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The Efficient Computation of Polyhedral Invariant Sets for Linear Systems with Polytopic Uncertainty

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Abstract— **In this paper the concept of maximal admissable set (MAS), introduced by Gilbert et al [3] for linear time-invariant systems, is extended to linear systems with polytopic uncertainty under linear state feedback. It is shown that by constructing a tree of state predictions using the vertices of the uncertainty polytope and by imposing state and input constraints on these predictions a polyhedral robust invariant set can be constructed. The resulting set is proven to be the maximal admissable set. The number of constraints defining the invariant set is shown to be finite if the closed loop system is quadratically stable (i.e. has a quadratic Lyapunov function). An algorithm is also proposed that efficiently computes the polyhedral set without exhaustively exploring the entire prediction tree. This is achieved through the formulation of a more general invariance condition than that proposed in Gilbert et al (1991) and by the removal of redundant constraints in intermediate steps. The efficiency and correctness of the algorithm is demonstrated by means of a numerical example.**

I. INTRODUCTION

The notion of invariant sets arises in many problems concerning analysis of dynamical systems, controller design and the construction of Lyapunov functions. An overview of the concept of set invariance and many references can be found in the overview paper [2]. A systematic way for constructing polyhedral sets for linear systems was initially proposed in [3]. The proposed algorithm constructs an invariant set by iteratively adding additional constraints until invariance is obtained. This paper extends these results towards linear systems with polytopic model uncertainty.

A number of contributions have been published in this direction. In [1] describes the construction of controllability sets for linear systems with polytopic model uncertainty and polytopic disturbances. These sets do not take a given controller into account, but rather guarantee that for each state inside the set, some control action exists that steers the system further inside the set with a given convergence rate. In [4] and related works, theoretical results related to invariant sets for uncertain systems with disturbances are discussed, but no general algorithms for the setting considered in this paper have been proposed. In [6] a method is proposed to construct low-complexity robust polyhedral invariant sets for uncertain linear systems driven by a linear feedback controller. A set defined by component-wise bounds in a similarly transformed state space is considered and invariance is imposed by demanding that the Perron-Frobenius norm of the closed loop system matrices is smaller than 1. However, this leads to conservative invariant sets and in some cases no invariant set can be obtained. Another, but also conservative, approach is the construction of ellipsoidal invariant sets. We refer to [5] and [7] for recent results in this direction.

This paper proposes an efficient algorithm that constructs the maximal admissable set [3] for linear systems with polytopic model uncertainty, which are controlled by a linear feedback controller and are subject to linear state and input constraints. A more general invariance condition than that proposed in [3, p. 1010, Theorem 2.2] is proposed, leading to an increased efficiency of our algorithm compared to [3, p. 1011, Algorithm 3.2] and enabling the extension towards linear systems with polytopic uncertainty.

This paper is organised as follows. In Section II the problem is formulated, after which, in Section III, a theoretical approach is taken towards the construction of the polyhedral invariant set. In Section IV a generalized invariance condition is formulated leading to an efficient algorithm for constructing a solution satisfying this invariance condition. The new algorithm is then applied to a numerical example in Section V demonstrating the efficcacity of the proposed algorithm. Section VI then gives some conclusions and Section VII concludes by pointing out several areas of future research.

II. PROBLEM FORMULATION

Consider the linear time-varying (LTV) system

$$x_{k+1} = \Phi(k)x_k \tag{1a}$$
$$y_k = Cx_k \tag{1b}$$

with $x_k \in \mathbb{R}^{n_x}$ denoting the state of the system at discrete time k, $y_k \in \mathbb{R}^{n_y}$ denoting the output of the system. The time-varying matrix $\Phi(k)$ belongs to a given uncertainty polytope

$$\Omega = \left\{ \Phi \in \mathbb{R}^{n_x \times n_x} \middle| \Phi = \sum_{i=1}^{L} \lambda_i \Phi_i \, , \sum_{i=1}^{L} \lambda_i = 1, \lambda_i \geq 0 \right\}. \tag{2}$$

The output is subject to linear constraints

$$y_k \in \mathcal{Y} = \{y | A_y y \leq b_y\}, \qquad k = 0, \ldots, \infty, \tag{3}$$

with $0 \in \mathcal{Y}$, which is equivalent with $b_y \geq 0$. We will assume that the system is robustly asymptotically stable

$$\lim_{n \to \infty} \left(\max_{\substack{\Phi(k) \in \Omega, k=0,\ldots,n-1 \\ \|x_0\|=1}} \|x_n\|_2 \right) = 0. \tag{4}$$

The aim is to find the set $\mathcal{S}$ of initial states x_0 for which all corresponding outputs $y(0), \ldots, y(\infty)$ satisfy the output constraints $\mathcal{Y}$.

