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Stabilization of Nonlinear Systems with State and Control Constraints Using Lyapunov-Based Predictive Control *

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Abstract— **This work considers the problem of stabilization of nonlinear systems subject to state and control constraints. We propose a Lyapunov-based predictive control design that guarantees stabilization and state and input constraint satisfaction from an explicitly characterized set of initial conditions. An auxiliary Lyapunov-based analytical bounded control design is used to characterize the stability region of the predictive controller and also provide a feasible initial guess to the optimization problem in the predictive controller formulation. For the case when the state constraints are soft, we propose a switched predictive control strategy that reduces the time for which state constraints are violated, driving the states into the state and input constraints feasibility region of the Lyapunov-based predictive controller. We demonstrate the application of the Lyapunov-based predictive controller designs through a chemical process example.**

Key words: State constraints, soft constraints, input constraints, model predictive control, bounded Lyapunov-based control, stability region, feasibility region.

I. INTRODUCTION

Control systems are often subject to constraints on their manipulated inputs and state variables. Input constraints arise as a manifestation of the physical limitations inherent in the capacity of control actuators (e.g., bounds on the magnitude of valve opening), and are enforced at all time (hard constraints). State constraints, on the other hand, arise either due to the necessity to keep the state variables within acceptable ranges, to avoid, for example, runaway reactions (in which case they need to be enforced at all times, and treated as hard constraints) or due to the desire to maintain them within bounds determined by performance considerations (in which case they may be relaxed, and treated as soft constraints). Constraints impose limitations on our ability to steer the dynamics of the closed-loop system, and can cause deterioration in the nominal closed-loop performance and may even lead to closed-loop instability if not explicitly taken into account at the stage of controller design.

Currently, model predictive control (MPC), also known as receding horizon control (RHC), is one of the few control methods for handling state and input constraints within an optimal control setting and has been the subject of numerous research studies that have investigated the stability properties of MPC (e.g., see [1], [15] for extensive surveys of various MPC formulations). In MPC formulations the stability guarantees are typically based on an assumption of initial feasibility of the optimization problem, and the set of initial conditions, starting from which a given MPC formulation is guaranteed to be feasible, is not explicitly characterized. The problem of state constraints satisfaction has also been extensively studied [8], [24], [9], [3], [25], [2], [22], and has typically been analyzed with the understanding that state constraints may be relaxed. In the minimum time approach [21], the smallest time beyond which the state constraints can be satisfied on an infinite horizon is identified, and the state constraints are relaxed up-to that time. While possible for linear systems, the computation of such a time, beyond which state constraints are satisfied, is a more difficult task for nonlinear systems. In other approaches, they are relaxed for all times, and only incorporated in the objective function as appropriate penalties on state constraint violation ('softening' of state constraints). In either approaches, the problem of providing explicitly the set of initial conditions starting from where stabilization can be achieved and state and input constraints are guaranteed to be feasible has not been addressed.

The desire to implement control approaches that allow for an explicit characterization of their stability properties has motivated significant work on the design of stabilizing control laws, using Lyapunov techniques (e.g., see [14], [4], [5]; the reader may refer to [12] for a survey of results in this area) that provide an explicit characterization of the region of guaranteed closed–loop stability. These controllers, however, are not guaranteed to be optimal with respect to an arbitrary performance criterion.

In a recent work, [17], we proposed a Lyapunov-based model predictive control formulation that provided guaranteed stability from an explicitly characterized set of initial conditions in the presence of input constraints. In this work we propose a Lyapunov-based model predictive control design for stabilization of nonlinear systems with state and input constraints. The design of the Lyapunov-based MPC uses a bounded controller, with its associated region of stability, as an auxiliary controller that is used to analyze the stability properties of the Lyapunov-based MPC. The proposed Lyapunov-based MPC is shown to possess an explicitly characterized set of initial conditions, starting from where it is guaranteed to be feasible, and hence stabilizing, while enforcing the state and input constraints at all times. For the case when the state constraints are soft, we propose a switched predictive control strategy that reduces the time for which state constraints are violated, driving the

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states into the state and input constraints feasibility region of the Lyapunov-based predictive controller. We demonstrate the application of the Lyapunov-based predictive controller designs through a chemical process example.

