t_0 \pm x_{20} + \frac{\sqrt{\pm 4(x_{10} - x_{1f}) + 2(x_{20}^2 + x_{2f}^2)}}{2} \quad (17)$$

$$J = \pm(x_{20} + x_{2f}) + \sqrt{\pm 4(x_{10} - x_{1f}) + 2(x_{20}^2 + x_{2f}^2)} \quad (18)$$

It can be verified that the optimal cost (14) and (18) satisfy the sufficient condition characterized by the HJB equation:

$$1 + \frac{\partial J}{\partial x_1} x_2 - \left| \frac{\partial J}{\partial x_2} \right| = 0 \quad , \quad J(x_f, t_f) = 0,$$

thus it is indeed the optimal solution.

Here we observe that this procedure consists of two steps: first it employs the necessary conditions to derive the candidate optimal control and the corresponding optimal trajectory, and then it checks the sufficient condition by introducing this candidate solution into the HJB equation.

On the other hand, it is extremely difficult to solve the HJB partial differential equation (PDE) directly. In fact, observing the stationarity of the Hamiltonian with respect to time, we know that $\partial J/\partial t = 0$. Then, we split the case into $\partial J/\partial x_2 > 0$ and $\partial J/\partial x_2 < 0$, which results in two linear first order PDEs. The traditional method of characteristics yields the optimal cost as an arbitrary function of specific arguments. That is,

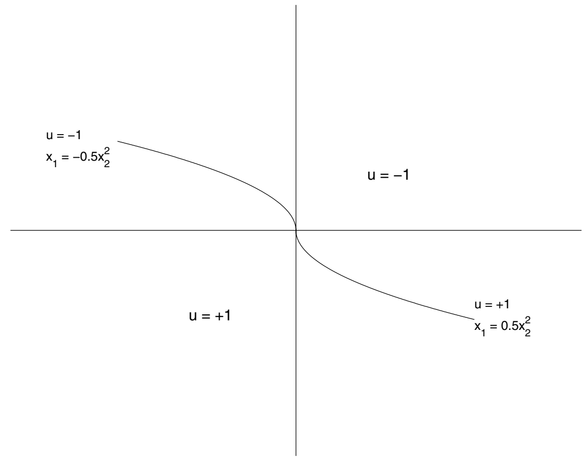

Fig. 1. Time-Optimal Control Logic for Double Integrator System (Terminal Condition at the Origin)

$$J = \pm x_2 + f_\pm \left(x_1 \pm \frac{1}{2} x_2^2 \right)$$

However, it turns out that we cannot determine $f_\pm$ due to the inherent singularity of the boundary conditions:

$$J(0) = 0 = f_\pm(0)$$

If we could find the 'true' optimal control satisfying both necessary and sufficient conditions by a single process, it would greatly facilitate the whole solution procedure and yield the optimal trajectory with greater ease. As an attempt towards that goal, we use our recently developed technique based on the theory of canonical transformations and generating functions.

C. Evaluation of Optimal Solution by Generating Function

Instead of resorting to the conventional two step process in the previous section, we use our recently developed single step procedure based on Theorem 2.2. Our goal is to find F_1, which provides the optimal cost and the associated optimal control logic by Theorem 2.2. F_1 is, by definition, the solution to the associated HJ equation

$$\frac{\partial F_1}{\partial t} + H\left(x, \frac{\partial F_1}{\partial x}, t\right) = 0$$

To solve the HJ equation for a generating function requires, at the least, that the functional form of the generating function be specified at some epoch[2]. It turns out that for our canonical transformation we cannot define *a priori* such a functional form for F_1 at any epoch, whereas we can for the related generating function F_2. Recall that for our canonical transformation the old and new coordinates are equal when

[2]Refer to [4], [5], and [11] for more detailed description of the solution of HJ equations along with their boundary conditions.

