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Constrained MPC using feedback linearization for systems with unstable inverse dynamics

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Abstract— **Earlier work used partial invariance to identify regions in state space where feedback linearization can be used despite the presence of unstable inverse dynamics. Such regions can be used as terminal regions in MPC with obvious advantages. Considering SISO bilinear systems, this paper exploits the fact that feedback linearization steers the state to the kernel of the output map. By restricting attention to this kernel, the paper develops results allowing for significant enlargement of the terminal region. Expressions are also given for maximal partially invariant sets, the recursive use of which leads to significant further enlargement.**

I. Introduction

The consensus in linear MPC [1] employs a dual mode prediction paradigm which splits the prediction horizon into two modes. The near future comprises a finite horizon in which control moves are treated as degrees of freedom. For the subsequent infinite prediction horizon a terminal feedback law is assumed. To ensure constraints satisfaction this is defined to be feasible within a terminal set (usually an invariant ellipsoid). In the interest of optimality it is usual to choose the terminal control law as the LQ-optimal for a given cost penalizing deviations from steady state.

The dual prediction mode paradigm carries over to the nonlinear case but computation of an optimal terminal law is not generally tractable. Input-Output Feedback Linearization (IOFL) provides an attractive remedy and has been used in nonlinear MPC (e.g. [2]), and in particular its use as a terminal control has been exploited (e.g. [3]). However the existence of unstable equilibria as well as issues of feasibility and variable relative degree limit the size of potential invariant sets and in some instances preclude the definition of such sets altogether. It then becomes necessary to use suboptimal control laws, but the use of optimal IOFL (OIOFL) can still be beneficial in their design. This issue was addressed in [4] in the context of SISO bilinear systems, where sets were defined which, relative to a given terminal set, are Partially Invariant and Feasible (PIF) under IOFL. PIF implies that under OIOFL the state at the next instant enters a terminal set which is invariant and feasible (IF) under the terminal control law (which itself could be the OIOFL, if that is possible, or some other law otherwise).

The computation of IF sets for nonlinear systems is non-trivial and to reduce computational complexity, [4] considered the use of low-complexity polytopes. However even for SISO bilinear systems, the conditions for invariance are nonlinear, making the problem of designing IF polytopes of maximum volume nonconvex. To avoid this difficulty, [4] used sufficient conditions based on over-bounding auxiliary polytopes. Auxiliary polytopes were also used to define low-complexity polytopes which, under IOFL, are PIF with respect to given IF polytopes. The use of low-complexity polytopes restricts the size of PIF sets and therefore limits the benefits of OIOFL control laws.

The present paper also considers SISO bilinear systems and proposes extensions that increase significantly the PIF set. This is achieved by exploiting the fact that OIOFL causes the next state to lie in the kernel of the output map C. By restricting attention to IF sets in the kernel of C, sets are obtained which are necessarily larger than the intersections of the earlier IF sets with the kernel of C. Correspondingly, this analysis allows for the definition of larger PIF sets. The paper then considers the problem of defining maximal PIF sets and through these proposes a procedure for enlarging IF sets in the kernel of C, and consequently for enlargement of the maximal PIF sets. PIF sets (or more conveniently suitably inscribed polytopic sets) can be used as terminal sets in the dual prediction mode paradigm. To illustrate the results we use a simple example allowing easy visualization of the benefits of the procedures developed in the paper.

II. Earlier work

Consider the SISO bilinear system with model:

$$x(k+1) = Ax(k) + (B + Fx(k))u(k),\ y(k) = Cx(k),\ x \in \mathbb{R}^n \quad (1)$$

and input constraints $u(k) \in U = \{u : |u| \le \bar{u}\}$, for which the OIOFL control law

$$u = -\frac{CAx}{CB + CFx}, \quad CB + CFx \ne 0 \quad (2a,b)$$

minimizes the cost

$$J = \sum_{k=0}^{\infty} y^2(k)\,. \quad (3)$$

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This is the usual regulation MPC cost, with the difference that it does not penalize control activity; in the past penalizing control has been used as a device for avoiding constraint violations. However this device is not needed here because input constrains are accounted for explicitly.

