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Model Predictive Control in Nap-of-Earth Flight using Polynomial *Control Basis Functions*

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Abstract – **This paper presents analysis using a new set of design tools to compute finite horizon optimal controls specifically for onboard model predictive control based trajectory synthesis. The optimizer must produce a finite-horizon control trajectory that will enable a UAV to track a low-altitude, high-speed (Nap-of-Earth flight) reference trajectory. The optimizer must synthesize the finite-horizon controls based on a suitable fidelity plant model. This can rapidly become a high-dimensional, non-convex optimization search, particularly if the dynamic model is nonlinear and the horizon long compared to the control bandwidth. To reduce the scope of the optimization problem we constrain the finite-horizon controls to a scaleable set *of control basis functions (CBF)*. We also use these *CBFs* to identify a linear perturbation model around a nominal realizable trajectory. In this paper, we focus on polynomial *CBFs* such as Laguerre and Legendre. We compare results to a baseline of those obtained without any simplifying approximations, and to a repeating sequence of tent functions that were introduced in previous publications. Our analysis indicates that Laguerre polynomials pose a good choice of *CBFs* to design the optimal controls for NOE type of experiments; a fourth order Laguerre polynomial supplies 5 distinct and characteristic polynomial terms and is adequate for good tracking performance. The performance is equivalent to using 10 tent functions. However, since only 5 Laguerre polynomials need to be manipulated to form the optimal control, the optimization speed is greatly enhanced. It is significantly faster than solving a full order MPC problem.**

I. INTRODUCTION

New and increasingly complex control algorithms are being developed that enhance the capability and expand the operational space of autonomously controlled air vehicles beyond the near-trim flight regimes of canonical, closed-loop autonomous systems. New nonlinear, dynamics-based guidance and control laws will command and execute agile flight with rapid maneuvering capability, large thrust, and closer approach to stall boundaries for fast deceleration and rapid turning than previously possible. Such performance improvements expand the operational regime of UAVs. This paper introduces mathematical simplifications that enable realizable, high capability flight that reduce onboard algorithmic computational complexity with novel control transformations. Specifically, it analyzes the use of polynomial bases in MPC-based control synthesis designs to reduce mathematical complexity and enable significantly faster online optimal control computation.

In our approach to Nap-of-Earth flight (flying altitudes <10m above ground level) for a UAV as outlined in[11], we formulate a guidance and control problem whereby the algorithm receives precise terrain information in short asynchronous bursts in flight from an onboard obstacle detection sensor. This terrain information has short finite range. A NOE flight control objective defines a highly constrained tracking control problem. The control synthesis algorithm must ensure that the solution trajectory tracks a reference and is dynamically feasible; we meet this requirement by suitably transforming the problem into the control space to synthesize realizable control trajectories that minimize the transformed tracking error, the optimizer synthesizes (feasible) controls such that the resulting trajectory is optimal and satisfies mapping constraints from the finite look-ahead terrain data. Model Predictive Control provides a natural control theoretic approach to this problem to synthesize short-horizon terrain-tracking controls since MPC's finite horizon control planning approach blends well with the short-interval reference defined over the short-range view provided by the sensor. Because MPC derives from optimal control theory, it predicts the optimal dynamic control sequence that minimizes terrain-tracking error. Our particular interest includes operation near threat (ground collision) boundaries and control beyond the trim manifold of the vehicle; therefore, we retain the nonlinear plant model.

Predictive control was first utilized in terrain following control in 1989 [1]. Applications of model predictive control (MPC) have spanned trajectory optimization and obstacle avoidance, [2] – [4]. In [3], the authors applied non-linear MPC (NMPC) to combine trajectory generation and tracking problems. They define a quadratic cost function with an output trajectory tracking error term, a control term, and an additional term introduced to bound internal state variables. This cost is minimized subject to input constraints to ensure physically realizable trajectories.