$t = t_0$. Using the associated state-costate relations for F_2, we can show that $F_2(x, \lambda_0, t = t_0, t_0) = x^T \lambda_0$ generates this identity transformation

$$x_0 = \frac{\partial F_2}{\partial \lambda_0} = x \quad , \quad \lambda = \frac{\partial F_2}{\partial x} = \lambda_0.$$

Therefore, given the Hamiltonian of a system, we can solve the HJ equation for F_2 from the initial time and then evaluate the F_1 through the Legendre transformations [9][10][11]:

$$F_1 = F_2 - x_0^T \lambda_0 \tag{19}$$

Now consider the HJ equation for F_2 instead of F_1:

$$\frac{\partial F_2}{\partial t} + H\left(x, \frac{\partial F_2}{\partial x}, t\right) = 0$$

From the stationarity of Hamiltonian and the transversality condition (12), we see that $\partial F_2/\partial t \equiv 0$, which results in

$$1 + \frac{\partial F_2}{\partial x_1} x_2 - \left|\frac{\partial F_2}{\partial x_2}\right| = 0 \ , \ F_2(x, \lambda_0, t = t_0, t_0) = x_1\lambda_{10} + x_2\lambda_{20}$$

We split the case into $\partial F_2/\partial x_2 > 0$ and $\partial F_2/\partial x_2 < 0$. Then, the HJ equation becomes

$$\frac{\partial F_2}{\partial x_1} x_2 \mp \frac{\partial F_2}{\partial x_2} = -1.$$

Noting that the PDE is linear, we can solve it by the method of characteristics:

$$\frac{dx_1}{x_2} = \frac{dx_2}{\mp 1} = \frac{d\lambda_{10}}{0} = \frac{d\lambda_{10}}{0} = \frac{dF_2}{-1}$$

$$\Rightarrow \quad \begin{cases} \frac{-1}{dF_2} = \frac{\mp 1}{dx_2} \Rightarrow F_2 \mp x_2 = c_1 \\ \frac{x_2}{dx_1} = \frac{-1}{dx_2} \Rightarrow x_1 + \frac{1}{2}x_2^2 = c_2 \\ -d\lambda_{10} = 0 \Rightarrow \lambda_{10} \equiv \text{constant} \\ -d\lambda_{20} = 0 \Rightarrow \lambda_{20} \equiv \text{constant} \end{cases}$$

$$\Rightarrow \quad F_2 = \pm x_2 + f_{\mp}\left(x_1 \pm \frac{1}{2}x_2^2, \lambda_{10}, \lambda_{20}\right)$$

where $f_{\mp}$ is an arbitrary function of the given arguments. Now introducing the initial boundary condition yields

$$x_{10}\lambda_{10} + x_{20}\lambda_{20} = \pm x_{20} + f_{\mp}\left(x_{10} \pm \frac{1}{2}x_{20}^2, \lambda_{10}, \lambda_{20}\right)$$

$$\Rightarrow \quad f_{\mp}\left(x_{10} \pm \frac{1}{2}x_{20}^2, \lambda_{10}, \lambda_{20}\right) = x_{10}\lambda_{10} + x_{20}\lambda_{20} \mp x_{20}$$

where x_{10} and x_{20} are constants. Hence, F_2 is

$$F_2 = \pm(x_2 - x_{20}) + x_{10}\lambda_{10} + x_{20}\lambda_{20}$$

With the aid of the Legendre transformation (19), we obtain F_1 in the case where no switching occurs:

$$F_1(x_f, x_0, t_f, T) = \mp(x_{20} - x_{2f}).$$

It is seen that $J = -F_1$ is equivalent to (14) in the previous section.

It is in the indirect procedure of determining F_1 where the advantage of our method lies. Unlike the HJB equation for the optimal cost, the HJ equation for F_2 always has a non-trivial boundary condition defined by the identity transformation, which helps us solve for F_2. Then F_1 can be obtained by the Legendre transformation, directly giving us the optimal cost and the corresponding optimal control logic by Theorem 2.2.