The law of (2) results in zero predicted cost and is thus optimal, but it may not converge to desired equilibrium points. The usual remedy is to deploy instead the LQR optimal for the linearization of (1) about a given equilibrium point. However such a terminal control law is clearly suboptimal and in may also result in a small terminal IF set. Instead [4] proposes a bilinear controller:

$$u = -\frac{Kx}{1+\mu^T x} \tag{4}$$

with K,μ chosen to maximize the volume of the low-complexity polytope

$$\Pi(V) = \{x \mid \|Vx\|_\infty \le 1\}, \quad V \in \mathbb{R}^{n\times n} \tag{5}$$

under the IF conditions

$$\left\| V\left[A - \frac{(B+Fx)K}{1+\mu^T x} \right] x \right\|_\infty \le 1 - \varepsilon|Cx|, \quad |Kx| \le \bar{u}\left|1+\mu^T x\right| \tag{6}$$

The parameter ε is a tuning knob that can be used to compromise between the need to maximize the volume of $\Pi(V)$ and dynamic performance (given that ε controls the upper bound on convergence rate within $\Pi(V)$. For simplicity in the sequel it will be assumed that $\varepsilon = 0$. However even with this assumption, (6a) is nonlinear and to reduce complexity [4] introduces an auxiliary polytope $\Pi(\hat{V})$ which is made to over-bound $\Pi(V)$ and then invokes a sufficient condition for (6) in terms of the vertices $w_i, \hat{w}_i$ of $\Pi(V)$, $\Pi(\hat{V})$, respectively:

$$\begin{aligned} &\left\| V\left[(1+\mu^T w_i)A\hat{w}_j - (B+Fw_i)K\hat{w}_j \right] \right\|_\infty \le (1+\mu^T w_i) \\ &|Kw_i| \le (1+\mu^T w_i)\bar{u} \\ &\|\hat{V}w_i\|_\infty \le 1 \end{aligned} \tag{7a-c}$$

where (7a) ensures invariance, (7b) feasibility, and (7c) the over-bounding of $\Pi(V)$ by $\Pi(\hat{V})$. With this formulation it is possible to perform the volume maximization for $\Pi(V)$ recursively via a succession of convex optimizations which give a monotonic increase in volume.

The OIOFL law (2) is optimal with respect to (3) whereas (4) is not. The only reason for using (4) instead of (2) is to avoid instability, e.g. in the neighbourhood of an unstable equilibrium point. The question therefore arises whether there are regions of state space where (2) can be used while preserving convergence to the equilibrium point. The answer is given in [4] in terms of sets which, under (2), are PIF w.r.t. a set that is IF under (4). Thus for any initial condition in the PIF set, under (2) the state will either remain in the PIF set, in which case (2) remains applicable, or will enter the IF set in which case the controller can switch to (4). Sufficient conditions for a set $\Pi(\tilde{V})$ with vertices $\tilde{w}_j$ to be PIF w.r.t. $\Pi(V)$ are:

$$\left\| V\left[(CB+CF\tilde{w}_i)A\tilde{w}_j - (B+F\tilde{w}_i)CA\tilde{w}_j \right] \right\|_\infty \le |CB+CF\tilde{w}_i| \tag{8}$$

and that $x \in \Pi(V)$ satisfies either the conditions

$$-C(A+\bar{u}F)x < \bar{u}CB, \; C(A-\bar{u}F)x < \bar{u}CB, \; -CFx < CB, \tag{9a}$$

or the conditions

$$C(A+\bar{u}F)x < -\bar{u}CB, \; -C(A-\bar{u}F)x < -\bar{u}CB, \; CFx < -CB. \tag{9b}$$

Like (6), (8) is nonlinear and to reduce computational complexity, [4] makes use of an over-bounding polytope with vertices $\breve{w}_j$, allowing use of a sufficient condition for (8) in which $\tilde{w}_j$ is replaced by $\breve{w}_j$. This condition is linear in $\tilde{w}_i$ and hence enables successive maximization of the volume of the PIF polytope through recursion of a convex optimisation problem. It is noted that, depending on which of (9a) and (9b) holds, (8-9) lead to the definition of two separate sets denoted by $\Pi^+(\tilde{V})$ and $\Pi^-(\tilde{V})$ respectively and it is the union of these two sets which is PIF.