This problem includes the problem of finding the set of allowable initial conditions x_0 for which a linear system $x_{k+1} = A(k)x_k + B(k)u_k$ with polytopic model uncertainty $[A(k)\ B(k)] \in \Omega'$, controlled by a linear state feedback controller $u_k = Fx_k$, satisfies linear state and input constraints $\mathcal{X} = \{x | A_x x \leq b_x\}$ and $\mathcal{U} = \{u | A_u u \leq b_u\}$. This can be seen by replacing $\Phi(k), C, A_y$ and b_y in (1)-(3) with $A(k) + B(k)F, I_{n_x \times n_x}, [A_x^{\mathrm{T}}\ (A_u F)^{\mathrm{T}}]^{\mathrm{T}}$ and $[b_x^{\mathrm{T}}\ b_u^{\mathrm{T}}]^{\mathrm{T}}$.

We first give a formal definition of the concept of robust positive invariance and then formalize the problem that is solved in this paper.

Definition 1 (Robust Positive Invariance): *Given a system* (1)-(2) *satisfying* (4) *then $\mathcal{S} \in \mathbb{R}^{n_x}$ is a robust positive invariant set if*

$$\Phi x \in \mathcal{S}, \quad \forall x \in \mathcal{S}, \forall \Phi \in \Omega. \tag{5a}$$

Definition 2 (Feasibility): *An invariant set $\mathcal{S}$ for a system* (1)-(2) *is* feasible *with respect to constraints* (3) *if*

$$Cx \in \mathcal{Y}, \quad \forall x \in \mathcal{S}. \tag{6}$$

It is clear that if a set is invariant and feasible, all initial states x_0 within that set guarantee that all corresponding future outputs will stay within the imposed constraint set $\mathcal{Y}$.

The problem tackled in this paper is the following :

Problem 1 (P1): *Given a system* (1)-(2) *satisfying* (4)*, state constraints* (3)*, find a feasible and robust positive invariant set $\mathcal{S}$ of polyhedral form*

$$\mathcal{S} = \{x \in \mathbb{R}^{n_x} | A_{\mathcal{S}} x \leq b_{\mathcal{S}}\}. \tag{7}$$

In the following sections we refer to this problem as **P1**.

Remark 1: Strictly speaking, polyhedrons are 3-dimensional polytopes and hence both terms cannot be interchanged. However, due to the broad use of the term *polyhedral invariant sets*, we will use it throughout this paper to denote polytopic invariant sets of arbitrary dimension.

III. THEORETICAL APPROACH

The following theorem, though not practically executable, directly provides a solution to **P1**.

Theorem 1: *The set $\mathcal{S}$ defined as $\mathcal{S} = \bigcap_{i=0}^{\infty} \mathcal{S}_i$ with $\mathcal{S}_i = \{x | A_{\mathcal{S}_i} x \leq b_{\mathcal{S}_i}\}$, where*

$$A_{\mathcal{S}_0} = A_y C, \qquad b_{\mathcal{S}_0} = b_y, \tag{8}$$

and for $i = 1, \ldots, \infty$

$$A_{\mathcal{S}_i} = \begin{bmatrix} A_{\mathcal{S}_{i-1}} \Phi_1 \\ \vdots \\ A_{\mathcal{S}_{i-1}} \Phi_L \end{bmatrix}, \qquad b_{\mathcal{S}_i} = \begin{bmatrix} b_{\mathcal{S}_{i-1}} \\ \vdots \\ b_{\mathcal{S}_{i-1}} \end{bmatrix}, \tag{9}$$

is a valid solution to ***P1****.*

Proof: It follows from the definition of $\mathcal{S}_0$ and the fact that by definition $\mathcal{S} \subset \mathcal{S}_0$, that $Cx \in \mathcal{Y}, \forall x \in \mathcal{S}$, which shows that $\mathcal{S}$ is *feasible*. In a next step we prove that if $x_0 \in \mathcal{S}$ that $y_k \in \mathcal{Y}$ for $k = 1, \ldots, \infty$. This can be seen by observing that $x_0 \in \mathcal{S}_k$ and by then making a convex combination of the L matrix blocks in $A_{\mathcal{S}_k}$ and $b_{\mathcal{S}_k}$ with $\lambda_i(k-1), i = 1, \ldots, L$ as weights, with $\lambda_i(k)$ defined as in $\Phi(k) = \sum_{i=1}^{L} \lambda_i(k) \Phi_i$. By recursively making convex combinations of the L matrix blocks of the resulting two matrices A and b with weights $\lambda_i(k-2)$ down to $\lambda_i(0)$, we eventually get the set of inequalities

$$A_{\mathcal{S}_0} \Phi(k-1) \Phi(k-2) \ldots \Phi(1) \Phi(0) x_0 \leq b_{\mathcal{S}_0}. \tag{10}$$