It remains to evaluate F_1 when there is a switching in control logic. Consider first the case of $u = -1 \to +1$. If we integrate the system for $u = -1$ with the initial condition and $u = +1$ with the terminal condition separately, we can obtain the optimal trajectories (15)-(16). Forcing continuity at the switching time t_s, we have the coordinates (x_{1s}, x_{2s}):

$$\begin{aligned} x_{1s} &= \frac{x_{10} + x_{1f}}{2} + \frac{x_{20}^2 - x_{2f}^2}{4} \\ x_{2s} &= -\frac{\sqrt{4(x_{10} - x_{1f}) + 2(x_{20}^2 + x_{2f}^2)}}{2} \end{aligned}$$

Then, integrating system (10) from t_0 to t_s yields

$$t_s = t_0 + x_{20} + \frac{\sqrt{4(x_{10} - x_{1f}) + 2(x_{20}^2 + x_{2f}^2)}}{2}$$

Also noting that the total elapsed time is simply the addition of both elapsed times before and after the switching time, we have[3].

$$\begin{aligned} t_f - t_0 &= |x_{20} - x_{2s}| + |x_{2f} - x_{2s}| \\ &= (x_{20} + x_{2f}) + \sqrt{4(x_{10} - x_{1f}) + 2(x_{20}^2 + x_{2f}^2)} \end{aligned}$$

Now note that we have two constraints: 1) $\lambda_2(t_s) = 0$ from the switching condition and 2) $H(t_f) = 0$ from the transversality condition. Introducing these conditions into (11) determines c_1 and c_2:

$$\begin{aligned} c_1 &= \frac{2}{\sqrt{4(x_{10} - x_{1f} + 2(x_{20}^2 + x_{2f}^2)}} \\ c_2 &= 1 + \frac{2(t_0 + x_{20})}{\sqrt{4(x_{10} - x_{1f} + 2(x_{20}^2 + x_{2f}^2)}} \end{aligned}$$

Then the state-costate relations for F_1 yield

$$\begin{aligned} \lambda_{10} &= -\frac{\partial F_1}{\partial x_{10}} = c_1 \\ \lambda_{20} &= -\frac{\partial F_1}{\partial x_{20}} = -c_1 t_0 + c_2 \\ \lambda_{1f} &= \frac{\partial F_1}{\partial x_{1f}} = c_1 \\ \lambda_{2f} &= \frac{\partial F_1}{\partial x_{2f}} = -c_1 t_f + c_2 \end{aligned}$$

[3] In fact, for the minimum time problem we do not necessarily evaluate F_1 for the switching case since the elapsed time $t_f - t_0$ itself is the cost function. Once we find the optimal cost function, we can directly evaluate the optimal control from the optimal cost function. However, we still derive how to find F_1 for the switching case since it is useful for other formulations where the cost function to be minimized is not the terminal time.

Integrating with respect to each dependent variable $(x_{10}, x_{1f}, x_{20}, x_{2f})$ sequentially yields

$$F_1 = -(x_{20} + x_{2f}) - \sqrt{4(x_{10} - x_{1f}) + 2(x_{20}^2 + x_{2f}^2)} \qquad (u = -1 \to +1)$$

Also, similar arguments lead to F_1 for $u = +1 \to -1$:

$$F_1 = +(x_{20} + x_{2f}) - \sqrt{-4(x_{10} - x_{1f}) + 2(x_{20}^2 + x_{2f}^2)} \qquad (u = +1 \to -1)$$

It can be easily seen that the cost function for the switching case is equivalent to (18).

All these results serve as an explicit example of the fact that our previous results on the relation between the optimal cost function and the F_1 generating function hold regardless of control constraints. Also note that we evaluated F_1 for the switching control logic solely based on F_1 for the two simple cases of no switching. Our methodology is a stand-alone technique of solving optimal control problems, which satisfies both necessary and sufficient conditions.