III. Restrictions in the Kernel of C

A major obstacle to use of IOFL is non-minimum phase characteristics, which imply potential cancellation of unstable internal dynamics. To overcome this problem [4] introduced the bilinear control law (4), however as demonstrated by example, non-minimum phaseness does not preclude the possibility of convergence to stable equilibrium points. Below we consider this case first, extensions to the general case (through the use of stabilizing controllers such as (4)) are straightforward and are considered later in this section. Without loss of generality it will be assumed that the stable equilibrium point is at the origin; this can always be achieved through the shift transformations $z = x - x^o$, $v = u - u^o$ where x^o, u^o denote the equilibrium values, so that (1) becomes

$$z(k+1) = A^o z(k) + (B^o + Fz(k))v(k), \; A^o = A + u^o F, \; B^o = B + Fx^o \tag{10}$$

The starting point here is the following simple observation.

Lemma 3.1 Provided the current state is in the feasible set defined by $\Gamma = \{x : |CAx| \le \bar{u}|CB + CFx|\}$, the optimal IOFL steers the next state into the kernel of C, denoted $\mathcal{K}$.

Proof: For $x(k) \in \Gamma$, under optimal OIOFL we have

$$x(k+1) = Ax(k) - [B + Fx(k)]\frac{CAx(k)}{CB + CFx(k)} \tag{11}$$

so that $Cx(k+1) = 0$. Note that, for $x(k) \in \Gamma$, (2) tends to a limit less than $\bar{u}$ in modulus as $CB + CFx \to 0$. □

As explained in Section 2, (8) is a sufficient condition for $\Pi(\tilde{V})$ to be PIF under (2) w.r.t. $\Pi(V)$, but to reduce computational complexity, this condition is replaced with

$$\left\| V[(CB+CF\tilde{w}_i)A\breve{w}_j - (B+F\tilde{w}_i)CA\breve{w}_j \right\|_\infty \le |CB+CF\tilde{w}_i| \tag{12}$$

where $\breve{w}_j$ denote vertices of a polytope $\Pi(\breve{V})$ that over-bounds $\Pi(\tilde{V})$. However this condition is only sufficient and can lead to conservative results. Lemma 3.1 suggests a method for enlarging of the PIF set since it implies that under OIOFL, the PIF property applies not w.r.t. $\Pi(V)$, but rather its intersection with $\mathcal{K}$. However polytopic sets, though computationally convenient, implies a degree of conservatism, with the implication that $\Pi(V)\bigcap\mathcal{K}$ may not represent the largest IF polytope in $\mathcal{K}$. Note that the IF property in [4] was defined w.r.t. (4), but in the case of stable equilibrium points under OIOFL, IF can be defined w.r.t. (2). Clearly, restricting attention to $\mathcal{K}$ produces IF polytopes at least as large as $\Pi(V)\bigcap\mathcal{K}$, but which can potentially be larger, with a consequent further enlargement effect on the associated PIF sets (as demonstrated by the example in Section 6).

Before giving conditions for the definition of IF sets in the kernel of C, first note that polytopes in $\mathcal{K}$ are defined

$$P(\Psi)=\{x=Mr: r\in\Pi(\Psi)\},\quad \Pi(\Psi)=\{r:\|\Psi r\|_\infty\le 1\} \quad (13)$$

where the columns of M span $\mathcal{K}$, $\Psi\in\mathbb{R}^{n-1\times n-1}$ is full-rank, and for convenience we assume that $M^TM=I_{n-1}$. Clearly $\Pi(\Psi)$ defines a low-complexity polytope in $\mathbb{R}^{n-1}$ and its vertices are denoted ω_i. Accordingly the vertices of $P(\Psi)$ are $M\omega_i$ and this distinguishes them from the vertices w_i of the low-complexity polytope $\Pi(V)$ in $\mathbb{R}^n$.

Theorem 3.1 The polytope $P(\Psi)\in\mathcal{K}$ with vertices $M\omega_i$ is IF under OIOFL if

$$|CAM\omega_i|\le\bar{u}\,|CB+CFM\omega_i| \quad (14)$$

and there exists $\Pi(\tilde{\Psi})$ with vertices $\tilde{\omega}_i$ such that

$$\|\tilde{\Psi}\omega_i\|_\infty\le 1 \quad (15)$$

for which either of the two conditions below holds true

$$\left\|\Psi M^T\left[A-\frac{(B+FM\tilde{\omega}_j)CA}{C(B+FM\tilde{\omega}_j)}\right]M\omega_i\right\|_\infty\le 1, \quad (16a)$$

$$\left\|\Psi M^T\left[A-\frac{(B+FM\omega_j)CA}{C(B+FM\omega_j)}\right]M\tilde{\omega}_i\right\|_\infty\le 1. \quad (16b)$$