The complete, constrained trajectory plan is synthesized onboard; thus the trajectory synthesis algorithm must solve a high-dimensional (due to the several terrain-inspired constraints) nonlinear, dynamics-based optimization problem in bounded real-time. This research presents new mathematical tools to design the NOE flight trajectory in bounded-time. It extends our previous research, [11] where an MPC-based optimizer synthesizes trajectory and outer-loop control commands necessary to navigate between waypoints at low altitude in terrain. We present dynamics analysis tools that simplify and reduce the dimension of the nonlinear, constrained optimization problem by introducing

[1] Formerly Draper Lab Fellow. Results in paper are based on Ms. Lapp's M.S. research conducted at Draper Laboratory

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a minimal set of nonlinear, polynomial control basis functions that approximately capture the nonlinear dynamics in a scaleable way around an operating manifold. These bases can be superimposed like linear functions such that the scale factors map in a known parametric way to the series polynomial solution of the nonlinear differential equation. This is a powerful tool for control engineers because it approximates complex nonlinear input-output models with a scaleable set of functions. The *CBFs* are also used to approximate the input-output model; the optimizer computes the minimizing control scale factors to these basis functions. We present three different set of basis functions considered: the baseline is a set of tent bases described in [4, 11]. We compare results from tent *CBFs* with Laguerre and Legendre polynomials, [12]. [12] formally defines series solutions to differential equations including Laguerre and Legendre polynomial functions. In this paper, we also present sensitivity and robustness analysis results of our MPC control synthesis solutions using the *CBFs*.

II. REDUCED-ORDER OPTIMIZATION USING MODEL PREDICTIVE CONTROL

Model predictive control (MPC) is a repeating, finite horizon optimal control scheme which uses an internal model of the plant (prediction model) to predict vehicle response, thus minimizing the current and predicted tracking error over a short prediction interval, and optimizing the predicted trajectory within the prediction horizon [6, 7]. MPC produces a trajectory of controls over a finite prediction horizon $T_H = T_s*H$, where T_H is the prediction horizon in seconds, T_s is the sample rate, and H the number of time intervals over the prediction horizon, to optimize the system states. In this MPC formulation, the cost function trades off minimizing terrain tracking error and control effort to determine the optimal control sequence $\mathbf{u_k}$ in the H step prediction horizon:

$$J_i = \sum_{k=i}^{i+H-1} \left(y_{k+1} - r_{k+1}\right)^T Q_k \left(y_{k+1} - r_{k+1}\right) + u_k{}^T R_k u_k , \qquad (2.1)$$

where $\mathbf{y_{k+1}}$ is the output state at time t_{k+1}, $\mathbf{r_{k+1}}$ is the trajectory reference at time t_{k+1}, and Q_k and R_k are time-referenced output and control weighting matrices over the horizon. Once the control sequence has been determined, the first N+1 (where N is a subinterval of H) inputs $\mathbf{u_i}$ through $\mathbf{u_{i+N}}$ are applied to the plant, the system allowed to respond, and the calculation repeated. N * T_s thus determines the MPC update rate.

A. PERTURBATIONAL LINEARIZATION

We use MPC to compute trajectory controls for a 9-degree-of-freedom helicopter model. In many approaches that use MPC for control calculation, the optimization function is cast into a linear, quadratic, or nonlinear program from suitable dynamics transition matrices. (For nonlinear systems, researchers have used perturbational linearization to form a perturbational transition function e.g.[4, 11]). We form a linearized prediction model for MPC by perturbational linearization of the 9-DOF non-linear model about a nominal control and state trajectory. The nominal control ($\mathbf{u_0}$) profile is initialized on the optimal finite-horizon control solution obtained from the optimizer in its previous iteration. The nominal output trajectory ($\mathbf{y_0}$) over the prediction horizon T_H, is calculated using the nonlinear plant model using the nominal control and initializing at the present state.

Perturbational linearization about this nominal trajectory produces the local linearized dynamic model:

$$\dot{x} = F_1(\dot{x}_0) + \delta\dot{x} = F_1(x_0, u_0) + \left.\frac{\partial F_1}{\partial x}\right|_{x_0,u_0} \delta x + \left.\frac{\partial F_1}{\partial u}\right|_{x_0,u_0} \delta u \qquad (2.2)$$