Finally, we state F_2 for completeness:

$$F_2(x_f, \lambda_0, t) = \begin{cases} \pm x_{2f} + x_{10}\lambda_{10} + x_{20}\lambda_{20} \mp x_{20}\ , \ (u = \mp 1) \\ \frac{\pm\lambda_{20}^2 - 2\lambda_{20} \mp 1}{2\lambda_{10}} \mp x_{2f} + x_{1f}\lambda_{10} \mp \frac{x_{2f}^2 \lambda_{10}}{2}, \\ (u = \mp 1 \to \pm 1) \end{cases}$$

Here x_{10} and x_{20} can be considered as free variables, which are independent of x_f and λ_0. It can be easily verified that F_1 and F_2 are linked via the Legendre transformation.

IV. Application to Singular Optimal Control Problem

As a second illustration, we consider a generic problem in singular control. Here we again show that the solution satisfying all the necessary conditions for singular control can be found by first solving a different HJ equation with non-trivial boundary conditions and then transforming to the F_1 generating function. The general sufficient condition for a singular control problem has not been found.

A. Problem Formulation

Consider minimizing a quadratic cost function

$$J = \frac{1}{2}x^T(t_f)Q_f x(t_f) + \frac{1}{2}\int_{t_0}^{t_f} x^T(t)Q(t)x(t)dt$$

subject to the linear system with initial condition

$$\dot{x} = Ax + Bu\ , \quad x(t_0) = x_0$$

Here $Q, Q_f, A \in \Re^{n\times n}$, and $B \in \Re^{n\times m}$. We assume that Q_f and $Q(t)$ are positive semidefinite and that the terminal time t_f is fixed. If we define the pre-Hamiltonian as

$$\bar{H} = \frac{1}{2}x^T Qx + \lambda^T(Ax + Bu), \tag{20}$$

then the costate equations and transversality conditions are, respectively, from the 1st order necessary conditions

$$\dot{\lambda} = -Qx - A^T\lambda, \quad \lambda(t_f) = Q_f x(t_f)$$

Also the optimality condition is

$$H_u = \lambda^T B = 0 \tag{21}$$

Note that the optimality condition yields a singular arc; it does not contain the control variables and does not provide the control law in itself. In order to derive an expression for the control variable, we take time derivatives of H_u until the control variable appears:

$$\begin{aligned} & \frac{d}{dt}H_u = \dot{\lambda}^T B = 0 \Rightarrow -(\lambda^T A + x^T Q)B = 0 & (22) \\ & \frac{d^2}{dt^2}H_u = -\dot{\lambda}^T AB - \dot{x}^T QB = 0 & \\ \Rightarrow \quad & (\lambda^T A + x^T Q)AB - (x^T A^T + u^T B^T)QB = 0 & \\ \Rightarrow \quad & u = -(B^T QB)^{-1}B^T[(QA - A^T Q)x - A^T A^T \lambda], & \\ & \text{if } \ B^T QB \ \text{ is nonsingular} & (23) \end{aligned}$$

(20), (21), and (22) represent $2m+1$ equations which define the locus of possible singular arcs in the $2n$-dimensional (x, λ) space. (23) is the linear feedback control law that one obtains on a singular arc. The Kelley condition [7]

$$-\frac{\partial}{\partial u}\left[\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial u}\right)^T\right] = B^T QB \geq 0$$

is satisfied since we assume that Q is positive semidefinite.