Proof: Condition (14) is necessary and sufficient for $P(\Psi)\subseteq\Gamma$ i.e. so that (2) is feasible for all $x\in P(\Psi)$. Condition (15) is necessary and sufficient for $\Pi(\tilde{\Psi})\supseteq\Pi(\Psi)$. Finally, by Lemma 3.1 it is known that under OIOFL, $x(k)$ will be steered to $x(k+1)\in\mathcal{K}$ and its coordinates within $\mathcal{K}$ will be $r(k+1)=M^Tx(k+1)$. Therefore by (11) for $x(k)=Mr(k)$ the necessary and sufficient condition for invariance becomes

$$\left\|\Psi M^T\left[A-\frac{(B+FMr(k))CA}{C(B+FMr(k))}\right]Mr(k)\right\|_\infty\le 1 \quad (17)$$

As in Section 2, the vertices of $\Pi(\Psi)$ and $\Pi(\tilde{\Psi})$ can be used to replace (17) (which is nonlinear in terms of $r(k)$) by the sufficient condition of (16a) or (16b). □

Together with (14) and (15), either condition (16a) or (16b) ensures IF. However (16a,b) are not equivalent, and hence the condition resulting in larger $\Pi(\Psi)$ and $P(\Psi)$ should be selected. The formulation presented in Theorem 3.1 allows for the maximization of the volume of $\Pi(\Psi)$ to be performed through successive convex optimizations.

Algorithm 3.1 Initialize the process by selecting a small $\Pi(\tilde{\Psi})$ and maximize over the vertices ω_i the volume of $\Pi(\Psi)$. Scale up the optimal $\Pi(\Psi)$ and use the resulting polytope as the new over-bounding polytope $\Pi(\tilde{\Psi})$ and repeat. Stop the procedure when improvement in the volume of $\Pi(\Psi)$ is below a given threshold.

Corollary 3.1 Algorithm 3.1 can always be made to give a polytope $\Pi(\Psi)$ no smaller than the intersection of any given IF polytope $\Pi(V)$ with $\mathcal{K}$.

Proof: This is obvious if one uses the intersection of the corollary to initialise the algorithm. □

We next define sets which, under OIOFL, are PIF w.r.t. $P(\Psi)$ rather than $\Pi(V)$. It was earlier agued that under OIOFL, PIF w.r.t. $\Pi(V)$ is essentially the same as PIF w.r.t. $\Pi(V)\bigcap\mathcal{K}$. The implication of Corollary 3.1 is that Algorithm 3.1 results in a set $P(\Psi)$ larger than $\Pi(V)\bigcap\mathcal{K}$. As will be illustrated in the examples section this can have a significant effect on the size of the PIF set.

Before considering PIF however, consideration is given to the general case in which the state is required to converge to an unstable equilibrium point (or all equilibrium points under OIOFL are unstable). There may not then exist sets which are IF under OIOFL, so that suboptimal controllers such as the bilinear controller (4) must be employed. The simplest way to handle this is to choose C' satisfying

$$K=\frac{1}{C'B}C'A,\quad \mu^T=\frac{1}{C'B}C'F \quad (18)$$

then all the results of the section carry over since the control law (4) under (18) is the OIOFL law for (1) with respect to a new output $y'=C'(x-x^o)$. Such an output is "synthetic" in the sense of [5] since then $x\to x^0$ implying convergence of the actual output $y=Cx$ to the desired steady state value Cx^o. Of course (18) limits the degrees of freedom available in the definition of $\Pi(V)$ and this may significantly affect the size of the associated IF polytopic set. Thus it may be necessary to use (4) without the constraints of (18). It is nevertheless still possible to benefit from the use of the enlarged $P(\Psi)$ provided that it is no longer required to be IF but rather PIF under (4).