$$y = F_2(x_0, u_0) + \delta y \qquad (2.3)$$

Consistent with perturbational control methods, we write the perturbed model as: $\delta\dot{x} = F_a \delta x + G_a \delta u$. We introduce control bases (*CBFs*) to define the (possibly reduced) space of the perturbational controls δu and thus quasi-statically and numerically identify the linearized dynamics matrices F_a, G_a. We constrain the optimizer to compose the optimal perturbational controls by scaling a fixed set of pre-defined bases that span the T_H prediction interval. Using reduced bases reduces the optimization dimension to the small number of basis functions defined over the perturbational control space; the optimizer's objective is to compose the control sequence by optimally scaling these pre-defined *CBFs* only. In contrast, when solving the full-scale optimization problem, the optimizer must explicitly compute each of the H time index of controls in the prediction horizon. Therefore, for many problems, constraining the search to a low-dimensional subspace spanned by a reduced parameter set significantly reduces the optimization dimension. In [11], we used an overlapping sequence of tent basis functions.

B. LAGUERRE BASIS FUNCTIONS

In this subsection, we introduce a finite set of Laguerre polynomials to define the perturbational control space. The output perturbation is computed given this polynomial choice of perturbational inputs; the optimizer solves for the suitable scale factors to these polynomial functions to yield the best output trajectory for a give reference. Laguerre polynomials of the n^{th} order are computed from the following Rodriguez Formula[12]:

$$b_n = \frac{e^t}{n!} \frac{d^n}{dt^n} (t^n e^{-t}) \qquad (2.4)$$

The first five series expansions for n=0:4 are:

$$b_0 = 1 \qquad b_1 = (1 - t)$$

$$b_2 = (1 - 2t + 0.5t^2) \quad b_3 = (1 - 3t + 1.5t^2 - \tfrac{1}{6}t^3)$$

$$b_4 = (1 - 4t + 3t^2 - \tfrac{2}{3}t^3 + \tfrac{1}{24}t^4)$$

Figure 1 depicts these first five basis functions, normalized to the [0,1] interval.

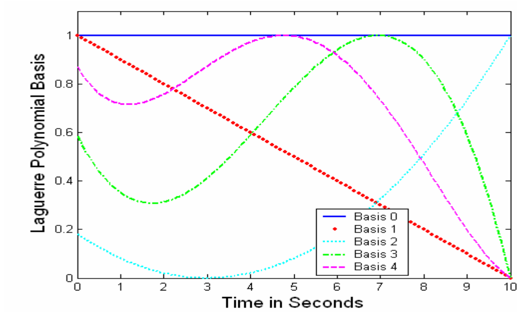

Figure 1. Figure shows the five Laguerre functions of order n=0:4 that constitute the perturbational control bases; these bases are used to compose the controls for each independent control of the system.

The vector $\delta \mathbf{u}$ over the T_H interval is simply a linear combination of the column basis functions:

$$\begin{bmatrix} \delta u_k \\ \vdots \\ \delta u_{k+H-1} \end{bmatrix} = \begin{bmatrix} b_0 & b_1(kT_s) & b_2(kT_s) & b_3(kT_s) & b_4(kT_s) \end{bmatrix}_{k=0:H} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix} \quad (6)$$

$$\delta \mathbf{u} = B.\alpha \quad (6A)$$

The perturbational input-output model is linear and identified as described. The input sequence in B is applied to each input channel sequentially (obtained as a selection function by the explicit choice of scale factors: $\alpha(0)=[1,0,0,0,0]^T$, $\alpha(1)=[0,1,0,0,0]^T$, ... , $\alpha(4)=[0,0,0,0,1]^T$) Each basis perturbs the nominal control on each input channel: e.g., $\mathbf{U}^{thrust}(i) = \mathbf{U_0}^{thrust} + \mathbf{B}.\alpha(i)$. To identify the associated linearized model we augment the known nominal control sequence with the perturbed control sequence per control channel to produce the predictive perturbed control signal: $[\mathbf{U}^{thrust}, \mathbf{U_0}^{pitch}, \mathbf{U_0}^{roll}, \mathbf{U_0}^{tail}]$, where a 0 subscript denotes nominal control. This elementally perturbed sequence drives the internal nonlinear differential model. The corresponding output associated with each control perturbation is compared to the nominal output and the difference stored as the associative perturbed output matrix for the particular input perturbation. Because the perturbational system is assumed linear about the nominal trajectories, scaling the perturbational control polynomials produces an associated scaling of the perturbational outputs. Thus as $\delta \mathbf{u} = B\alpha$, $\delta \mathbf{y} = S\alpha$ where $S = [\mathbf{s_1}, \mathbf{s_2} \dots \mathbf{s_5}]$ is the corresponding output perturbation matrix.