This proves for $k = 1, \ldots, \infty$ that $x_k \in \mathcal{S}_0$ and hence that $y_k \in \mathcal{Y}$ if $x_0 \in \mathcal{S} \subset \mathcal{S}_k$. We now prove that $x_1 \in \mathcal{S}, \forall x_0 \in \mathcal{S}$. Assume that $x_0 \in \mathcal{S}$ and that $x_1 \notin \mathcal{S}$. This would mean that there exists a $k \geq 0$ for which $A_{\mathcal{S}_k}$ and $b_{\mathcal{S}_k}$ contain a row defining a linear inequality that is violated by x_1, which would in turn mean that for certain values of $\Phi(i) \in \Omega', i = 1, \ldots, k$ the corresponding output y_{k+1} would violate the output constraint. This contradicts the fact that was previously proven that all future outputs satisfy the output constraints. We therefore must conclude that also $x_1 \in \mathcal{S}$. This proves *robust positive invariance* of $\mathcal{S}$, which proves that $\mathcal{S}$ is a valid solution to **P1**. ■

Remark 2: By means of a similar argumentation it is possible to prove that the above set $\mathcal{S}$ is the largest possible feasible and robust positive invariant set for the given system and constraints. Indeed, one can see that any state outside $\mathcal{S}$ leads to a future output that violates the output constraints for at least one realization of the uncertainty, which then means that any other feasible robust positive invariant set $\mathcal{S}'$ cannot contain any states outside $\mathcal{S}$ and that therefore $\mathcal{S}' \subset \mathcal{S}$.

Remark 3: Although Theorem 1 constructs $\mathcal{S}$ using linear inequalities it is not guaranteed that the set $\mathcal{S}$ is polyhedral, since an infinite number of constraints is used. Only when a finite number of constraints is sufficient, $\mathcal{S}$ will be polyhedral. The following theorem shows when $\mathcal{S}$ can be described by a finite number of constraints.

Theorem 2: *Considering the following definitions*

$$a = \|A_{\mathcal{S}_0}\|, \tag{11a}$$

$$b_{\min} = \min_i b_{\mathcal{S}_0}(i), \tag{11b}$$

$$\phi_{\max} = \max_i \|\Phi_i\|, \tag{11c}$$

$$c = \max_x \|x\| \quad \text{s.t.} \quad x \in \mathcal{S}_0, \tag{11d}$$

with $b_{\mathcal{S}_0}(i)$ denoting the i-th element of vector $b_{\mathcal{S}_0}$ and assuming $\phi_{\max} < 1$, then the sets $\mathcal{S}$ (as in Theorem 1) and $\overline{\mathcal{S}}_n \triangleq \bigcap_{i=0}^{n} \mathcal{S}_i$, with $\mathcal{S}_i$ also defined as in Theorem 1 and n defined as

$$n = \left\lfloor \frac{\ln b_{\min} - \ln a - \ln c}{\ln \phi_{\max}} \right\rfloor, \tag{12}$$

are identical.

Proof: We prove that $\mathcal{S}_0 \subset \mathcal{S}_k, \forall k > n$, which then proves the theorem. Therefore we assume that $x_0 \in \mathcal{S} \subset \mathcal{S}_0$ and then calculate an upper bound to $\|A_{\mathcal{S}_k} x_0\|$ as a function of k :

$$\|A_{\mathcal{S}_k} x_0\| \leq \max_{i_0 \ldots i_{k-1}} \|A_{\mathcal{S}_0} \Phi_{i_{k-1}} \ldots \Phi_{i_0} x_0\| \tag{13a}$$

$$\leq \max_{i_0 \ldots i_{k-1}} \left(\|A_{\mathcal{S}_0}\| \cdot \|\Phi_{i_{k-1}}\| \cdot \ldots \cdot \|\Phi_{i_0}\| \cdot \|x_0\|\right) \tag{13b}$$