Theorem 3.2 Let $\Pi(V)$ denote a polytope which is IF under (4). Then under (4), the polytope $P(\Psi)\in\mathcal{K}$ with vertices $M\omega_i$ is PIF w.r.t. $\Pi(V)$ if

$$|KM\omega_i|\le\bar{u}\,|1+\mu^TM\omega_i| \quad (19)$$

and there exists $\Pi(\tilde{\Psi})$ with vertices $\tilde{\omega}_i$ such that

$$\left\|\tilde{\Psi}\omega_i\right\|_\infty \le 1 \quad (20)$$

and either of the two conditions below hold true.

$$\left\| V\left[A - \frac{(B+FM\tilde{\omega}_j)K}{1+\mu^T M\tilde{\omega}_j}\right] M\omega_i \right\|_\infty \le 1 \quad (21a)$$

$$\left\| V\left[A - \frac{(B+FM\omega_j)K}{1+\mu^T M\omega_j}\right] M\tilde{\omega}_i \right\|_\infty \le 1 \quad (21b)$$

The proof of this result is similar to that of Theorem 3.1 and will be omitted. We note that maximizing the volume of $P(\Psi)$ (under the conditions of Theorem 3.2) will in general result in a polytope larger than $\Pi(V)\bigcap\mathcal{K}$ and therefore use of $P(\Psi)$ instead of $\Pi(V)$ (or indeed of $\Pi(V)\bigcap\mathcal{K}$) can lead to significant enlargement in the volume of the PIF sets proposed in [4].

IV. Partially invariant sets w.r.t. polytopes in the kernel of C

To address the problem of unstable inverse dynamics, earlier work [4] considered sets which under OIOFL are PIF w.r.t. sets $\Pi(V)$ that are IF (either under OIOFL or a bilinear controller). A set Σ is said to be PIF under OIOFL w.r.t. a set Π if OIOFL steers every state in Σ into Π in one feasible move. In the sequel we refer to the set Σ, with respect to which Π is PIF, as the target set.

Below we obtain enlarged PIF sets through the restrictions to $\mathcal{K}$ described in Section III. However, rather than consider polytopes $P(\Psi)\in\mathcal{K}$ centred at the origin, here we consider target sets centred at Mc:

$$P(\Psi, Mc) = \{x = Mr \mid \left\|\Psi(r-c)\right\|_\infty \le 1\} \quad (22)$$

Theorem 4.1 Under OIOFL the maximum PIF set Σ of (1) w.r.t. $P(\Psi, Mc)$ is

$$\Sigma = \Gamma \bigcap \left[\bigcap_{1\le i\le N} (\Sigma_{i+} \bigcup \Sigma_{i-})\right] \quad (23)$$

where

$$\Sigma_{i+} = \left\{ x \left| \begin{matrix} x^T H_i x + g_{i-}^T x - f_{i+} < 0 \\ x^T H_i x + g_{i+}^T x - f_{i-} > 0 \\ CB + CFx > 0 \end{matrix} \right. \right\} \Sigma_{i-} = \left\{ x \left| \begin{matrix} x^T H_i x + g_{i-}^T x - f_{i+} > 0 \\ x^T H_i x + g_{i+}^T x - f_{i-} < 0 \\ CB + CFx < 0 \end{matrix} \right. \right\}$$

$$\begin{aligned} H_i &= F^T C^T \phi_i^T A - A^T C^T \phi_i^T F \\ g_{i-}^T &= CB\phi_i^T A - \phi_i^T BCA - \phi_i^T cCF - CF, \; f_{i+} = \phi_i^T cCB + CB \quad (24) \\ g_{i+}^T &= CB\phi_i^T A - \phi_i^T BCA + \phi_i^T cCF - CF, \; f_{i-} = \phi_i^T cCB - CB \end{aligned}$$

with ϕ_i^T denoting the i^{th} row of ΨM^T.

Proof: A necessary and sufficient condition for Σ to be PIF w.r.t. $P(\Psi, Mc)$ is that for every $x\in\Sigma$ we have:

$$\left\| \Psi\left(M^T \left[A - \frac{(B+Fx)}{CB+CFx} CA\right] x - c\right)\right\|_\infty \le 1 \quad (25)$$

which gives the conditions of (24) after re-arranging. Clearly since either $CB + CFx > 0$ or $CB + CFx < 0$, the PIF set must comprise the union of Σ_{i+} and Σ_{i-}, but these must hold for all i implying the intersection over i in (23). Finally that intersection must also be intersected with Γ in order to ensure that Σ is feasible. □

It is noted that due to symmetry, the definition of Σ_{i+}, Σ_{i-} can be achieved using only half the number of inequalities given in (24). It is also clear that due to the quadratic nature of these inequalities, the boundaries of Σ_{i+}, Σ_{i-} can be ellipsoidal (as illustrated in the examples section) or hyperbolic depending whether or not the matrices $F^T C^T \phi_i^T A - A^T C^T \phi_i^t F$ are sign-definite.