Fig. 2 shows the perturbed response of forward flight speed to the independent basis perturbations in rotor pitch for a Yamaha R-50 helicopter around a trim flight condition at 30m/s steady level flight. These traces form the columns of S in response to each pitch collective input basis.

Similarly, Fig. 3 above shows the yaw perturbation produced in response to the *CBFs* used to perturb the tail

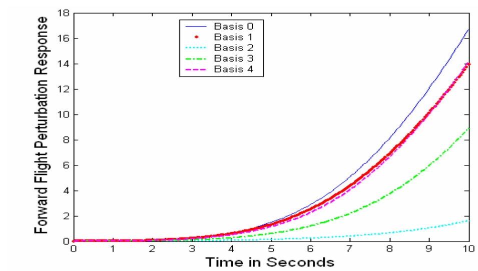

Figure 2. Forward Flight to pitch collective perturbation. Speed response to n=0:4 Laguerre basis perturbations of nominal pitch collective.

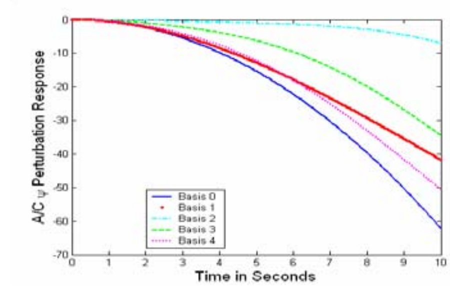

Figure 3. Yaw perturbation to Laguerre perturbations to tail rotor thrust.

rotor input. The outputs show the heading rate changes in heading over the 10 second prediction horizon compared to the nominal yaw states.

C. LEGENDRE POLYNOMIAL FUNCTIONS

Similarly, we have also analysed the system using Legendre Polynomial series to perturb the nominal controls. The Legendre Polynomials are produced from the following Rodriguez Formula:

$$b_n^{Leg} = \frac{1}{2^n n!} \frac{d^n}{dt^n} (t^2 - 1)^n$$

Fig. 4 shows the Legendre polynomial basis functions.

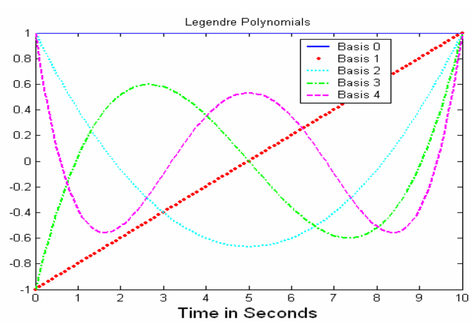

Figure 4. 4th order Legendre polynomials defined in the interval [0:10s]. Note that Legendre polynomials are normalized to the interval [-1:1].

Output perturbations induced by this choice of control basis functions around steady, level forward flight at 30m/s are presented in Fig. 5.

Figure 5. Forward velocity to pitch control perturbations on n=0:4 **Legendre** polynomial variations. Note the small pitch-collective output gain for the n=2:4 functions.

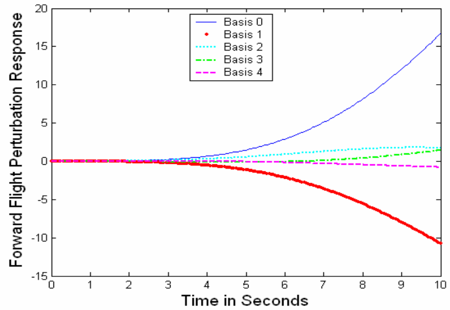

Figure 5 shows the response of forward flight speed to variations generated by Legendre polynomials perturbations to nominal rotor pitch. Note that forward speed only shows sizeable sensitivity to the zero[th] and first Legendre *CBFs*, the output gain from the higher order Legendre perturbational control functions is significantly lower than those seen with Laguerre polynomials. Similarly, the yaw response to the control bases was calculated and sensitivity to the n=2:4 polynomials was also seen to be very low.