II. PRELIMINARIES

In this work, we consider the problem of stabilization of continuous-time nonlinear systems with state and input constraints, with the following state-space description:

$$\dot{x}(t) \quad = \quad f(x(t)) + g(x(t))u(t) \tag{1}$$

$$u \in U \quad , \quad x \in X \tag{2}$$

where $x = [x_1 \cdots x_n]' \in \mathbb{R}^n$ denotes the vector of state variables, $u = [u^1 \cdots u^m]' \in U$ denotes the vector of manipulated inputs, $U \subseteq \mathbb{R}^m, X \subseteq \mathbb{R}^n$ denote the constraints on the manipulated inputs and the state variables, respectively, $f(\cdot)$ is a sufficiently smooth $n \times 1$ nonlinear vector function, and $g(\cdot)$ is a sufficiently smooth $n \times m$ nonlinear matrix function. Without loss of generality, it is assumed that the origin is the equilibrium point of the unforced system (i.e. $f(0) = 0$). Throughout the paper, the notation $\|\cdot\|$ will be used to denote the Euclidean norm of a vector, while the notation $\|\cdot\|_Q$ refers to the weighted norm, defined by $\|x\|_Q^2 = x'Qx$ for all $x \in \mathbb{R}^n$, where Q is a positive–definite symmetric matrix and x' denotes the transpose of x. The notation $L_f V$ denotes the standard Lie derivative of a scalar function $V(\cdot)$ with respect to the vector function $f(\cdot)$ and $\limsup_{t\to\infty} f(x(t)) = \lim_{t\to\infty} \{\sup_{\tau \geq t} f(x(\tau))\}$. In order to provide the necessary background for our results in Sections III and IV, we will briefly review in the remainder of this section the design procedure for, and the stability properties of, a bounded control design, which will be used to characterize the feasibility region of a Lyapunov-based MPC formulation in Section III. Throughout the manuscript, we assume that for any $u \in \mathbf{U}$ the solution of the system of Eq.1 exists and is continuous for all t, and we focus on the state feedback problem where measurements of $x(t)$ are assumed to be available for all t.

A. Bounded Lyapunov-based control

Consider the system of Eq.1 for which a control Lyapunov function, V, exists. Using the results in [14] (see also [4]), the following bounded control law can be constructed:

$$u(x) \quad = \quad \left\{ \begin{array}{ccc} -k(x)\beta(x) & , & \|\beta(x)\| \neq 0 \\ 0 & , & \|\beta(x)\| = 0 \end{array} \right\} := u_b(x) \tag{3}$$

$$k(x) \quad = \quad \frac{\alpha(x) + \sqrt{(\alpha(x))^2 + (u^{max}\|\beta(x)'\|)^4}}{\|\beta(x)'\|^2 \left[1 + \sqrt{1 + (u^{max}\|\beta(x)'\|)^2}\right]} \tag{4}$$

where $\beta(x) = L_g V(x)'$, $L_g V(x) = [L_{g^1} V \cdots L_{g^m} V]$ is a row vector, where g^i is the ith column of g, $\alpha(x) = L_f V + \rho_c V$ and $\rho_c > 0$, and u^{max} is a real positive number such that for all $u \in U$, $\|u\| \leq u^{max}$. For the above controller, one can show, using a standard Lyapunov argument, that whenever the closed–loop state, x, evolves within the region described by the set:

$$\Phi_{x,u} \quad = \quad \{x \in X : \alpha(x) \leq u^{max}\|\beta(x)\|\} \tag{5}$$

then the controller satisfies the state and input constraints, and the time-derivative of the Lyapunov function is negative-definite. Assume that

$$\Omega_{x,u} \quad = \quad \{x \in \mathbb{R}^n : V(x) \leq c_{x,u}^{max}\} \subseteq \Phi_{x,u} \tag{6}$$

for some $c_{x,u}^{max} > 0$. $\Omega_{x,u}$ then provides an estimate of the stability region, starting from where the origin of the constrained closed–loop system, under the control law of Eqs.3-4, is guaranteed to be asymptotically stable and state and input constraints are satisfied.

The control law ensures that for all initial conditions in $\Omega_{x,u}$, the closed–loop state remains in $\Omega_{x,u}$ when the control action is implemented in a discrete (sample and hold) fashion with a sufficiently small hold time (Δ) and eventually converges to some neighborhood of the origin whose size depends on Δ. This robustness property, formalized in [17] for the problem of stabilization under input constraints (yielding Ω_u as the region of stability and input constraint satisfaction), carries over to the case of input and state constraints. This property will be exploited in the Lyapunov-based predictive controller design of Section III and is formalized in Proposition 1 below (the proof of the proposition is similar to that of Proposition 1 in [17], and is omitted for brevity). For further results on the analysis and control of sampled-data nonlinear systems, the reader may refer to [10], [11], [23].