$$\leq a \phi_{\max}^k c. \tag{13c}$$

The largest element of $A_{\mathcal{S}_k} x_0$ is therefore bounded above by $a\phi_{\max}^k c$. The smallest element of $b_{\mathcal{S}_k}$ is the same as the smallest element of $b_{\mathcal{S}_0}$ and is therefore equal to $b_{\min}$. Hence if k satisfies the following condition, it is guaranteed that all inequalities of $\mathcal{S}_k$ are satisfied if $x_0 \in \mathcal{S}_0$:

$$\phi_{\max}^k \leq \frac{b_{\min}}{a\ c}, \tag{14}$$

which is equivalent with

$$k \geq \frac{\ln b_{\min} - \ln a - \ln c}{\ln \phi_{\max}}. \tag{15}$$

The inversion of the inequality is necessary since $\phi_{\max} < 1$ and therefore $\ln \phi_{\max} < 0$. It is clear that, because $k \in \mathbb{N}$, (15) is satisfied if $k > n$, which proves the theorem. ∎

Remark 4: Theorem 2 shows that if the closed loop system satisfies a certain convergence condition then set $\mathcal{S}$ can be constructed with a finite number of constraints, which then guarantees that $\mathcal{S}$ is polyhedral. Furthermore, (12) shows that n increases proportional to $\frac{1}{1-\phi_{\max}}$ for values of $\phi_{\max}$ close to 1.

Remark 5: For systems with dynamics such that the eigenvalues of the different Φ_i lie strictly within the unit circle, but where $\phi_{\max} \geq 1$, an appropriate state transformation can also enable the use of Theorem 2 to calculate n. It can easily be verified that when the closed loop system is quadratically stable (i.e. has a quadratic Lyapunov function) an ellipsoidal invariant set $\mathcal{E} = \{x | x^{\mathrm{T}} Z^{-1} x \leq 1\}$ can be found and that the transformation $x' = Z^{-\frac{1}{2}} x$ enables the use of Theorem 2. This observation essentially indicates that $\mathcal{S}$ can be described by a finite number of constraints if the system (1)-(2) is quadratically stable.

Remark 6: Theorem 2 also provides a method to show that if $\phi_{\max} < 1$, the resulting set $\mathcal{S}$ is non-empty. It can easily be found that, under the same assumptions of Theorem 1, all states x with $\|x\| \leq b_{\min}/a$ will satisfy all constraints of $\mathcal{S}$, which proves the existence of $\mathcal{S}$ if $\phi_{\max} < 1$. This also shows that $c \geq b_{\min}/a$ will also hold and that therefore the numerator of (12) will always be negative, leading to the observation that n will always be positive.

Although Theorem 2 provides a more practical way to calculate $\mathcal{S}$ by reducing the number of inequality constraints to a finite number, the method provided by this theorom can still become computationally intractable, even for relatively small values of n, because of the fact that the number of constraints increases exponentially with n. Therefore a more practical algorithm is provided in Section IV.

IV. PRACTICAL APPROACH

In this section we first reformulate **P1** into a different but equivalent problem **P2**, for which we then propose an efficient algorithm.

We first define the $^-$-operator :

$$\mathcal{S}^- = \{x | \Phi x \in \mathcal{S}, \forall \Phi \in \Omega'\}. \tag{16}$$

$\mathcal{S}^-$ can be interpreted as the set of all *previous states* for which it is guaranteed that the *current state* lies inside $\mathcal{S}$. This now enables us to formulate a necessary and sufficient condition for positive robust invariance for a set.

Lemma 1: *A set $\mathcal{S}$ is a robust positive invariant set for the system* (1) *iff*

$$\mathcal{S} \subset \mathcal{S}^-. \tag{17}$$

Proof: If (17) is satisfied then if $x_0 \in \mathcal{S}$, also $x_0 \in \mathcal{S}^-$ and therefore also $x_1 \in \mathcal{S}$, which proves that (17) is a *sufficient* condition for robust positive invariance. On the other hand, if there exists a state $x \in (\mathcal{S} \setminus \mathcal{S}^-)$ then there exists $\Phi \in \Omega'$ such that $\Phi x \notin \mathcal{S}$, which proves that (17) is also a *necessary* condition. ∎

Remark 7: Lemma 1 is a generalisation of the invariance condition proven in [3, p. 1010, Theorem 2.2] for the case $L = 1$, stating that $\overline{\mathcal{S}}_n$ is invariant if

$$\overline{\mathcal{S}}_n = \overline{\mathcal{S}}_{n+1}. \tag{18}$$

This can be seen by observing that $\overline{\mathcal{S}}_{n+1} \equiv \overline{\mathcal{S}}_n \cap \mathcal{S}_{n+1} = \overline{\mathcal{S}}_n \cap \overline{\mathcal{S}}_n^-$ and by then rewriting (18) as $\overline{\mathcal{S}}_n \subset \overline{\mathcal{S}}_n^-$, which is clearly a special case of (17), which does not impose a specific structure on $\mathcal{S}$.