D. INCORPORATING CONTROL BASES IN RH COST FUNCTION

The cost function is defined in terms of the nominal and perturbed states: **y** = Sα + $\mathbf{y_0}$ and **u** = Bα + $\mathbf{u_0}$, yielding:

$$J_i = \sum_{k=i}^{i+H-1} \begin{bmatrix} \left(S\alpha + y_0 - r_{k+1}\right)^T Q_k \left(S\alpha + y_0 - r_{k+1}\right) \\ + \left(\mathrm{B}\alpha + u_0\right)^T R \left(\mathrm{B}\alpha + u_0\right) \end{bmatrix} \quad (7)$$

One such expression exists for each output state variable in consideration, over the T_H second optimization horizon. Note that ***y*** and $\boldsymbol{y_0}$ are both defined over the T_H second optimization horizon. Here, the reference designates {$\mathbf{r_k}$, k=i:I+H-1} forward position, lateral position, vertical flight, aircraft speed, and heading angle states over the interval.

Using Laguerre polynomials to linearize the dynamics about a 30m/s level flight nominal trim trajectory, we produce a rank 9 matrix for the forward flight response matrix (S^TQS) in response to the full set of control perturbations (i.e,. all control perturbations over all the controls). Similarly, the lateral velocity response matrix (S^TQS) has rank 13 and yaw response matrix is rank 6. In contrast, using the Legendre Polynomials, while the forward flight state response matrix is also rank 9, matrix ranks for the lateral and yaw response matrices are 9 and 4 respectively. Therefore, Legendre polynomials admit smaller controllable subspace do than the Laguerre; a fact that was observed in simulation experiments.

In Section IV, we collect some performance results obtained by comparing the performance obtained using the Legendre and Laguerre polynomials respectively to synthesize the control profile.

III. MISSION OBJECTIVE FOR TRACKING CONTROL

Our optimization objective is to meet a desired flight speed and altitude AGL over variable terrain between waypoints. We specify a reference trajectory to follow at the desired altitude between the way-points. The reference trajectory, **R**, is defined between the vehicle's present position and a goal waypoint which is far enough away that it is beyond the prediction horizon. To compute **R** we traverse the terrain between waypoints with a spline that elicits the lowest gradient over the prediction horizon, and at the desired AGL. We penalize lateral and altitude trajectory tracking errors through a relative weighting parameter, ω, in the state weighting matrix, Q. Within Q_k, set for each time-step k along the prediction horizon, the lateral or x, y and Ψ error weights are set to 1 and the altitude or z weight is set to ω: $Q_k = [1, 1, \omega, 1]$. A high value for ω here will allow little deviation from the set altitude about ground while allowing the xy-track to meander. A smaller value emphasizes lateral tracking while not attending as highly to maintaining the precise distance above ground. It is worth noting additionally that any performance objectives can be incorporated, making MPC-based online optimization a versatile and effective way to deal with real-time reconfiguration and control requirements. When the cost function described in (7) is minimized subject to constraints, the resulting scaled controls produce the optimal admissible tracking trajectory, subject to any state and control constraints.

IV. RESULTS

A. Vehicle Model

We simulated this *CBF*-based MPC design for the Yamaha R-50 helicopter. We used standard helicopter non-linear equations of motion with appropriate mass properties, [13]. These model equations were perturbation-linearized about 30m/s level flight trajectory for use with MPC as described in section II-A. In simulation, the resulting MPC trajectory/constraint generator and controller were applied to the non-linear dynamics. The MPC loop ran at 2 Hz to compute the 20 Hz controls. We assume full state feedback, eliminating the need for an estimator.

The control constraints that were placed on the vehicle (and converted to constraints on the scaling factors, α) are as follows:

0*G*	≤	Main Rotor Thrust	≤	3½*G*
-20˚	≤	Pitch Collective	≤	20˚
-20˚	≤	Roll Cyclic	≤	20˚
-½*G*	≤	Tail Rotor Thrust	≤	½*G*

General tracking performance for a range of terrain segments using MPC was presented in [12].