B. Evaluation of Optimal Solution by Generating Function

Introducing the feedback control law into the Hamiltonian and the state and adjoint equations yields

$$\begin{aligned} H &= \frac{1}{2}x^T Qx + \lambda^T[A - B(B^T QB)^{-1}B^T(QA - A^T Q)]x \\ &\quad + \lambda^T B(B^T QB)^{-1}B^T A^T A^T \lambda \\ \dot{x} &= [A - B(B^T QB)^{-1}B^T(QA - A^T Q)]x \\ &\quad + [B(B^T QB)^{-1}B^T A^T A^T + AAB(B^T QB)^{-1}B^T]\lambda \\ \dot{\lambda} &= -Qx - [A^T - (QA - A^T Q)^T B(B^T QB)^{-1}B^T]\lambda \end{aligned}$$

Considering the condition $H_u \equiv 0$ and $\dot{H}_u \equiv 0$ simultaneously, we have

$$\begin{bmatrix} B^T \\ B^T A^T \end{bmatrix}\lambda = \begin{bmatrix} 0 \\ -B^T Q \end{bmatrix}x$$

Assuming the pre-multiplied matrix $[B\ \ AB]^T$ are square (that is, $2m = n$) and invertible, we have

$$\lambda = \begin{bmatrix} B^T \\ B^T A^T \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ -B^T Q \end{bmatrix}x = Kx \tag{24}$$

Introducing (24) into (23) yields the singular optimal strategy:

$$\begin{aligned} & u = -(B^T QB)^{-1}B^T[(QA - A^T Q) - A^T A^T K]x, \\ & \text{if } \ B^T QB \ \text{ is nonsingular} \end{aligned} \tag{25}$$

Now in order to study the relation between the singular optimal cost function and the F_1 generating function, we first evaluate $F_2(x, \lambda_0, t; t_0)$, as in regular optimal control problems. Observing that the Hamiltonian is quadratic, F_2 can be expressed as a quadratic form:

$$F_2(x, \lambda_0, t; t_0) = \frac{1}{2}\begin{bmatrix} x \\ \lambda_0 \end{bmatrix}^T \begin{bmatrix} F_{xx}(t, t_0) & F_{x\lambda_0}(t, t_0) \\ F_{\lambda_0 x}(t, t_0) & F_{\lambda_0\lambda_0}(t, t_0) \end{bmatrix}\begin{bmatrix} x \\ \lambda_0 \end{bmatrix}$$

The quadratic Hamiltonian in matrix form is

$$
\begin{aligned}
H &= \frac{1}{2}\begin{bmatrix} x \\ \lambda \end{bmatrix}^T \begin{bmatrix} H_{xx} & H_{x\lambda} \\ H_{\lambda x} & H_{\lambda\lambda} \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} x \\ \lambda \end{bmatrix}^T \begin{bmatrix} Q & [A - B(B^TQB)^{-1}B^T(QA - A^TQ)]^T \\ A - B(B^TQB)^{-1}B^T(QA - A^TQ) & B(B^TQB)^{-1}B^TA^TA^T + AAB(B^TQB)^{-1}B^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} x \\ \lambda_0 \end{bmatrix}^T \begin{bmatrix} I & F_{xx} \\ 0 & F_{\lambda_0 x} \end{bmatrix} \begin{bmatrix} \cdots \end{bmatrix} \begin{bmatrix} I & 0 \\ F_{xx} & F_{x\lambda_0} \end{bmatrix} \begin{bmatrix} x \\ \lambda_0 \end{bmatrix}
\end{aligned}
$$

where the following relation has been used:

$$\lambda = \frac{\partial F_2}{\partial x} = \begin{bmatrix} F_{xx} & F_{x\lambda_0} \end{bmatrix} \begin{bmatrix} x \\ \lambda_0 \end{bmatrix}$$

