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On the point-to-point and traveling salesperson problems for Dubins' vehicle

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Abstract— **In this paper we study the length of optimal paths for Dubins' vehicle, i.e., a vehicle constrained to move forward along paths of bounded curvature. First, we obtain an upper bound on the optimal length in the point-to-point problem. Next, we consider the corresponding Traveling Salesperson Problem (TSP). We provide an algorithm with worst-case performance within a constant factor approximation of the optimum. We also establish an asymptotic bound on the worst-case length of the Dubins' TSP.**

I. INTRODUCTION

The Traveling Salesperson Problem (TSP) with its variations is one of the most widely known combinatorial optimization problems. While extensively studied in the literature, these problems continue to attract great interest from a wide range of fields, including Operations Research, Mathematics and Computer Science. The Euclidean TSP (ETSP) [1], [2] is formulated as follows: given a point set Λ in $\mathbb{R}^2$, find the minimum-length tour of Λ. Exact algorithms, heuristics as well as constant factor approximation algorithms with polynomial time requirements are available for the Euclidean TSP, see [3], [4], [5]. Another interesting geometric version of the TSP is studied in [6].

The focus of this paper is the TSP for Dubins' vehicle; we shall refer to it as DTSP. Dubins' vehicle is a classic basic model for mobile robots and aerial vehicles. We note here that though the DTSP has a clear geometric interpretation, it is impossible to formulate it as a finite dimensional combinatorial optimization problem, unlike the ETSP. A fairly complete picture is available for the minimum-time point-to-point path planning problem for Dubins' vehicle, see [7] and [8]. Bounded curvature paths in environments with obstacles are also widely studied, see [9] and references therein. Based on the algorithms for the ETSP and on the algorithm for the point-to-point problem for Dubins' vehicle, it is easy to devise heuristics for the DTSP. Here we want to establish some bounds on the DTSP in comparison with the ETSP and on the performance of a heuristic.

The motivation to study the DTSP arises in robotics and uninhabited aerial vehicles (UAVs) applications, see e.g., [10], [11]. In particular we envision applying our algorithm to the setting of a UAV monitoring a collection of spatially distributed targets. From a purely scientific viewpoint, it also appears to be of general interest to bring together the work on Dubins' vehicle and that on ETSP.

The DTSP is a static optimization problem. It is our contention that this problem is of interest from a control viewpoint. Indeed, although we do not do so here, we intend to use our algorithm for DTSP in a receding horizon scheme, where the aerial vehicle is required to visit a dynamically changing set of targets. Performance guarantees for the DTSP translate directly into robust control guarantees of the receding horizon scheme.

The main contributions of this paper are three. First, we propose an algorithm for the DTSP through a pointset Λ, called the Alternating Algorithm, based on the solution to the ETSP over Λ together with an alternating heuristic to assign target orientations at each target point. Second, as an intermediate step in the analysis of our algorithm, we provide an upper bound on the point-to-point minimum length of Dubins' optimal paths. Third and last, we obtain some worst-case bounds on the performance of the proposed Alternating Algorithm and on the solutions of the DTSP as compared to each other and the corresponding ETSP.

II. THE TSP FOR DUBINS' VEHICLE AND THE ALTERNATING ALGORITHM

A *Dubins' vehicle* is a planar vehicle that is constrained to move along paths of bounded curvature, without reversing direction. Accordingly, we define *feasible curve for Dubins' vehicle* or *Dubins' path*, as a curve $\gamma : [0, T] \to \mathbb{R}^2$ that is twice differentiable almost everywhere, and such that the magnitude of its curvature is bounded above by $1/r$, where $r > 0$ is the minimum turn radius. Let $l(\gamma) = \int_0^T \|\gamma'(t)\| dt$ be the length of the path γ. We represent the vehicle configuration by the triplet $(x, y, \psi) \in SE(2)$, where (x, y) are the Cartesian coordinates of a reference point on the longitudinal axis of the vehicle and the heading ψ is the angle formed by such axis with a fixed direction in the plane. Let (d, θ) be the polar coordinates of (x, y). We shall state this equivalence as $(x, y) \leftrightarrow (d, \theta)$.

Let Λ be a set of n points in a compact region $\mathcal{Q} \subset \mathbb{R}^2$ and Λ_n be the collection of all point sets $\Lambda \subset \mathcal{Q}$ with cardinality n. Let $\mathrm{ETSP}(\Lambda)$ denote the cost of the Euclidean TSP over Λ, i.e., the length of the shortest closed path through all points in Λ. Correspondingly, let $\mathrm{DTSP}(\Lambda, r)$ denote the cost of the Dubins' TSP over Λ,

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i.e., the length of the shortest closed Dubins' path through all points in Λ.