Remark 4.1 Computational complexity can be reduced (albeit at the cost of a certain degree of conservatism) by defining PIF polytopes Σ with vertices w_i through the use of over-bounding polytopes with vertices $\tilde{w}_i$. Thus replacing (25) by either of the following two conditions

$$\begin{aligned} &\left\| \Psi\left(M^T \left[A - \frac{(B+F\tilde{w}_i)}{CB+CF\tilde{w}_i} CA\right] w_j - c\right)\right\|_\infty \le 1, \\ &\left\| \Psi\left(M^T \left[A - \frac{(B+Fw_i)}{CB+CFw_i} CA\right] \tilde{w}_j - c\right)\right\|_\infty \le 1 \end{aligned} \quad . \quad (26)$$

and invoking this condition on either side of the hyper-plane $CB + CFx = 0$ defines two PIF polytopes: Σ_+, Σ_-.

Remark 4.2 The PIF conditions above are stated for the case of a target set in the kernel of C. However they also apply w.r.t. general polytopes $\Pi(V)$; all that is required is that Ψ is replaced by V, M is replaced by the identity matrix, and ϕ_i^T is replaced by v_i^T, the i^{th} row of V.

V. Dual predictive mode MPC and terminal sets

Sections 3 and 4 provide a means of computing enlarged regions where it is possible to benefit from the optimality of OIOFL despite the presence of non-minimum phase dynamics. In addition however, the IF and PIF regions defined in Theorems 3.1, 3.2, 4.1, and Remark 4.1 can be used in dual prediction mode MPC. Thus, if convergence to a stable OIOFL equilibrium point can be achieved, then the intersection of the PIF region Theorem 4.1 and Remark 4.1 with the IF region of Theorem 3.1 could be used as a terminal region; the assumed terminal control law is OIOFL and the corresponding predicted cost over mode 2 would be zero. Clearly the implied constraint comprises simple linear inequalities. On the other hand, if it is desired to converge to an unstable equilibrium under OIOFL, then it is possible to use as terminal region the intersection of the PIF region of Theorem 4.1 and Remark 4.1 with the IF region of [4]. Again the implied stability constraint comprises linear inequalities whereas the implied terminal control law is

OIOFL within the PIF set and (4) otherwise. Use of OIOFL incurs a predicted cost of zero, whereas the cost-to-go associated with (4) can be computed explicitly (see [4]).

Although the terminal constraints described above are linear, the feasibility constraints over mode 1 are nonlinear and generally nonconvex. Therefore it may be advantageous to use the PIF regions of Theorem 4.1 without the modification of Remark 4.1. This leads to nonlinear and possibly nonconvex terminal constraints, but direct use of (24) can allow a very significant expansion of the terminal region as demonstrated in Section 6.

Given a polytope $P(\Psi)$ which is IF under (2), it is possible to invoke (24) to define the maximal set Σ which, under (2), is PIF w.r.t. $P(\Psi)$. It follows therefore that every point in $\Sigma \cap \mathcal{K}$ will be steered by OIOFL into $P(\Psi)$ and subsequently remain in $P(\Psi)$. Using conditions for PIF w.r.t. $\Sigma \cap \mathcal{K}$, a region Σ' can therefore be defined which, though much larger than Σ, contains points at which it is possible to use the optimal law of (2) without concern for non-minimum phase difficulties (in the sense that the state will always remain within Σ' and will converge to the equilibrium point). However $\Sigma \cap \mathcal{K}$ need not be convex and to simplify computation below we require PIF w.r.t. an inscribed polytope in place of $\Sigma \cap \mathcal{K}$. This procedure can be applied recursively as in the computation of controllable sets [7,8], however here we are invoking the PIF property with reference to a particular control law, namely OIOFL.

Algorithm 5.1 Step 0: For $i=0$ set $P(\Psi^{(i)}) = P(\Psi)$.

Step 1: Use Theorem 4.1 in order to define the maximal set $\Sigma^{(i)}$ which is PIF w.r.t. $P(\Psi^{(i)})$; inscribe in $\Sigma^{(i+1)}$ a polytope $P(\Psi^{(i+1)}) \supseteq P(\Psi^{(i)})$.

Step 2: If $P(\Psi^{(i)}) - P(\Psi^{(i+1)})$ is smaller than a given threshold, set $\Sigma^* = \Sigma^{(i)}$ and stop; otherwise increment i and return to Step 2.

The following two results are stated without proof.