Lemma 1 enables us to reformulate problem **P1** into the following problem.

Problem 2 (P2): *Given a system* (1)-(2) *satisfying* (4) *and given the constraints* (3)*, find matrices $A_{\mathcal{S}}$ and $b_{\mathcal{S}}$ such that the set $\mathcal{S} = \{x \in \mathbb{R}^{n_x} | A_{\mathcal{S}} x \le b_{\mathcal{S}}\}$ satisfies*

$$\mathcal{S} \subset \mathcal{S}^- \equiv \{x \in \mathbb{R}^{n_x} | A_{\mathbb{S}^-} x \le b_{\mathbb{S}^-}\}, \tag{19a}$$

$$Cx \in \mathcal{Y}, \ \forall x \in \mathcal{S}, \tag{19b}$$

with $A_{\mathbb{S}^-} \equiv [A_{\mathcal{S}}\Phi_1; \ldots; A_{\mathcal{S}}\Phi_L]$ and $b_{\mathbb{S}^-} \equiv [b_{\mathcal{S}}; \ldots; b_{\mathcal{S}}]$.

In the rest of this paper we refer to this problem as **P2**.

We can now formulate an algorithm for solving **P2** that starts with the set $\mathcal{S}_0$ and then iteratively adds constraints from $\mathcal{S}_1, \mathcal{S}_2, \ldots$ in order to satisfy (17).

Algorithm 1: Given a linear system (1)-(2) satisfying (4) and given the constraints (3).

1) Set the initial values for $A_{\mathcal{S}}$ and $b_{\mathcal{S}}$ to

$$A_{\mathcal{S}} := A_y C \qquad b_{\mathcal{S}} := b_y. \tag{20}$$

2) Initialize the index $i := 1$.
3) Perform the following steps iteratively while i is not strictly larger than the number of rows in $A_{\mathcal{S}}$:
   a) Select row i from $A_{\mathcal{S}}$ and $b_{\mathcal{S}}$:

   $$a = (A_{\mathcal{S}})_{(i,:)}, \qquad b = (b_{\mathcal{S}})_{(i,:)}. \tag{21}$$

   b) Check whether adding any of the constraints $a\Phi_i x \le b, \ i = 1, \ldots, L$ to $A_{\mathcal{S}}, b_{\mathcal{S}}$ would decrease the size of $\mathcal{S}$, by solving the following LP for $i = 1, \ldots, L$:

   $$c_i = \max_x \quad a\Phi_i x - b \tag{22a}$$

   $$\text{s.t.} \quad A_{\mathcal{S}} x \le b_{\mathcal{S}} \tag{22b}$$

   For each $i = 1, \ldots, L$, if $c_i > 0$, then add the constraint $a\Phi_i x \le b$ to $A_{\mathcal{S}}, b_{\mathcal{S}}$ as follows :

   $$A_{\mathcal{S}} := \begin{bmatrix} A_{\mathcal{S}} \\ a\Phi_i \end{bmatrix}, \qquad b_{\mathcal{S}} := \begin{bmatrix} b_{\mathcal{S}} \\ b \end{bmatrix}. \tag{23}$$

   c) Increment i :

   $$i := i + 1. \tag{24}$$

We now prove correctness and convergence of Algorithm 1.

Lemma 2 (Correctness): *If Algorithm 1 terminates in a finite number of iterations then the resulting matrices $A_{\mathcal{S}}, b_{\mathcal{S}}$ are a valid solution to **P2**.*

Proof: From the initialization step 1) and the fact that the algorithm only adds constraints and never removes constraints, it is clear that the resulting set $\mathcal{S}$ will satisfy (19b). Satisfaction of (19a) after termination of the algorithm follows directly from the observation that after step 3b) row i of $A_{\mathcal{S}}$ and $b_{\mathcal{S}}$ (denoted with a and b) satisfies the property $\{x | A_{\mathcal{S}} x \le b_{\mathcal{S}}\} \subset \{x | a\Phi_i x \le b, i = 1, \ldots, L\}$. Since constraints are only added to $A_{\mathcal{S}}, b_{\mathcal{S}}$ and never removed, it is clear that this property will also still hold for the first $i-1$ rows. Hence, after termination of the algorithm and due to the termination condition in step 3), this property will hold for all the rows of $A_{\mathcal{S}}$ and $b_{\mathcal{S}}$, which is identical to satisfaction of (19a) and concludes the proof. ■