Since the optimal path between two configurations of a Dubins' vehicle has been completely characterized in [7], a solution for the DTSP consists of (i) determining the order in which the Dubins' vehicle visits the given set of points, and (ii) assigning headings for the Dubins' vehicle at the points. In the following, we will describe an algorithm that approximates the solution of the DTSP problem, with an additive guarantee on the cost penalty. The algorithm builds on the knowledge of the optimal solution of the ETSP for the same point set, and provides a sub-optimal DTSP tour.

Let $A = (a_1, \ldots, a_n)$ be an ordered set of points that is a permutation of Λ. Let $\Psi = \{\psi_1, \ldots, \psi_n\}$ be a set of headings of the Dubins' vehicle at the n points $a_1, \ldots, a_n$. Therefore the configuration of Dubins' vehicle at a_i is (x_i, y_i, ψ_i) where (x_i, y_i) are the coordinates of a_i, for $i = 1, \ldots, n$. The algorithm we propose, which we will call ALTERNATING ALGORITHM, works as follows. Compute an optimal ETSP tour of Λ and label the edges on the tour in order with consecutive integers. A DTSP tour can be constructed by retaining all odd-numbered edges (except the n^{th}), and replacing all even-numbered edges with minimum-length Dubins' paths preserving the point ordering. We illustrate the output of the ALTERNATING ALGORITHM in Figure 1. The algorithm can be formally stated as follows.

Name: ALTERNATING ALGORITHM
Goal: To determine an ordering A and a set of headings Ψ for the DTSP through Λ
Requires: An algorithm ETSP-ALGO to compute optimal ETSP ordering of a pointset

```
set A := ETSP-ALGO(Λ)
set ψ_1 := orientation of segment from a_1 to a_2
for i = 2 to n − 1 do
  if i is even then
    set ψ_i := ψ_{i−1}
  else
    set ψ_i := orientation of segment from a_i to a_{i+1}
  end if
end for
if n is even then
  set ψ_n := ψ_{n−1}
else
  set ψ_n := orientation of segment from a_n to a_1
end if
```

In due course of the paper, we will first obtain an upper bound on the length of point-to-point Dubins' path i.e., an upper bound on the length of the path that a Dubins' vehicle will have to travel while making a transition from any arbitrary initial configuration to any arbitrary final configuration. To this effect, we shall show that: for $(x, y) \in \mathbb{R}^2$, $(x, y) \leftrightarrow (d, \theta)$ and $\psi \in [0, 2\pi[$, there exists a constant, $\kappa < 2.658$ such that the length of the Dubins' path from a configuration of $(0, 0, 0)$ to a configuration of (d, θ, ψ) is always less than or equal to $d + \kappa\pi r$. Using this result, we shall show that the optimal cost for the TSP for Dubins' vehicle is bounded according to the relation:

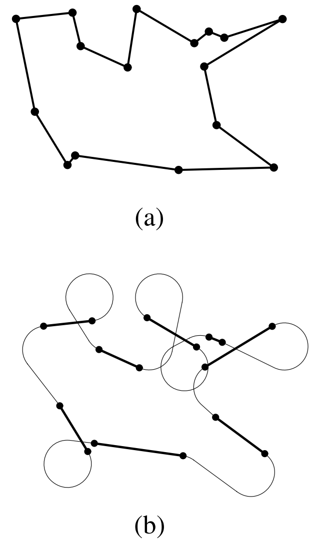

Fig. 1. An application of the ALTERNATING ALGORITHM: (a) A graph representing the solution of ETSP over a given Λ (b) A graph representing the solution given by the ALTERNATING ALGORITHM on Λ where the alternate segments of ETSP are retained

$$\mathrm{ETSP}(\Lambda) \leq \mathrm{DTSP}(\Lambda, r) \leq \mathrm{ETSP}(\Lambda) + \kappa\lceil n/2\rceil\pi r.$$

Following this, among other results, we shall establish some measure on the worst case performance of the ALTERNATING ALGORITHM as compared to DTSP. Let $\mathrm{L_{AA}}(\Lambda, r)$ be the length of the closed path over Λ as given by the ALTERNATING ALGORITHM. Formally, we shall show that: as $n \to +\infty$, then

$$\sup_{\Lambda\in\Lambda_n} \mathrm{DTSP}(\Lambda, r) \leq \sup_{\Lambda\in\Lambda_n} \mathrm{L_{AA}}(\Lambda, r) \leq \frac{\kappa}{2} \sup_{\Lambda\in\Lambda_n} \mathrm{DTSP}(\Lambda, r).$$

III. ON THE OPTIMAL POINT-TO-POINT LENGTH FOR DUBINS' VEHICLE

In order to obtain an upper bound on the length of Dubins' vehicle while executing the ALTERNATING ALGORITHM, we first obtain an upper bound on the length of the optimal path that a Dubins' vehicle has to travel while making transition from any arbitrary initial configuration, $(x_{\text{initial}}, y_{\text{initial}}, \psi_{\text{initial}})$ to any arbitrary final configuration, $(x_{\text{final}}, y_{\text{final}}, \psi_{\text{final}})$.