Finally introducing the above quadratic expression to the Hamilton-Jacobi equation for F_2 and considering each submatrix component, we have the following matrix differential equations for $F_{xx}(t,t_0)$, $F_{x\lambda_0}(t,t_0)$, and $F_{\lambda_0\lambda_0}(t,t_0)$

$$
\begin{aligned}
0 &= \dot{F}_{xx} + H_{xx} + F_{xx}H_{\lambda x} + H_{x\lambda}F_{xx} + F_{xx}H_{\lambda\lambda}F_{xx} \\
0 &= \dot{F}_{x\lambda_0} + H_{x\lambda}F_{x\lambda_0} + F_{xx}H_{\lambda\lambda}F_{x\lambda_0} \\
0 &= \dot{F}_{\lambda_0\lambda_0} + F_{\lambda_0 x}H_{\lambda\lambda}F_{x\lambda_0}
\end{aligned}
$$

with initial conditions derived from the identity transformation $F_2(x, \lambda_0, t = t_0; t_0) = x^T\lambda_0$:

$$F_{xx}(t_0,t_0) = 0 \ , \ F_{x\lambda_0}(t_0,t_0) = I \ , \ F_{\lambda_0\lambda_0}(t_0,t_0) = 0$$

We can now evaluate F_1 from the Legendre transformation, along with the relation $\lambda = Kx \rightarrow \lambda_0 = Kx_0$:

$$
\begin{aligned}
F_1 &= F_2 - x_0^T\lambda_0 \\
&= \frac{1}{2}x^TF_{xx}x + x^TF_{x\lambda_0}Kx_0 + x_0^T(\frac{1}{2}K^TF_{\lambda_0\lambda_0}K - K)x_0
\end{aligned}
$$

From the state-costate relations for F_2, we have

$$x_0 = \frac{\partial F_2}{\partial \lambda_0} = F_{\lambda_0 x}x + F_{\lambda_0\lambda_0}\lambda_0$$

Introducing $\lambda = Kx$ or $\lambda_0 = Kx_0$, we can rearrange this equation for x:

$$x = F_{\lambda_0 x}^{-1}(I - F_{\lambda_0\lambda_0}K)x_0$$

where the relevant matrices are assumed to be invertible. Finally introducing this into (26) yields F_1 as a function of initial condition only:

$$
\begin{aligned}
F_1 = \ & \frac{1}{2}x_0^T[(I - K^TF_{\lambda_0\lambda_0})F_{x\lambda_0}^{-1}F_{xx}F_{\lambda_0 x}^{-1}(I - F_{\lambda_0\lambda_0}K) \\
& +2(I - K^TF_{\lambda_0\lambda_0})K + (K^TF_{\lambda_0\lambda_0}K - 2K)]x_0
\end{aligned}
$$

Again from Theorem 2.2, we have $J = -F_1$, and is the same as the (candidate) optimal cost from the traditional method [8].

So far we have demonstrated how to solve the linear quadratic singular optimal control problem by using generating functions. Deriving the singular optimal control logic from the optimality condition and its auxiliary conditions, we determine the quadratic Hamiltonian for states and costates. Noting that it is difficult to solve the HJ equation for F_1 directly, we solve for F_2 and convert it into F_1 via the Legendre transformation. Then by 2.2, we determine the (candidate) optimal cost function.

Under the assumption of analyticity of the cost function and the system, we can expand the Hamiltonian as a Taylor series in the states and adjoints. Then, this procedure can be generalized and applied to a nonlinear system with (possibly) non-quadratic performance index by expanding the generating functions in Taylor series form. For a more detailed description of such higher order problems, we refer to Park and Scheeres [5] and Guibout and Scheeres [11].

V. CONCLUSION

We have studied the application of Hamiltonian dynamical system theory to the optimal control problem. We have shown that our proposed method can be extended to more general problems with control constraints and singular optimal control problems. In spite of these generalizations, a fundamental relation still holds between the optimal cost function and the F_1 generating function. Then, employing this relation, we have investigated some novel solution procedures. For the problem with control constraints, our procedure provided the optimal solution satisfying both necessary and sufficient conditions for optimality simultaneously. For the singular optimal control problem, it reproduces the solution from traditional methods satisfying both the 1st order necessary conditions and the generalized Legendre-Clebsche condition. Though we consider a specific system for each application, our results imply that our solution procedure can be applied to more general optimal control problems with control constraints and singular intervals.

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