F	Alg. 1a n_c	Alg. 1a T (sec.)	Alg. 1b n_c	Alg. 1b T (sec.)	Alg. 1c n_c	Alg. 1c T (sec.)
$[-0.3 - 0.1]$	118	31.6	30	53.9	30	14.4
$[-0.5 - 0.3]$	44	5.5	14	8.8	14	5.9

TABLE I

NUMBER OF CONSTRAINTS AND CALCULATION TIMES FOR THE INVARIANT SETS DEPICTED IN FIGURES 1 AND 2 FOR THREE DIFFERENT VARIANTS OF ALGORITHM 1 (1A : ALGORITHM 1 WITHOUT ADDITIONAL GARBAGE COLLECTION, 1B : ALGORITHM 1A WITH ADDITIONAL GARBAGE COLLECTION AFTER TERMINATION, 1C : ALGORITHM 1B WITH ADDITIONAL GARBAGE COLLECTION AFTER EVERY 10 ITERATIONS).

Remark 8: Correctness of Algorithm 1 can also be proven in alternative ways, for example by considering the operator $\overline{\mathcal{S}} = \mathcal{S}^- \cap \mathcal{S}$ and then showing that the algorithm converges to $\mathcal{S} = \overline{\mathcal{S}}$, but this is left up to the reader.

Lemma 3 (Convergence): *Under the same conditions as Theorem 2, Algorithm 1 will terminate in a finite number of iterations.*

Proof: Theorem 2 states that $\mathcal{S}$ from Theorem 1 and $\overline{\mathcal{S}}_n \triangleq \bigcap_{i=0}^{n} \mathcal{S}_i$ are identical and therefore, by virtue of Lemma 1, $\overline{\mathcal{S}}$ also satisfies (17). Since step 3b) of Algorithm 1 only adds constraints also found in $A_{\mathcal{S}_i}, b_{\mathcal{S}_i}, i = 1, \ldots, \infty$ and in the same order as they are found in these matrices for increasing i, Algorithm 1 will never add any constraints from $A_{\mathcal{S}_i}, b_{\mathcal{S}_i}, i = n{+}1, \ldots, \infty$ and therefore the maximum number of rows in $A_{\mathcal{S}}$ as constructed by Algorithm 1 is bounded by a finite number, namely the number of constraints in $A_{\mathcal{S}_i}, b_{\mathcal{S}_i}, i = 0, \ldots, n$. Since i is incremented in each iteration, Algorithm 1 must therefore also reach the termination condition of step 3) in a finite number of iterations, which proves the lemma. ■

Remark 9: After termination of Algorithm 1 it is advised to check whether any of the constraints in $A_{\mathcal{S}}, b_{\mathcal{S}}$ are redundant, meaning that they can be removed without increasing the size of $\mathcal{S}$. This can occur when constraints added in later iterations modify $\mathcal{S}$ in such a way that previously added constraints become irrelevant. A way of checking the redundance of a constraint is by solving an LP similar to (22). This process of '*garbage collection*' can also be incorporated in the algorithm itself in order to speed up the solution of (22) in each iteration. This does not invalidate the arguments used in the proofs of Lemma's 2 and 3 since $\mathcal{S}$ itself is not modified by the removal of the redundant constraints.

V. EXAMPLE

In this section, a numerical example is presented in order to show the validity of the theory and the effectiveness of the