Let us now provide some useful preliminary definitions. Without loss of generality, we shall assume $(x_{\text{initial}}, y_{\text{initial}}, \psi_{\text{initial}}) = (0, 0, 0)$ and we let $(x_{\text{final}}, y_{\text{final}}) \leftrightarrow (d, \theta)$ and $\psi_{\text{final}} = \psi$. Let $\mathcal{C}_r : \overline{\mathbb{R}}_+ \times [0, 2\pi[\times [0, 2\pi] \to \overline{\mathbb{R}}_+$ associate to (d, θ, ψ), where $(x, y) \leftrightarrow, (d, \theta)$ the minimum length $\mathcal{C}_r(d, \theta, \psi)$ from the initial configuration $(0, 0, 0)$ to the final configuration (x, y, ψ) for a Dubins' vehicle. Let $F_0 :]0, \pi[\times]0, \pi[\to]0, \pi[$, $F_1 :]0, \pi[\to \mathbb{R}$ and $F_2 :]0, \pi] \to \mathbb{R}$ be defined by

$$F_0(\psi, \theta) = 2\tan^{-1}\Big(\frac{\sin(\psi/2) - 2\sin(\psi/2 - \theta)}{\cos(\psi/2) + 2\cos(\psi/2 - \theta)}\Big), \quad (1)$$

$$F_1(\psi) = \psi + \sin\Big(\frac{F_0(\psi, \psi/2 - \alpha(\psi))}{2}\Big) + 4\cos^{-1}\Big(\frac{\sin((\psi - F_0(\psi, \psi/2 - \alpha(\psi)))/2)}{2}\Big) \quad (2)$$

$$F_2(\psi) = 2\pi - \psi + 4\cos^{-1}\Big(\frac{\sin(\psi/2)}{2}\Big), \quad (3)$$

where $\alpha(\psi) = \pi/2 - \cos^{-1}(\frac{\sin(\psi/2)}{2})$. We are now ready to state the main result of this section.

Theorem 3.1 (Upper bound on optimal length): For $\psi \in [0, 2\pi[$, $(x, y) \in \mathbb{R}^2$, $(x, y) \leftrightarrow (d, \theta)$ and $r > 0$,

$$\mathcal{C}_r(d, \theta, \psi) \leq d + \kappa\pi r, \quad (4)$$

where $\kappa \in [2.657, 2.658]$ is defined by

$$\kappa = \frac{1}{\pi}\max\{F_2(\pi), \sup_{\psi\in]0,\pi[}\min\{F_1(\psi), F_2(\psi)\}\}.$$

A. Dubins' classification of optimal curves

Following [7], the minimum length feasible curve for Dubins' vehicle is either (i) an arc of a circle of radius r, followed by a line segment, followed by an arc of a circle of radius r, or (ii) a sequence of three arcs of circles of radius r, or (iii) a subpath of a path of path type (i) or (ii). To specify the *type* of these minimum length feasible curves for Dubins' path we follow the notations used in [8]. Three elementary motions are considered: turning to the left, turning to the right (both along a circle of radius r), and straight line motion S. Three operators are introduced: L_v (for left/counterclockwise turn of length $v > 0$), R_v (for right/clockwise turn of length $v > 0$), S_v (for straight motion of length $v > 0$). The operators L_v, R_v, and S_v, transform an arbitrary configuration $(x, y, \psi) \in SE(2)$ into its corresponding image point in $SE(2)$ by

$$(x + \sin(\psi + v) - \sin\psi, y - \cos(\psi + v) + \cos\psi, \psi + v),$$
$$(x - \sin(\psi - v) + \sin\psi, y + \cos(\psi - v) - \cos\psi, \psi - v),$$
$$(x + v\cos\psi, y + v\sin\psi, \psi),$$

respectively. Thus, the *Dubins' set* $\mathcal{D}$ which is the domain for the type of the minimum length feasible curve for a Dubins' vehicle between a given initial and final configuration is given by $\mathcal{D} = \{LSL, RSR, RSL, LSR, RLR, LRL\}$. One may refer to [7] for a detailed discussion on the construction of these path types between a given initial and final configuration. One may note that there are sets of initial and final configurations for which all the path types may not be feasible between those configurations.