Lemma 5.1 The sets $\Sigma^{(i)}$ of Algorithm 5.1 have the nested property that $\Sigma^{(i)} \subseteq \Sigma^{(i-1)}$ and $P(\Psi) \subseteq \Sigma^{(i)}$ for all i.

Theorem 5.1 $\Sigma^{(i)}$ of Algorithm 5.1 are IF under (2).

Remark 5.1 Just as condition (24) of Theorem 4.1 allows for the definition of sets Σ_{i+}, Σ_{i-} with non-zero centres, so in Algorithm 5.1 it is possible to use target sets

$$P(\Psi^{(i)}, Mc^{(i)}) = \{x = Mr \mid \|\Psi^{(i)}(r - c^{(i)})\|_\infty \le 1\}$$

with non-zero centres $Mc^{(i)}$. This contributes further to the very significant enlargement effect of the recursive procedure of Algorithm 5.1.

Algorithm 5.1 may be computationally demanding but is implemented offline. Use of Σ^* rather than Σ implies no increase in online computation, and this can be reduced through the use of polytopic inscriptions of Σ^*. Implicit in the application of Algorithm 5.1 is the assumption that $P(\Psi)$ is IF, which implies that under OIOFL there exists at least one stable equilibrium point. If this is not the case, then recursion can still be used to bring about significant terminal set enlargements, but it has to be applied with respect to a stabilizing controller such as that of (4) rather than (2).

VI. ILLUSTRATIVE EXAMPLES

For the purpose of visualization we restrict attention to bilinear systems with only two states. Furthermore for simplicity we also restrict attention to an example that has one stable and one unstable equilibrium point under OIOFL; as explained the case of two unstable equilibrium points can be covered by the extension of the results of sections 3-5 to the bilinear control law of (4) in place of (2). The matrices of the model (given in the form of (1)) are:

$$A = \begin{bmatrix} 0.22 & 0.6 \\ 0.4 & 0.6 \end{bmatrix}, B = \begin{bmatrix} 0.9 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0.3 \end{bmatrix}, F = \begin{bmatrix} 0.4 & -0.2 \\ 0.2 & -0.4 \end{bmatrix}$$

and for these values, under IOFL the system has a stable equilibrium point at the origin and an unstable equilibrium point at $(-0.3007, 1.0025)$.

Figure 1 gives a comparison of the procedures of [4] and the enlargements possible through the results presented in

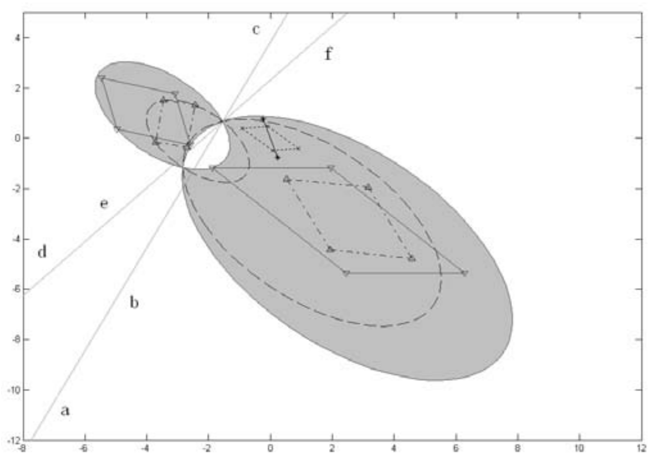

Fig. 1. Comparison of IF polytopes

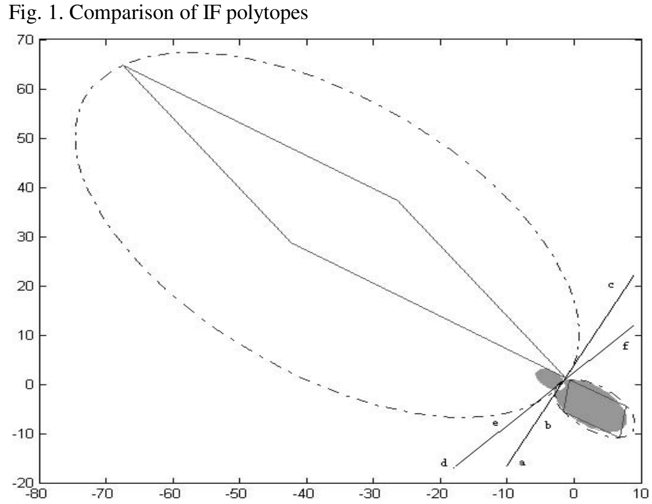

Fig. 2. IF regions computed using recursive PIF (Algorithm 5.1)