Proposition 1: *Consider the constrained system of Eq.1, under the bounded control law of Eqs.3–4 with $\rho_c > 0$ and let $\Omega_{x,u}$ be the stability region estimate under continuous implementation of the bounded controller. Let $u(t) = u(j\Delta)$ for all $j\Delta \leq t < (j+1)\Delta$ and $u(j\Delta) = u_b(x(j\Delta))$, $j = 0, \cdots, \infty$, where $u_b(\cdot)$ was defined in Eq.3. Then, given any positive real number d, there exist positive real numbers Δ^* and δ' such that if $\Delta \in (0, \Delta^*]$ and $x(0) := x_0 \in \Omega_{x,u}$, then $x(t) \in \Omega_{x,u} \subseteq X$ and $\limsup_{t\to\infty} \|x(t)\| \leq d$. Also, if $V(x)(t) \leq \delta'$ then $V(x(\tau)) \leq \delta'\ \forall\ \tau \in [t, t+\Delta)$ and if $\delta' < V(x)(t) \leq c_{x,u}^{max}$ then $\dot{V}(x(\tau)) < 0\ \forall\ \tau \in [t, t+\Delta)$.*

III. LYAPUNOV-BASED MODEL PREDICTIVE CONTROL

Consider model predictive control of the system of Eq.1 with hard state and input constraints. We present here a Lyapunov–based MPC formulation (see Remark 1 for a discussion on this formulation and its relationship to other Lyapunov-based formulations) that guarantees feasibility of the optimization problem subject to hard constraints on the state and input and hence constrained stabilization of the closed–loop system from an explicitly characterized set of initial conditions. For this MPC design, the control action at state x and time t is obtained by solving, on-line, a finite horizon optimal control problem of the form:

$$P(x,t) \quad : \quad \min\{J(x,t,u(\cdot)) | u(\cdot) \in S,\ x \in X\} \tag{7}$$

$$s.t.\ \dot{x} \quad = \quad f(x) + g(x)u \tag{8}$$

$$\dot{V}(x(\tau)) < 0 \;\; if \;\; V(x(t)) > \delta^{'},\; \tau \in [t, t+\Delta) \quad (9)$$

$$V(x(\tau)) \leq \delta^{'} \;\; if \;\; V(x(t)) \leq \delta^{'},\; \tau \in [t, t+\Delta) \quad (10)$$

where $S = S(t,T)$ is the family of piecewise continuous functions (functions continuous from the right), with period Δ, mapping $[t, t+T]$ into U and T is the specified horizon. Eq.8 is the nonlinear model describing the time evolution of the state x, V is the Lyapunov function used in the bounded controller design and $\delta^{'}$ is defined in Proposition 1. Eq.9 is a strict inequality constraint, requiring that the value of the Lyapunov function decrease during the first time step, while Eq.10 is an inequality constraint requiring the Lyapunov function value to stay below $\delta^{'}$, once it has decreased to a value less than $\delta^{'}$. A control $u(\cdot)$ in S is characterized by the sequence $\{u[j]\}$ where $u[j] := u(j\Delta)$ and satisfies $u(t) = u[j]$ for all $t \in [j\Delta, (j+1)\Delta)$. The performance index is given by

$$J(x,t,u(\cdot)) \;\; = \;\; \int_t^{t+T} \left[\|x^u(s;x,t)\|_Q^2 + \|u(s)\|_R^2\right] ds \quad (11)$$

where R and Q are strictly positive definite, symmetric matrices and $x^u(s;x,t)$ denotes the solution of Eq.1, due to control u, with initial state x at time t. The minimizing control $u^0(\cdot) \in S$ is then applied to the plant over the interval $[j\Delta, (j+1)\Delta)$ and the procedure is repeated indefinitely. This defines an implicit model predictive control law:

$$M(x) \;\; := \;\; argmin(J(x,t,u(\cdot))) \;\; := u_1 \quad (12)$$

Closed–loop stability and state and input constraint feasibility properties of the closed–loop system under the Lyapunov–based predictive controller are inherited from the bounded controller under discrete implementation and are formalized in Proposition 2 below.