In the remaining part of the paper we will need to frequently use the curves of type LRL and RLR starting with the initial configuration $(0, 0, 0)$ and the final configuration $(0, 0, \psi)$. We introduce additional notation to facilitate the presentation. For $\psi \neq 0$, let $C_{p_1}(\psi)$ be a circle with center $O_{C_{p_1}} \equiv (0, r)$ and radius r, and let $C_{p_2}(\psi)$ be a circle with center $O_{C_{p_2}} \equiv (-r\sin\psi, r\cos\psi)$ and radius r. Note that $\psi \neq 0$ implies that $C_{p_1}(\psi) \cap C_{p_2}(\psi)$ is either a single point or two points. Then let $C_{m_1}(\psi)$ and $C_{m_2}(\psi)$ be two circles with radius r that are tangent to both $C_{p_1}(\psi)$ and $C_{p_2}(\psi)$, see Figure 2 and Figure 3. By construction,

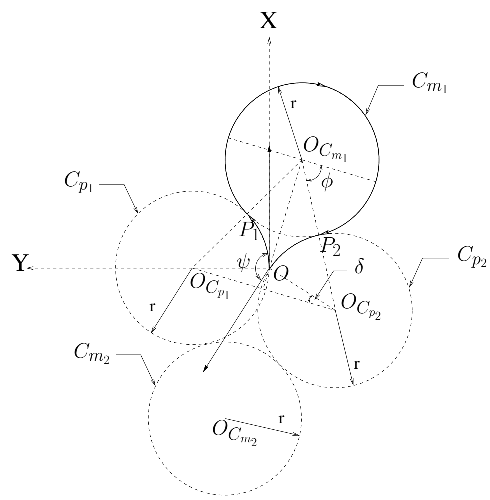

Fig. 2. LRL curves returning to the origin for the case when $\psi \in [0, \pi]$.

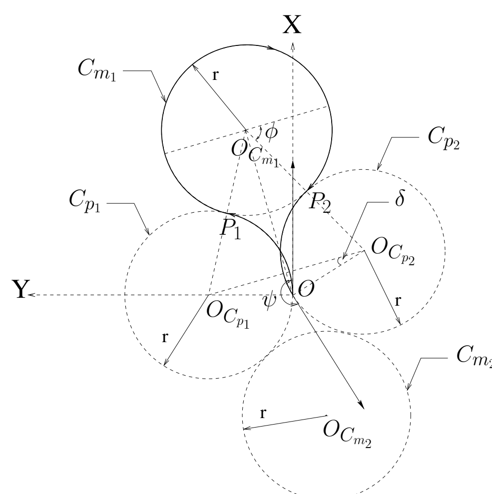

Fig. 3. LRL curves returning to the origin for the case when $\psi \in]\pi, 2\pi[$.

$C_{p_1}(\psi)$ intersects $C_{m_1}(\psi)$ and $C_{m_2}(\psi)$ at one point each: let $P_1(\psi)$ be the first of these two points that is reached moving left from the origin O along $C_{p_1}(\psi)$. Without loss of generality, assume $P_1(\psi) \in C_{m_1}(\psi)$. Let $O_{C_{m_1}}$ be the center of C_{m_1}. Let $P_2(\psi) = C_{m_1}(\psi) \cap C_{p_2}(\psi)$. In order to remove ambiguity, we shall pick that heading of the tangent line to a circle at a given point which is consistent with the orientation of that circle to be *the* orientation of the tangent to that circle at that point. Let the orientation of Dubins' vehicle at P_1 be along the orientation of the tangent to C_{p_1} at P_1. Similarly, let the orientation of Dubins' vehicle at P_2 be along the orientation of the tangent to C_{p_2} at P_2. Let the configuration of Dubins' vehicle at P_1 and P_2 be denoted by $J_{p_1}, J_{p_2} \in SE(2)$, respectively. Let t_1, t_2, t_3 be such that $L_{t_1}(0,0,0) = J_{p_1}, R_{t_2}(J_{p_1}) = J_{p_2}$ and $L_{t_3}(J_{p_2}) = (0,0,\psi)$. Let $LRL_O(\psi)$ and $RLR_O(\psi)$ be the minimum length curves of types LRL and RLR respectively from the configuration $(0,0,0)$ to the configuration $(0,0,\psi)$.

For $\psi \neq 0$, we define *forbidden cones* $V_1, V_2 : [0, 2\pi[\to \mathbb{R}^2$ to be the open, positive cones with symmetry axes $(d, \psi/2)_{d\in\mathbb{R}_+}$ and $(d, \pi + \psi/2)_{d\in\mathbb{R}_+}$, respectively, and half angle for both of them given by $\alpha(\psi) = \pi/2 - \cos^{-1}(\frac{\sin(\psi/2)}{2})$. Recall that, given a set Z, we let Z^c be its complement set. Hence $V_1^c(\psi) = \mathbb{R}^2 \setminus V_1(\psi)$ and $V_2^c(\psi) = \mathbb{R}^2 \setminus V_2(\psi)$.