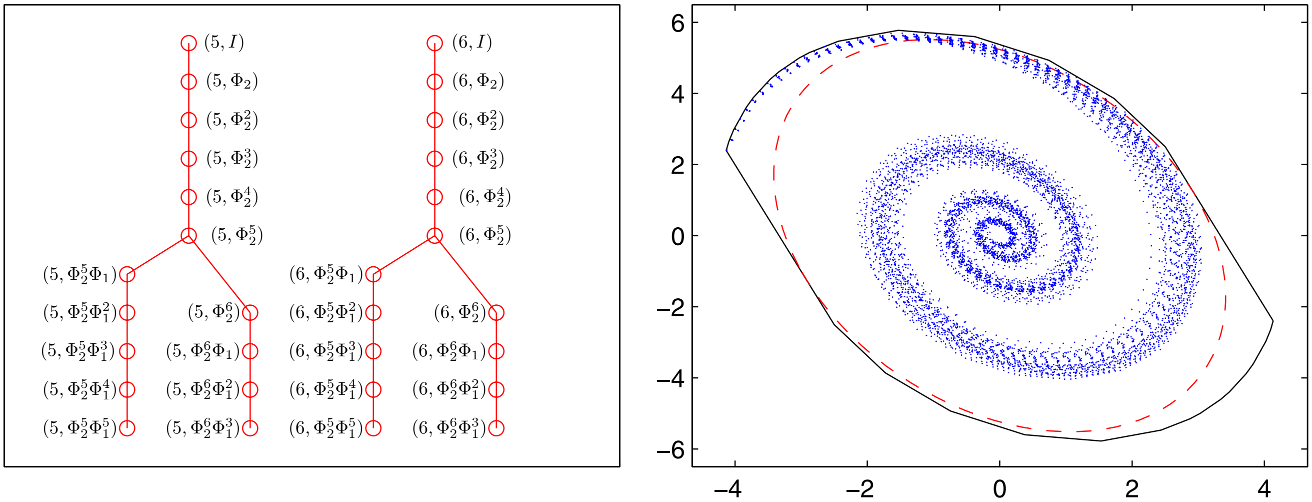

Fig. 1. Invariant polyhedral and ellipsoidal invariant set for the closed loop system formed by (25) and the feedback law $u = [-0.3 \ -0.1]x$. **Left :** Structure of the constraints defining the invariant set. The notation (i, M) denotes the constraint $a_i^{\mathrm{T}} Mx \leq b_i$, with a_i^{T}, b_i denoting the i-th rows of $A_{\mathcal{Y}}$ and $b_{\mathcal{Y}}$ respectively. **Right :** Shape of the invariant set. 50 state trajectories starting from the leftmost vertex of the polyhedral invariant set are depicted in dotted lines.

algorithm presented in the previous section. We consider a linear uncertain system representing a double integrator with an uncertainty polytope defined by the following two vertices :

$$A_1 = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{25a}$$

$$A_2 = \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 \\ 1.5 \end{bmatrix}. \tag{25b}$$

The system is subject to state and input constraints $[-10 \ -10]^{\mathrm{T}} \leq x_k \leq [10 \ 10]^{\mathrm{T}}$ and $-1 \leq u_k \leq 1, k = 0, \ldots, \infty$.

Figures 1 and 2 depict polyhedral invariant sets computed with Algorithm 1. Redundant constraints are not depicted. A comparison with the largest ellipsoidal invariant set is also made indicating a significantly larger area for the polyhedral invariant set, especially when a high feedback gain is used. To verify the invariance of the polyhedral sets, 50 trajectories are calculated, with the initial state situated at the leftmost vertex of the invariant set and with the system matrices $[A(k) \ B(k)]$ randomly chosen from $[A_1 \ B_1]$ and $[A_2 \ B_2]$ at each time instant. Both figures confirm that the polyhedral sets are indeed positively invariant.

The tree structures depicted in the figures indicate that not all possible predictions have to be included in the invariant sets. The maximum tree depths indicate that respectively 11-step and 7-step ahead predictions are needed to construct the invariant sets. However, constructing $\overline{\mathcal{S}}_{11} = \bigcap_{i=0}^{11} \mathcal{S}_i$ and $\overline{\mathcal{S}}_7 = \bigcap_{i=0}^{7} \mathcal{S}_i$ (cfr. Theorem 1) would take $6(2^{12} - 1) = 24570$ and $6(2^8 - 1) = 1530$ constraints, out of which only respectively 30 and 14 constraints are considered to be non-redundant according to Algorithm 1, as can be seen in the figures.

Table I shows calculation times for three variants of Algorithm 1. Garbage collection during *and* after the algorithm seems to be the best overall method in terms of calculation time. A significant decrease in calculation time is obtained for the set depicted in Figure 1, while only a small penalty in calculation time is observed for the set depicted in Figure 2. The mentioned calculation times are obtained on a P4-2GHz PC using Matlab 6.5.

VI. CONCLUSION

In this paper the construction of polyhedral robust positive invariant sets for linear systems with polytopic model uncertainty subject to linear constraints is explored. A theoretical approach is initially pursued after which a new invariance condition is proposed leading to a new efficient algorithm for the construction of the invariant set. The resulting set is shown to consist of a finite number of constraints if the system is quadratically stable and to be the maximal admissable set for the system.