B. *Comparison of Basis Polynomial Performance for Control synthesis*

The main focus of this paper is to compare the performance of an MPC-based UAV controller using the Legendre and

Laguerre polynomial perturbational controls and contrast the computational effort over (1) direct minimization with no linearization and (2) the baseline, tent function based linearization. Therefore, this comparative analysis constitutes the main results of this paper. Complete results are in [16]. We analyzed the following criteria to enable us to identify a suitable basis set for future control design:

(1) minimum cost produced by the optimizer over one prediction horizon as we vary the basis function family and the number of bases,
(2) comparison of the polynomial *CBF*-based trajectory produced over the prediction horizon compared to the true optimal result,
(3) calculation times required to solve the optimization problem for suitable numbers of bases.

In each experiment, we define a reference trajectory that corresponds to a particular terrain swath with a step change in altitude (we do not in practice ever command step changes in altitudes; the point of this experiment was to exercise the selection against a worst case profile). MPC runs for one cycle over its 10-second prediction horizon during which it must accommodate that jump.

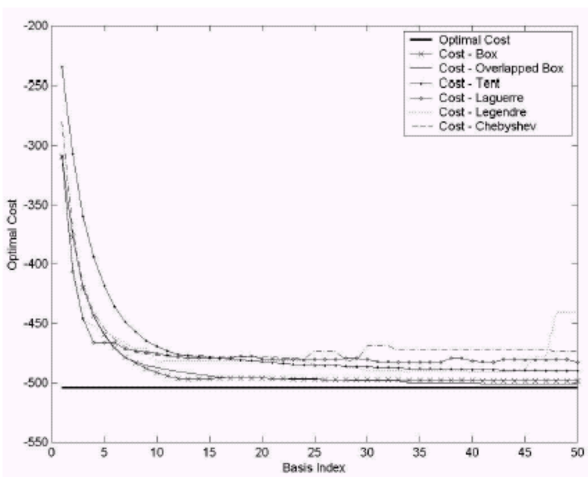

Figure 6. Minimum cost vs various orders of polynomial and polytopic functions.

Figure 6 compares the optimum values of the cost function when MPC operates with a host of basis function families and different basis orders, in each optimization published. We observe that optimization performance stabilizes in the limit over the number of bases to a particular distinct characteristic value for each considered polytopic family (tents, ramps, boxes). However, increasing *polynomial* orders beyond n=~20 produces non-monotonic behavior; this indicates that the higher order polynomial functions either over-parameterize or poorly approximate the optimum control solutions. In this particular calculation, costs using Laguerre function polynomials drop most rapidly with increasing order; results using 4^{th} order Laguerre polynomials are equivalent to those obtained using 10 tent functions and 6^{th} order Legendre polynomials. The Legendre polynomials also appear to be a promising candidate approach and have been widely considered in the guidance and control literature, particularly in space system applications. However, we have found that when we include terrain constraints, the optimizer had significant trouble computing the *feasible controls* from Legendre basis functions. Chebyshev

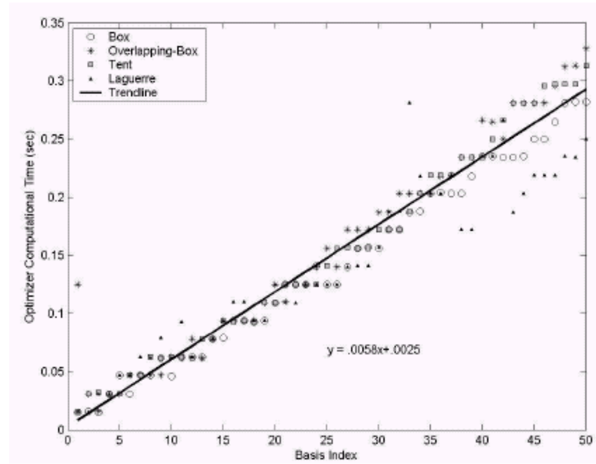

Figure 7. Computation times vs. number of bases.

polynomials demonstrated similar characteristics.