B. Proof of Theorem 3.1

We begin with some preliminary results.

Lemma 3.2: (Length of LRL and RLR curves returning to the origin) Given $\psi \in]0, 2\pi[$ and $r > 0$, then

(i) $l(LRL_O(\psi)) = r\psi + 4r\cos^{-1}(\frac{\sin(\psi/2)}{2})$, and
(ii) $l(RLR_O(\psi)) = r(2\pi - \psi) + 4r\cos^{-1}(\frac{\sin(\psi/2)}{2}$.

Due to lack of space, instead of stating the whole proof, we refer the reader to Figs. 2 and 3.

Lemma 3.2 has the following direct consequence.

Lemma 3.3: (Upper bound on the length of minimal length curves returning to the origin) For all $\theta \in [0, 2\pi[$, $\psi \in [0, 2\pi[$ and $r > 0$

$$\mathcal{C}_r(0, \theta, \psi) \leq \mathcal{C}_r(0, \theta, \pi) = \frac{7}{3}\pi r.$$

Now we start to analyze the general case where $d \neq 0$.

Lemma 3.4: (Upper bound on optimal length via LRL_O and RLR_O) For $\psi \in]0, 2\pi[$, and $(x, y) \leftrightarrow (d, \theta)$, we have

(i) if $(x, y) \in V_1^c(\psi)$, then

$$\mathcal{C}_r(d, \theta, \psi) \leq d + l(LRL_O(\psi)),$$

(ii) if $(x, y) \in V_2^c(\psi)$, then

$$\mathcal{C}_r(d, \theta, \psi) \leq d + l(RLR_O(\psi)).$$

Proof: Let us prove part (i); part (ii) is proved by similar considerations. We recall the construction used for $LRL_O(\psi)$ curves. We define two additional circles $\overline{C}_{m_1}$ and $\overline{C}_{p_1}$ of radii r and whose respective centers $O_{\overline{C}_{m_1}}$ and $O_{\overline{C}_{p_2}}$ are given by

$$O_{\overline{C}_{m_1}} = O_{C_{m_1}} + (d\cos\theta, d\sin\theta),$$
$$O_{\overline{C}_{p_2}} = O_{C_{p_2}} + (d\cos\theta, d\sin\theta).$$

Let $\overline{C}_{m_1}$ be oriented clockwise and let $\overline{C}_{p_2}$ be oriented counter-clockwise. Then, there always exists an oriented segment, say M, tangent to C_{m_1} and $\overline{C}_{m_1}$ with the property that a Dubins' vehicle can make transition from C_{m_1} to $\overline{C}_{m_1}$ through M. Let $P_3 = M \cap C_{m_1}$, $\overline{P_3} = M \cap \overline{C}_{m_1}$, $\overline{P_2} = P_2 + (d\cos\theta, d\sin\theta)$ and $\overline{O} = O + (d\cos\theta, d\sin\theta)$. It is easy to see from the construction that, provided the point P_3 lies in the clockwise arc P_1P_2 along the circle C_{m_1}, the path consisting of (in order) OP_1 along C_{p_1}, P_1P_3 along C_{m_1}, $P_3\overline{P_3}$ along M, $\overline{P_3P_2}$ along $\overline{C}_{m_1}$, $\overline{P_2O}$ along $\overline{C}_{p_2}$ is a feasible curve for Dubins' vehicle from O to $\overline{O}$, see Figure 4. With a slight abuse of notation, we shall denote

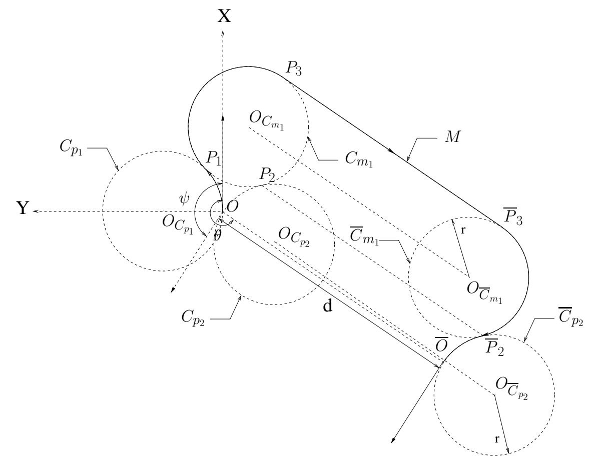

Fig. 4. A suboptimal path from $(0,0,0)$ to (d,θ,ψ), $(x,y) \leftrightarrow (d,\theta)$ for $(x,y) \in V_1^c(\psi)$.