The resulting sets are shown to be larger than ellipsoidal invariant sets, especially if the invariant set can be represented with a small number of constraints. Additionally the elimination of redundant constraints (*garbage collection*) in the set description during and after the construction of the invariant set is shown to significantly improve the computation speed.

Another advantage that is worth mentioning is the fact that polyhedral invariant set can easily deal with non-symmetric constraints, whereas ellipsoidal sets can only deal with suc constraints in a very conservative way. It is important to note that quadratic Lyapunov functions as in [5] can still be used within the polyhedrons constructed in this paper.

Due to the garbage collection and due to the fact that the algorithm is based on a more general invariance condition, it is also expected to have a lower computation time for

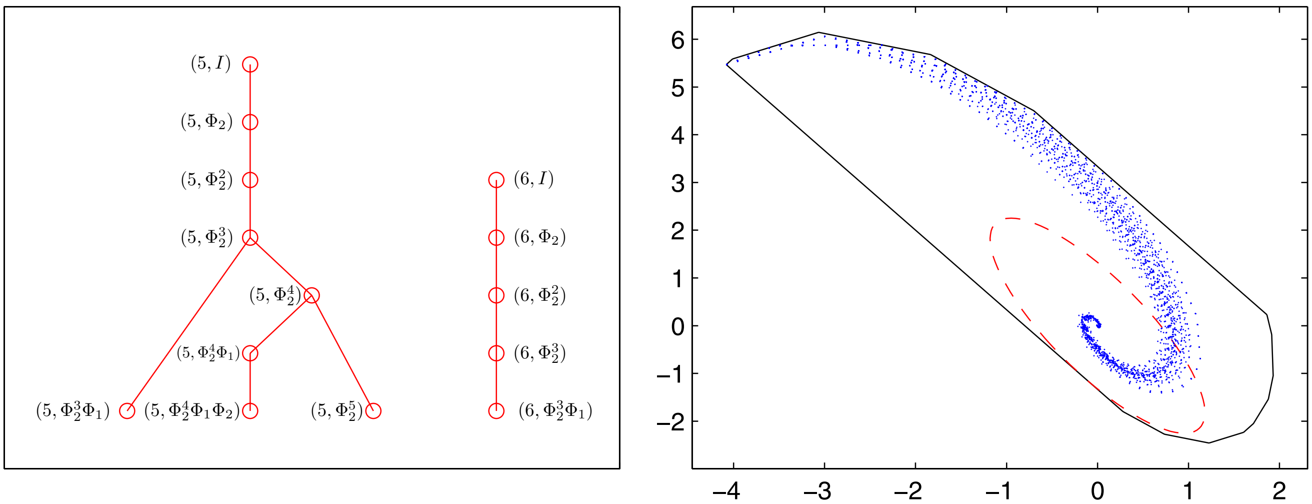

Fig. 2. Invariant polyhedral and ellipsoidal invariant set for the closed loop system formed by (25) and the feedback law $u = [-0.5 \;\; -0.3]x$. Input constraints were changed into $-0.4 \leq u \leq 1$. **Left :** Structure of the constraints defining the invariant set. The notation (i, M) denotes the constraint $a_i^{\mathrm{T}} Mx \leq b_i$, with a_i^{T}, b_i denoting the i-th rows of $A_{\mathcal{Y}}$ and $b_{\mathcal{Y}}$ respectively. **Right :** Shape of the invariant set. 50 state trajectories starting from the leftmost vertex of the polyhedral invariant set are depicted in dotted lines.

systems without uncertainty ($L = 1$) compared to the algorithm described in [3, p. 1011, Algorithm 3.2].

VII. Future Work

The results presented in this paper can be seen as an enabling technology for several future applications.

One possible future research direction is the use of robust invariant polyhedral sets in Modelbased Predictive Control, where invariant sets are generally used as a terminal state constraint in order to ensure stability and avoid infeasibilities in future time steps. Polyhedral invariant sets have the advantage of being larger that ellipsoidal invariant sets, can be non-symmetrical and that they can be imposed on a terminal state by means of linear inequality constraints instead of quadratic constraints.

Another possible research direction is the further reduction of the number of constraints at the cost of the volume of the invariant set.

Other interesting future research directions are the construction of Lyapunov functions induced by polyhedral robust invariant sets or robust controller synthesis based on polyhedral invariant sets, similar to [5] where ellipsoidal invariant sets are used.

Finally, inclusion of robustness with respect to disturbance inputs is also an interesting future research area.

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