Proposition 2: *Consider the constrained system of Eq.1 under the MPC law of Eqs.7–12 with $\Delta \leq \Delta^*$ where Δ^* was defined in Proposition 1. Then, given any $x_0 \in \Omega_{x,u}$, where $\Omega_{x,u}$ was defined in Eq.6, the optimization problem of Eq.7-12 is feasible for all times, $x(t) \in \Omega_{x,u} \subseteq X$ for all $t \geq 0$ and $\limsup_{t\to\infty} \|x(t)\| \leq d$.*

Proof of Proposition 2: The proof of this proposition is divided in three parts. In the first part we show that for all $x_0 \in \Omega_{x,u}$, the predictive control design of Eqs.7-12 is feasible. We then show that $\Omega_{x,u}$ is invariant under the predictive control algorithm of Eqs.7-12, and that the state and input constraints are satisfied for all times. Finally, we prove practical stability for the closed–loop system.

Part 1: Consider some $x_0 \in \Omega_{x,u}$ under the predictive controller of Eqs.7-12, with a prediction horizon $T = N\Delta$, where Δ is the hold time and $1 \leq N < \infty$ is the number of the prediction steps. The initial condition can either be such that $V(x_0) \leq \delta^{'}$ or $\delta^{'} < V(x_0) \leq c_{x,u}^{max}$.

Case 1: If $\delta^{'} < V(x_0) \leq c_{x,u}^{max}$, the control input trajectory under the bounded controller of Eqs.3–4 provides a feasible solution to the constraint of Eq.9 (see Proposition 1). A feasible initial guess for the optimization problem of Eqs.7-12 therefore exists, and, in particular, is given by $u(j\Delta) = u_b(j\Delta),\ j = 1, \cdots, N$. Note that if $u = u_b$ for $t = [0, \Delta]$, and $\Delta \in (0, \Delta^*]$, then $\dot{V} < 0 \;\forall\; t \in [0, \Delta]$ and $u_b \in U$ (since u_b is computed using the bounded controller of Eqs.3–4). Also, under discrete implementation of the bounded controller of Eqs.3–4, for any $x_0 \in \Omega_{x,u}$, $x(t) \in \Omega_{x,u} \;\forall\; t \geq 0$, therefore the constraint $x(t) \in X$ is also satisfied by this control trajectory (note that $\Omega_{x,u} \subseteq \Phi_{x,u} \subseteq X$).

Case 2: If $V(x_0) \leq \delta^{'}$, once again we infer from Proposition 1 that the control input trajectory provided by the bounded controller of Eqs.3–4 provides a feasible initial guess, given by $u(j\Delta) = u_b(j\Delta),\ j = 1, \cdots, N$ (recall from Proposition 1, that under the bounded controller of Eqs.3–4, if $V(x_0) \leq \delta^{'}$ then $V(x(t)) \leq \delta^{'} \;\forall\; t \geq 0$). This shows that for all $x_0 \in \Omega_{x,u}$, the Lyapunov based predictive controller of Eqs.7-12 is feasible.

Part 2: As shown in Part 1, for any $\delta^{'} < V(x_0) \leq c_{x,u}^{max}$, the constraint of Eq.9 in the optimization problem is feasible. Upon implementation, therefore, the value of the Lyapunov function decreases, and since $\Omega_{x,u}$ is a level set of V, the closed–loop state trajectory cannot escape out of $\Omega_{x,u}$. On the other hand, if $V(x_0) \leq \delta^{'}$, feasibility of the constraint of Eq.10 guarantees that the closed–loop state trajectory evolves such that $V(x(t)) \leq \delta^{'} \;\forall\; t \geq 0$. In both cases, $\Omega_{x,u}$ continues to be an invariant region under the Lyapunov based predictive controller of Eqs.7-12. Also, since $\Omega_{x,u} \subseteq \Phi_{x,u} \subseteq X$, we have that $x(t) \in X$ for all $t \geq 0$.

Part 3: Finally, consider an initial condition, x_0, such that $\delta^{'} < V(x_0) \leq c_{x,u}^{max}$. Since the optimization problem continues to be feasible for all $t \geq 0$, we have that $\dot{V} < 0$ for all $\delta^{'} < V(x(t)) \leq c_{x,u}^{max}$. All trajectories originating in $\Omega_{x,u}$, therefore converge to the set defined by $\Omega^t := \{x \in \mathbb{R}^n : V(x) \leq \delta^{'}\}$. For $V(x_0) \leq \delta^{'}$, the feasibility of the optimization problem of Eqs.7-12 implies $V(x(t)) \leq \delta' \;\forall\; t \geq 0$. Therefore, for all $x_0 \in \Omega_{x,u}$, $\limsup_{t\to\infty} V(x(t)) \leq \delta^{'}$. Then, since $V(x) \leq \delta^{'}$ implies $\|x\| \leq d$ (note that $V_k(\cdot)$ is a continuous function of the state, therefore one can find a finite, positive real number, $\delta^{'}$, such that $V(x) \leq \delta^{'}$ implies $\|x\| \leq d$), we have that $\limsup_{t\to\infty} \|x(t)\| \leq d$. This completes the proof of Proposition 2.