this curve as $LRL_{\overline{O}}(d, \theta, \psi)$. The condition that P_3 lies along the arc P_1P_2 along the circle C_{m_1} holds true when the orientation of the segment $M = P_3\overline{P_3}$ does not lie between the orientations of the tangents to C_{m_1} at P_1 and P_2. In summary we have:

$$\begin{aligned}&\text{orientation of } M = \text{orientation of } P_3\overline{P_3} = \theta,\\ &\text{orientation of tangent to } C_{m_1} \text{ at } P_1\\ &\qquad = \psi/2 - \pi/2 + \cos^{-1}(\sin(\psi/2)/2),\\ &\text{orientation of tangent to } C_{m_1} \text{ at } P_2\\ &\qquad = \psi/2 + \pi/2 - \cos^{-1}(\sin(\psi/2)/2).\end{aligned}$$

Therefore the above condition is satisfied when $\theta \notin]\psi/2 - \pi/2 + \cos^{-1}(\sin(\psi/2)/2), \psi/2 + \pi/2 - \cos^{-1}(\sin(\psi/2)/2)[$. It follows from the definition of $V_1(\psi)$ that this is true *if and only if* $(x, y) \in V_1^c(\psi)$.

Because $LRL_{\overline{O}}(d, \theta, \psi)$ is a suboptimal path, for $\psi \in]0, 2\pi[$, $(x, y) \in V_1^c(\psi)$ and $(x, y) \leftrightarrow (d, \theta)$, we have

$$\mathcal{C}_r(d, \theta, \psi) \leq l(LRL_{\overline{O}}(d, \theta, \psi)). \tag{5}$$

From Figure 2 and Figure 4,

$$l(LRL_{\overline{O}}(d, \theta, \psi)) = d + l(LRL_O(\psi)). \tag{6}$$

Combining (5) and (6) we get the final result. ■

One can prove that for $d = 0$, the minimal length feasible curve for Dubins' vehicle is of type LRL or RLR. This, along with Lemma 3.2, leads us to our next lemma which we state without any proof.

Lemma 3.5 (Optimal path length returning to the origin): Let $d = 0$ and $\theta \in [0, 2\pi[$.

(i) if $\psi \in]0,\pi]$, then $LRL_O(\psi)$ is the optimal path and

$$\mathcal{C}_r(0,\theta,\psi) = r\psi + 4r\cos^{-1}\Big(\frac{\sin(\psi/2)}{2}\Big),$$

(ii) if $\psi \in]\pi, 2\pi[$, then $RLR_O(\psi)$ is the optimal path and

$$\mathcal{C}_r(0,\theta,\psi) = r(2\pi - \psi) + 4r\cos^{-1}\Big(\frac{\sin(\psi/2)}{2}\Big).$$

Let

$$U_1 = \bigcup_{\psi\in]0,\pi]} V_1^c(\psi), \qquad U_2 = \bigcup_{\psi\in]\pi,2\pi[} V_2^c(\psi).$$

Lemma 3.6 (Relation between $\mathcal{C}_r(d,\theta,\psi)$ and $\mathcal{C}_r(0,\theta,\psi)$): For $(x,y) \leftrightarrow (d,\theta)$ and $(x,y) \in U_1 \cup U_2$,

$$\mathcal{C}_r(d,\theta,\psi) \le d + \mathcal{C}_r(0,\theta,\psi),$$

and, therefore,

$$\mathcal{C}_r(d,\theta,\psi) \le d + \frac{7}{3}\pi r.$$

Proof: The proof follows from Lemma 3.4 and Lemma 3.5. The second statement is a consequence of Lemma 3.3. ■

It now remains to obtain a bound on $C_r(d,\theta,\psi)$ when $(x,y) \in V_1(\psi)$ or $(x,y) \in V_2(\psi)$ where $(x,y) \leftrightarrow (d,\theta)$. To this effect let the vehicle start moving at time $t = 0$ at unit speed along C_{p_1} in the counterclockwise direction and keep updating the parameters d,θ,ψ as if the coordinate system was moving along with the vehicle. Consequently $V_1(\psi)$ keeps shrinking and there is a time instant $t = t^*$ when the final configuration is such that $(x,y) \notin V_1(\psi)$. The following lemma and its proof contain the details of this constructions and its implications.

Lemma 3.7: For $\psi \in]0,\pi[$, $(x,y) \in V_1(\psi)$, $(x,y) \leftrightarrow (d,\theta)$ and $r > 0$,

$$\mathcal{C}_r(d,\theta,\psi) \le d + rF_1(\psi).$$

The proof of this result needs an additional geometric construction and we omit it here for lack of space.

From the definition it follows that for $(x,y) \ne (0,0)$, $(x,y) \in V_1(\psi) \Longrightarrow (x,y) \in V_2^c(\psi)$. This observation along with part (ii) of Lemma 3.4 and part (ii) of Lemma 3.2 leads us to our next lemma which we state without any proof.