Figure 7 shows the computation times required per search as a function of basis order for all basis families. At this juncture, we eliminated the Legendre family from consideration because it failed to converge in some of our test suite of tracking scenarios. Based on these results, we note that 5^{th} order Laguerre is computationally and numerically similar to 10^{th} order tent bases. An additional analysis compared computation times using 10 tent bases, and n=0:4^{th} order Laguerre polynomial functions over multiple terrain swatches; both the mean and the standard deviation of computation time using the 5 Laguerre functions were significantly lower. Our results on a 1.2GHz Pentium with 512M RAM showed a mean convergence time of 0.2072s with the 10 tents, and 0.0812s using 5 Laguerre.

Finally, we present results showing tracking performance when the controller must track a reference step altitude change with these functions compared with MPC. In this case, when the optimizer computes the optimal controls over the 10sec horizon, we apply the first 0.5 second of controls, and then refine the previous MPC optimization output, all the way to the end. The final altitude tracking response is shown in Figure 8.

The reader will observe that MPC with 5 Laguerre functions are able to follow the terrain with smaller altitude excursions; the tents in contrast, provide a systemmatically faster tracking response and stabilize to the optimal steady state altitude reference. The Laguerre functions oscillate in a small, bounded range (<1m) about the desired altitude set point. While the box-based solutions eventually converge, they show large settling transients.

VI. Other Applications

Finally, we have also validated the usefulness of the polynomial CBFs in other applications. [14] describes NMPC to analyse real-time missile-avoidance control. Laguerre polynomial control solutions expanded the convergence regime compared to the 10 tent *CBFs*. The

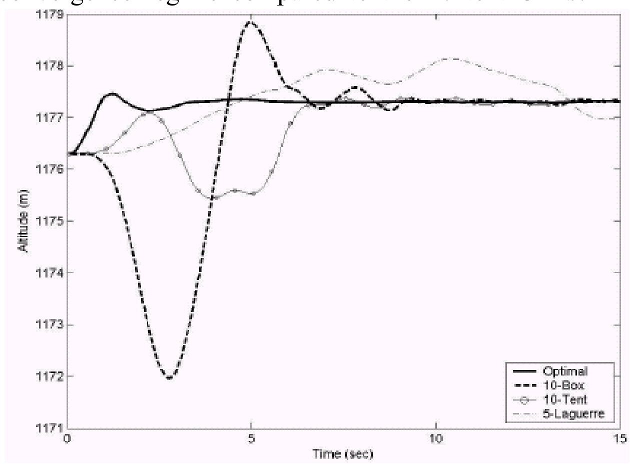

Figure 8. Reference tracking performance using tent, box, and Laguerre functions to synthesize optimal controls. The Laguerre are both quicker to converge and also demonstrate good boundedness of the output.

MPC objective in this research was to compute tactical controls that would manoeuvre an F-16 aircraft out of line-of-sight of an oncoming missile. Typically, this engagement requires high thrust, brake, or roll near the collision time. The tent bases served quite well to produce missile avoidance controls, however, as the missile seeker's capability space expanded, the tents were unable to find survivable, non-fatal controls. However, using 5^{th} order Laguerre *CBFs*, the optimizer was successfully able to produce admissible and feasible controls that dodge the missile. In one engagement scenario cited, the miss distance was approximately 150m compared with 0 (fatal) for tents. Additionally, using 5^{th} order Laguerre polynomials instead of 10 tents, convergence times greatly fell as outlined here.

V. Conclusions

This paper presents results using polynomial basis functions in control synthesis designs as a method to reduce the mathematical complexity and enable rapid, onboard computation in model predictive control to enhance the capabilities of autonomously controlled air vehicles. We introduce Laguerre polynomials as a way to (1) compute a perturbational linear dynamic model for MPC and (2) compute the perturbational control sequence from a reduced basis function set to span the admissible control space. This is a further enhancement over previously introduced *CBFs* that we formed from a repeating sequence of tent and ramp functions. The optimizer is constrained to compute the controls by scaling the finite set of polynomial control basis functions which is of low dimension compared with the true optimization solution. We presented results showing the efficacy of using the Laguerre polynomials; in particular we showed that Laguerre polynomials effectively produce similar tracking and optimum costs to those produced using the previously introduced tent functions. However, we require fewer Laguerre approximating functions (just 5 Laguerre compared to 10 tents) this reduces computational time required for the optimization. We also show that a judicious selection of *CBFs* is essential; Legendre polynomials were not a suitable choice for our dynamics.

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