Remark 1: Lyapunov-based predictive control approaches (see, for example, [13], [20]) typically incorporate a Lyapunov function decay constraint similar to that of Eq.9, albeit requiring the constraint to hold at the *end* of the prediction horizon as opposed to during the first time step. This may lead to the state trajectory going out of the level set (and therefore, possibly out of the state constraint satisfaction region, violating the state constraints), and motivates using the constraint of Eq.9 that requires the Lyapunov function to decrease during the first step.

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Remark 2: One of the key challenges that impact on the practical implementation of nonlinear MPC (NMPC) is the inherent difficulty of characterizing, *a priori*, the set of initial conditions starting from where a given NMPC controller is guaranteed to stabilize the closed–loop system. The Lyapunov-based predictive controller formulation guarantees initial and subsequent feasibility of the optimization problem for an explicitly characterized set of initial conditions. In addition, the optimization problem in the predictive controller is also initialized with a feasible initial guess, which reduces the computation burden. Note also that any other Lyapunov-based analytic control design that provides an explicit characterization of the state and input constrained stability region, and is robust with respect to discrete implementation, can be used as the auxiliary controller, and the choice is not limited to the bounded controller used in this paper.

IV. Handling soft state constraints via controller switching

Consider now the nonlinear system of Eq.1 where the state constraints represent desired bounds on the values of the state variables. In this case, the state constraints can be treated as soft constraints, allowing their violation for some period of time. It is important, nevertheless, to implement control action that reduces the time for which constraints are violated. We propose in this section a control design that uses two predictive formulations and switches between them. First we design a predictive controller that, while respecting input constraints, drives the state trajectory into the feasible region of the predictive controller formulation of Proposition 2, in a way that reduces the time for which the state constraints are violated, and then implement the predictive controller of Proposition 2 to achieve stabilization together with state and input constraint satisfaction for the rest of the time. To this end, we first cast the system of Eq.1 as a switched system of the form:

$$\dot{x} = f(x) + g(x)u_{i(t)}, \qquad i \in \{1, 2\} \tag{13}$$

where $i : [0, \infty) \to \{1, 2\}$ is the switching signal which is assumed to be a piecewise continuous (from the right) function of time, implying that only a finite number of switches is allowed on any finite–time interval. The index, $i(t)$, represents a discrete state that indexes the control input, u, with the understanding that $i(t) = 1$ if and only if $u_i(x(t)) = u_1$ (i.e., the Lyapunov-based MPC formulation of Eqs.7-12 is used) and $i(t) = 2$ if and only if $u_i(x(t)) = u_2$) (i.e., an MPC formulation designed to reduce the time of state constraint violation, is used). Theorem 1 below presents both the control law, u_2, and the switching law.

Theorem 1: *Consider the switched nonlinear system of Eq.13, for which there exists a control Lyapunov function V, and for a given pair of positive real numbers (d, ρ_c), Δ is chosen such that $\Delta \le \Delta^*$, where Δ^* was defined in Proposition 1. Given any initial condition $x_0 \in \Omega_u$, let T_b be the time it takes for the bounded controller of Eqs.3–4, under discrete implementation with a discretization step Δ, to achieve $x(T_b) \in \Omega_{x,u}$. Consider the following optimization problem:*

$$u \;=\; argmin(J) := u_2 \tag{14}$$

$$J = qV(x(t+\Delta)) + \int_t^{t+T} \left[\|u(s)\|_R^2\right] ds \tag{15}$$

where $q > 0,\ R > 0$, T is the prediction horizon given by $T = T_b - t$, subject to the following constraints:

$$\dot{x} \;=\; f(x) + g(x)u \tag{16}$$

$$\dot{V}(\tau) \le 0\ \forall\, \tau \in [t, t+T) \tag{17}$$

$$u \in U,\ x(t+T) \in \Omega_{x,u} \tag{18}$$

Let T_{switch} be the earliest time such that $x(T_{switch}) \in \Omega_{x,u}$, where $\Omega_{x,u}$ was defined in Eq.6, under the controller of Eqs.14-18. Then, the following switching law:

$$i(t) \;=\; \left\{ \begin{array}{lll} 2 & , & 0 \le t \le T_{switch} \\ 1 & , & t > T_{switch} \end{array} \right\} \tag{19}$$

ensures, for the closed–loop system, that $x(T_b) \in \Omega_{x,u}$, $T_{switch} \le T_b$, $x(t) \in \Omega_{x,u} \subseteq X\ \forall\ t > T_{switch}$ and $\limsup_{t\to\infty} \|x(t)\| \le d$.

Proof of Theorem 1: The proof of the theorem proceeds as follows: we first show that the optimization problem of Eqs.14–18 is feasible for all $0 \le t \le T_{switch}$, $x(T_{switch}) \in \Omega_{x,u}$ and that $T_{switch} \le T_b$. Then, we use the result of Proposition 2 to show that for $t > T_{switch}$, the controller of Eqs.7-12 ensures that $x(t) \in \Omega_{x,u} \subseteq X\ \forall\ t > T_{switch}$ and $\limsup_{t\to\infty} \|x(t)\| \le d$.

Case 1: Consider $x_0 \in \Omega_u \backslash \Omega_{x,u}$. Using the result of Proposition 1 in [17], we have that under discrete implementation of the bounded controller of Eqs.3–4 with $\Delta \le \Delta^*$, the state trajectory, starting from x_0, evolves such that $x(t) \in \Omega_u$ and $\dot{V} \le 0$ for all $t \ge 0$. From the definition of T_b, we have that $x(T_b) \in \Omega_{x,u}$. The optimization problem of Eqs.14–18 is guaranteed to be initially feasible, since a feasible initial guess can always be obtained using the control input trajectory under the bounded controller and is given by: $u(k\Delta) = u_b(k\Delta),\ k = 1, \cdots, T/\Delta$

Subsequently, the tail of the solution at the first time step: $u(k\Delta),\ k = 2, \cdots, T/\Delta$ is a feasible initial guess for the constraints in the optimization problem at the next time step (at the next time step, the horizon reduces from $T = T_b$ to $T = T_b - \Delta$). Under the implementation of the solution of the control move at the first time step, we get:

$\dot{V}(\tau) \le 0\ \forall\, \tau \in [t + (k-1)\Delta), t + k\Delta)\ k = 1, \cdots, T/\Delta$

Under the implementation of the tail, therefore:

$\dot{V}(\tau) \le 0\ \forall\, \tau \in [t + (k-1)\Delta), t + k\Delta)\ k = 2, \cdots, T/\Delta$

and also $x(T_b) \in \Omega_{x,u}$, which is the constraint that the optimization problem needs to enforce at the next time step. The optimization problem of Eqs.14–18, therefore, is guaranteed to be initially and successively feasible, and hence $x(T_b) \in \Omega_{x,u}$.

By definition of T_{switch}, if the state trajectory enters $\Omega_{x,u}$ before T_b, then T_{switch} is set to that value, hence $x(T_{switch}) \in \Omega_{x,u}$ where $T_{switch} \le T_b$. From Proposition 2, we get that for all $x(T_{switch}) \in \Omega_{x,u}$, $x(t) \in \Omega_{x,u} \subseteq X\ \forall\ t \ge T_{switch}$ and $\limsup_{t\to\infty} \|x(t)\| \le d$.

Case 2: For any initial condition $x_0 \in \Omega_{x,u} \subseteq \Omega_u$ we have that $T_b = T_{switch} = 0$ (since $x_0 \in \Omega_{x,u}$), and the switching law of Eq.19 dictates that the controller of Eqs.7-12 is implemented for all times. Since $x_0 \in \Omega_{x,u}$, from Proposition 2, we get that $x(t) \in \Omega_{x,u} \subseteq X \; \forall \; t \geq 0$ and $\limsup_{t\to\infty} \|x(t)\| \leq d$, completing the proof of Theorem 1.