Lemma 3.8: For $\psi \in]0,\pi]$, $(x,y) \in V_1(\psi)$, $(x,y) \leftrightarrow (d,\theta)$ and $r > 0$,

$$\mathcal{C}_r(d,\theta,\psi) \le d + rF_2(\psi).$$

Lemma 3.9: For $\psi \in]0,\pi[$, $(x,y) \in V_1(\psi)$, $(x,y) \leftrightarrow (d,\theta)$ and $r > 0$,

$$\mathcal{C}_r(d,\theta,\psi) \le d + r\min\{F_1(\psi), F_2(\psi)\}.$$

Therefore, for $\psi \in]0,\pi]$, $(x,y) \in V_1(\psi)$, $(x,y) \leftrightarrow (d,\theta)$ and $r > 0$,

$$\begin{aligned}\mathcal{C}_r(d,\theta,\psi) &\le d + r\max\{F_2(\pi), \sup_{\psi\in]0,\pi[} \min\{F_1(\psi), F_2(\psi)\}\} \\ &= d + \kappa\pi r.\end{aligned}$$

Proof: The first statement of the lemma follows from Lemma 3.7 and Lemma 3.8. This along with the consideration for the case of $\psi = \pi$ easily leads one to the second statement. ■

Similarly, one can prove that for $\psi \in]\pi, 2\pi[$, $(x,y) \in V_2(\psi)$, $(x,y) \leftrightarrow (d,\theta)$ and $r > 0$, $\mathcal{C}_r(d,\theta,\psi) \le d + \kappa r$. Combining this with Lemma 3.6 and the last statement of Lemma 3.9, we can state that for $\psi \in]0, 2\pi[$, $(x,y) \in \mathbb{R}^2$, $(x,y) \leftrightarrow (d,\theta)$ and $r > 0$

$$\mathcal{C}_r(d,\theta,\psi) \le d + \kappa r. \tag{7}$$

It now remains to prove a similar bound on $\mathcal{C}_r(d,\theta,0)$ for which we state the following lemma.

Lemma 3.10: For $(x,y) \in \mathbb{R}^2$, $(x,y) \leftrightarrow (d,\theta)$ and $r > 0$,

$$\mathcal{C}_r(d,\theta,0) \le d + 2\pi r.$$

The proof of this result requires the same setup as for the proof of Lemma 3.7 and we do not state it here for lack of space.

Lemma 3.10 combined with eqn. (7) gives the proof for Theorem 3.1. It is easy to check that for $\psi \in]0,\pi[$, $F_1(\psi)$ is a monotonically increasing function of ψ and $F_2(\psi)$ is a monotonically decreasing function of ψ. Therefore, there exists a unique ψ^* such that $F_1(\psi^*) = F_2(\psi^*)$. By numerical calculations one can find that $\kappa \simeq 2.6575$.

C. Numerical Results

The length of the optimal Dubins' path, $\mathcal{C}_r(d,\theta,\psi)$, was calculated for numerous sets of final configurations (d,θ,ψ) starting with an initial configuration of $(0,0,0)$ and a corresponding parameter k was evaluated for each of the instances according to the relation: $\mathcal{C}_r(d,\theta,\psi) = d + k\pi r$. The results suggest that the value of k is bounded by a quantity, say κ_{num} whose value is equal to $\frac{7}{3}$. Moreover, it appears that k achieves the value of κ_{num} only when the Dubins' vehicle makes a transition from a state of the form $(0,0,0)$ to a state of the form $(0,0,\pi)$ according to our setup. Hence, though we do not have an analytical proof to establish these empirical results exactly, our analysis gives a fairly good estimate of κ_{num}.

IV. ON THE TSP FOR DUBINS' VEHICLE

Once an upper bound is obtained on the length of the optimal point-to-point Dubins' path, this section now gives measure of performance of the ALTERNATING ALGORITHM and the optimal algorithm for DTSP. The aim of this section can be summarized through the following statement.

Problem 4.1: Given an upper bound on the length of the optimal point-to-point Dubins' path, find a measure of the general performance of DTSP and the worst case performance of the ALTERNATING ALGORITHM.

We now state the two important results of this section.