this paper. The two lines abc and def divide the plane into four sectors of which the two larger sectors define the feasible set Γ. The polytope with vertices marked "x" denotes the maximum volume low-complexity polytope which is IF under (2). The polytopes with vertices marked with upward-pointing triangles are the maximum volume polytopes obtained using the procedures of [4] and have a total area of 8.733. Restricting attention to the kernel of C allows one to define a set (in this case the line segment defined by the two points marked "*" in Figure 1) which intersects the IF box but also contains points outside the box. Using this as a target set, it is possible to obtain the maximum volume polytopes with vertices marked in Figure 1 by downward-pointing triangles. The total area of these is 20.5387, representing a 135% enlargement in area.

A comparable enlargement can also be observed in terms of the maximal PIF sets which, with the line segment as a target set, are given by the intersection of Γ with the shaded region bounded by the two sets of ellipsoids of Figure 1. The corresponding maximal PIF set w.r.t. the IF box is given by the intersection of Γ with the regions bounded by the dashed lines.

The significant benefits in terms of enlargement that can be brought about through the use of recursive PIF (i.e. through application of Algorithm 5.1) are shown in Figure 2 where the shaded areas are the same as those shown in Figure 1 whereas the sets bounded by the dashed lines are the PIF regions after 3 iterations of steps 1 and 2 of Algorithm 5.1. As remarked in Section 5, these regions are also IF and can be used as terminal regions in a dual prediction mode MPC scheme using as the terminal law the unconstrained optimal, namely OIOFL. Furthermore, the 2 polytopes in Figure 2 are inscribed in the PIF regions and are also contained in the feasible region Γ. These have a total volume of 852.602, which is almost 100 times the size of the maximum volume PIF polytopes obtained using the procedures of [4].

It is noted that use of the results of the paper in conjunction with the closed loop paradigm [6] allow the online optimization of dual prediction mode MPC to be performed explicitly for the special case of a single-step mode 1 horizon. According to the closed loop paradigm, feasibility is achieved by perturbing away from the optimal control law, but clearly in the interest of optimality such perturbations should be as small as possible. It is also possible to use the results of the paper to compute explicitly the boundaries of the stabilizable set for a single-step horizon. For the example under consideration this set, comprising the union of the light and dark shaded areas of Figure 3 can be seen to be a great deal bigger than the terminal regions of Figures 1 and 2. The light shaded area comprises all states for which a closed loop paradigm perturbation is necessary, whereas for all other points in the set the optimal OIOFL law can be used.

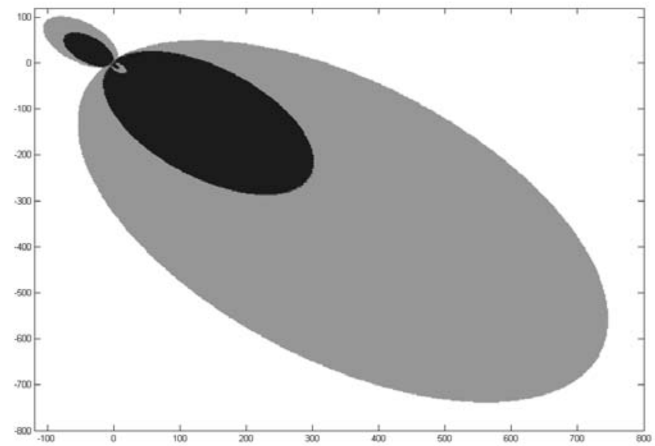

Fig. 3. Stabilizable set for single-step horizon MPC

VII. Conclusions

Optimal IOFL steers the state into the kernel of C and restriction of attention to this kernel allows for a significant enlargement of regions which, even in the presence of non-minimum phase characteristics, can be used as terminal sets in dual prediction mode MPC. Further enlargement is possible through the use of maximal PIF sets and their recursive computation. The benefits of these procedures were demonstrated by means of a simple illustrative example. For the case that all the equilibrium points under IOFL are unstable, the results can be extended to stabilizing sub-optimal controllers such as the bilinear control laws proposed in earlier work. However the question of how to minimize the effects of suboptimality in this case remains an open problem.

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