Remark 4: For linear systems, the problem of state constraints satisfaction is typically handled by relaxing the constraints, while appropriately penalizing the state constraint violation within the objective function [25], or solving a multi-objective problem [22] that minimizes both the duration and size of state constraint violation. While these approaches do away with any potential infeasibility due to the state constraints, there are no guarantees as to for how long the state constraints will continue to be violated. The use of the bounded controller, to obtain an estimate of the time within which the state trajectory can be driven inside $\Omega_{x,u}$, allows the use of this value in the constraint of Eq.18 and guarantees state constraint satisfaction by that time. Furthermore, since the objective function minimizes the Lyapunov function value itself at the next time instance, and the target set, $\Omega_{x,u}$, is a level set, it is likely that the resulting control action will drive the trajectory inside the feasible region faster, and result in a smaller time for which the state constraints are violated (see the simulation example for a demonstration).

Remark 5: The problem of implementing MPC with guaranteed stability regions was recently addressed for state [7] and output [16] feedback control of linear systems, in [6] for nonlinear systems, and in [18] for nonlinear systems subject to uncertainty and input constraints, by means of a hybrid predictive control structure that was used to embed the implementation of MPC within the stability region of a Lyapunov-based bounded controller via switching between the predictive controller and the MPC. In this work, unlike the hybrid predictive control structure, the switching takes place between two different predictive control formulations (the controller of Eqs.14–18 and the controller of Eqs.7–12), not to provide a fall back mechanism in the event of infeasibility (the Lyapunov-based predictive controller of Eqs.7–12 is guaranteed to be feasible from an explicitly characterized set of initial conditions), but rather to use the controller of Eqs.14–18 to guide the system trajectory into the state and input constrained stability region of the Lyapunov-based predictive control design of Eqs.7–12.

V. Application to a chemical process example

Consider a continuous stirred tank reactor where an irreversible, first-order exothermic reaction of the form $A \xrightarrow{k} B$ takes place. The mathematical model for the process takes the form:

$$\dot{C}_A = \frac{F}{V_l}(C_{A0} - C_A) - k_0 e^{\frac{-E}{RT_R}} C_A$$
$$\dot{T}_R = \frac{F}{V_l}(T_{A0} - T_R) + \frac{(-\Delta H)}{\rho c_p} k_0 e^{\frac{-E}{RT_R}} C_A + \frac{Q}{\rho c_p V_l} \qquad (20)$$

where C_A denotes the concentration of the species A, T_R denotes the temperature of the reactor, Q is the heat removed from the reactor, V_l is the volume of the reactor, k_0, E, ΔH are the pre-exponential constant, the activation energy, and the enthalpy of the reaction and c_p and ρ, are the heat capacity and fluid density in the reactor. The values of all process parameters can be found in [19]. The control objective is to stabilize the reactor at the unstable equilibrium point $(C_A^s, \; T_R^s) = (0.57 \; Kmol/m^3, 395.3 \; K)$, while keeping the state variables between $C_A^{min} = 0.41 \; Kmol/m^3 \leq C_A \leq 0.73 \; Kmol/m^3 = C_A^{max}$ and $T_R^{min} = 392.3 \; K \leq T_R \leq 398.3 \; K = T_R^{max}$ using the rate of heat input, Q, and change in inlet concentration of species A, $\Delta C_A = C_{A0} - C_{A0_s}$ as manipulated inputs with constraints: $|Q| \leq 0.167 \; KJ/min$ and $|\Delta C_{A0}| \leq 1 \; Kmol/m^3$. We construct a bounded controller of the form of Eq.3 using $V(x) = x'Px$ where $x = (C_A - C_A^s, \; T_R - T_R^s)$, $P = \begin{Bmatrix} 60.2 & 3.82 \\ 3.82 & 0.34 \end{Bmatrix}$ and $\rho_c = 0.001$ and compute its stability region estimate under input constraints Ω_u and that under state and input constraints $\Omega_{x,u}$, shown in Fig.1. The parameters in the objective function of Eq.11 are chosen as $Q = qI$, with $q = 1.0$, and $R = rI$, with $r = 1.0$ and those in the objective function of Eq.15 are chosen as $q = 10.0$ and $R = rI$, with $r = 0.01$. The constrained nonlinear optimization problem is solved using the MATLAB subroutine fmincon, and the set of ODEs is integrated using the MATLAB solver ODE45.

We first demonstrate the implementation of the Lyapunov-based predictive controller of Proposition 2 (Eqs.7-12) for the case when the state constraints are hard constraints and need to be satisfied at all times. To this end, we consider an initial condition that belongs to the state and input constrained stability region of the predictive controller, $\Omega_{x,u}$. As shown by the solid line in Fig.1, starting from the initial condition $(C_A, \; T_R) = (0.445 \; Kmol/m^3, \; 398.0 \; K)$, successful stabilization of the closed–loop system is achieved, together with state and input constraint satisfaction for all times. The corresponding state and input profiles are shown in Fig.2.