Theorem 4.2: (Bounds on the TSP for Dubins' vehicle) For any point set $\Lambda \in \Lambda_n$ with $n \ge 2$ and $r > 0$,

$$\mathrm{ETSP}(\Lambda) \le \mathrm{DTSP}(\Lambda, r) \le \mathrm{ETSP}(\Lambda) + \kappa\lceil n/2\rceil\pi r.$$

Furthermore, given $r > 0$, there exists a point set $\Lambda \in \Lambda_n$ such that

$$\mathrm{ETSP}(\Lambda)+2\lfloor\frac{n}{2}\rfloor\pi r \leq \mathrm{DTSP}(\Lambda, r) \leq \mathrm{ETSP}(\Lambda)+\kappa\lceil\frac{n}{2}\rceil\pi r.$$

Theorem 4.3: (Performance of the ALTERNATING ALGORITHM *in the worst case for Dubins' TSP)* For $n \geq 2$ and $r > 0$,

$$\begin{aligned}\sup_{\Lambda\in\Lambda_n} \mathrm{DTSP}(\Lambda, r) &\\ &\leq \sup_{\Lambda\in\Lambda_n} \mathrm{L}_{\mathrm{AA}}(\Lambda, r)\\ &\leq \frac{\mathrm{ETSP}(\Lambda)+\kappa\lceil n/2\rceil\pi r}{\mathrm{ETSP}(\Lambda)+2\lfloor n/2\rfloor\pi r} \sup_{\Lambda\in\Lambda_n} \mathrm{DTSP}(\Lambda, r).\end{aligned}$$

Furthermore, as $n \to +\infty$,

$$\begin{aligned}\sup_{\Lambda\in\Lambda_n} \mathrm{DTSP}(\Lambda, r) &\\ &\leq \sup_{\Lambda\in\Lambda_n} \mathrm{L}_{\mathrm{AA}}(\Lambda, r)\\ &\leq \frac{\kappa}{2} \sup_{\Lambda\in\Lambda_n} \mathrm{DTSP}(\Lambda, r).\end{aligned}$$

To prove these results, we begin with some preliminaries. It is fairly easy to see that $\mathrm{L}_{\mathrm{AA}}(\Lambda, r) \geq \mathrm{ETSP}(\Lambda)$. An immediate consequence of this observation and Theorem 3.1 when applied to the ALTERNATING ALGORITHM is the following lemma which we state without any proof.

Lemma 4.4: (Bounds on the performance of the ALTERNATING ALGORITHM *as compared to ETSP)* For any point set $\Lambda \in \Lambda_n$ and $r > 0$,

$$\mathrm{ETSP}(\Lambda) \leq \mathrm{L}_{\mathrm{AA}}(\Lambda, r) \leq \mathrm{ETSP}(\Lambda) + \kappa\lceil n/2\rceil\pi r.$$

Next, without giving a proof due to lack of space, we provide a worst-case lower bound on DTSP.

Lemma 4.5: (Worst-case lower bound on DTSP) Given $r > 0$, there exists a point set $\Lambda \in \Lambda_n$ such that

$$\mathrm{DTSP}(\Lambda, r) \geq \mathrm{ETSP}(\Lambda) + 2\lfloor\frac{n}{2}\rfloor\pi r.$$

The first statement in Theorem 4.2 follows from the facts that $\mathrm{DTSP}(\Lambda, r) \geq \mathrm{ETSP}(\Lambda)$ and $\mathrm{L}_{\mathrm{AA}}(\Lambda, r) \geq \mathrm{DTSP}(\Lambda, r)$ and Lemma 4.4. The second statement follows from Lemma 4.5. Similarly, the first statement in Theorem 4.3 follows from the simple fact that $\mathrm{L}_{\mathrm{AA}}(\Lambda, r) \geq \mathrm{DTSP}(\Lambda, r)$, Lemma 4.4 and Lemma 4.5. To prove the second part of Theorem 4.3, we state a result from [12]: for a set Λ of n points in the compact set $\mathcal{Q}$, there exists a finite constant $\beta(\mathcal{Q})$ such that

$$\mathrm{ETSP}(\Lambda) \leq \beta(\mathcal{Q})\sqrt{n}.$$

Taking the limit as $n \to +\infty$ in the first part of Theorem 4.3 and using the result stated above we prove the second part of Theorem 4.3.

Using the result in Theorem 4.2, one can prove that given a pointset, for small enough r, the order of points in the optimal path for the Euclidean TSP is same as the order of points in the optimal path for the TSP for Dubins' vehicle.

V. CONCLUSIONS

There exist results in literature which state that for a given compact set and a pointset Λ of n points, $\mathrm{ETSP}(\Lambda)$ belongs to $O(\sqrt{n})$. In this paper, we characterized the worst-case solutions to the point-to-point and to the traveling salesperson problem for Dubins' vehicle where we showed that in worst case, for any $\rho > 0$, $\mathrm{DTSP}(\Lambda, \rho)$ belongs to $O(n)$. We provide some results on the stochastic analysis of TSP for Dubins' vehicle in [13]. Open directions of research include (i) tightening the bounds we provided, and (ii) applying these results to task assignments and surveillance problems, see [14].

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