Consider now the case where the state constraints reflect desirable bounds on the state variables, and can be treated as soft constraints. In this case, the closed–loop state could be initialized in Ω_u, from initial conditions where state constraints are initially violated. Starting from an initial condition that violates the state constraint on the temperature, $(C_A, \; T_R) = (0.678 \; Kmol/m^3, 391.6 \; K)$, it takes 0.42 minutes for the state trajectory to enter $\Omega_{x,u}$ under the implementation of the bounded controller (see dashed lines in Figs.1-2). Setting $T_b = 0.42$ minutes, therefore and implementing the predictive controller of Theorem 1, we find that the controller is able to drive the state trajectory inside $\Omega_{x,u}$ at $t = 0.26$ minutes, reducing the time for which the soft state constraints are violated by about 40% (see dotted lines in Figs.1-2). After $T_{switch} = 0.26$ minutes the predictive controller of Proposition 2 is employed to successfully achieve stabilization in the presence of state and input constraints.

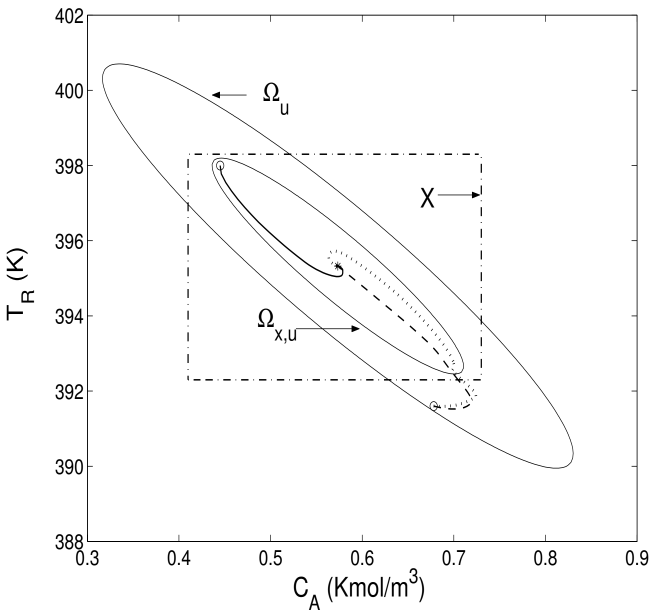

Fig. 1. Closed–loop state trajectory under the predictive controller of Proposition 2 (solid line), under the bounded controller (dashed line) and under the predictive controller of Theorem 1 (dotted line).

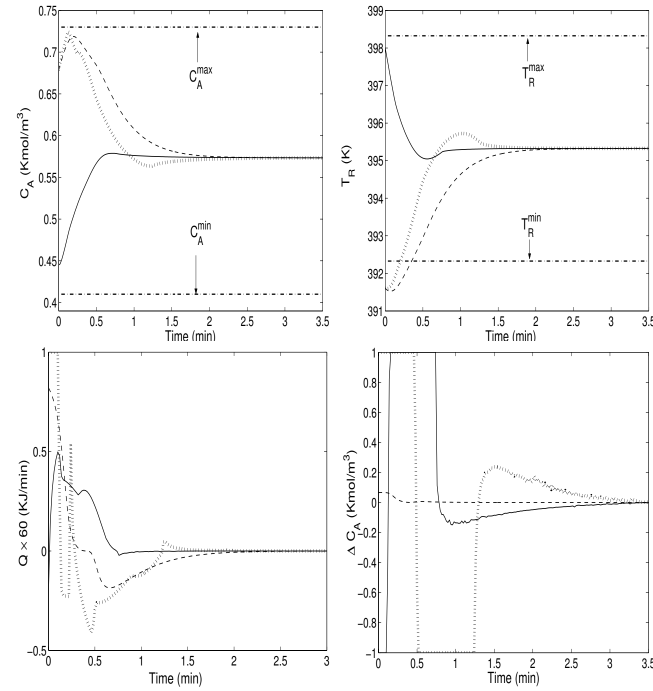

Fig. 2. Closed–loop state (top) and input (bottom) profiles under the predictive controller of Proposition 2 (solid lines), under the bounded controller (dashed lines) and under the predictive controller of Theorem 1 (dotted lines).

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