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From Input Shaping® and OATF to Vibration Suppression Shape Filter

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Abstract— **A control profile is generated which suppresses one or more resonant dynamics in a flexible dynamic system. This control profile can be used as a velocity profile, or as a shape filter to an arbitrary control command. The robustness can be arbitrarily improved, which brings about a smoother profile. The technique can be applied to both open-loop and closed-loop systems.**

I. Introduction

Control of flexible structures has been extensively studied in recent years. Flexible structures such as high-speed disk drive actuators require extremely precise positioning under very tight time constraints. Whenever a fast motion is commanded, residual vibration in the flexible structure is induced, which increases the settling time. One solution is to design a closed-loop controller to damp out vibrations caused by the command inputs and disturbances to the plant. However, the resulting closed-loop response may still be too slow to provide an acceptable settling time, and the closed-loop control is not able to compensate for high frequency residual vibration which occurs beyond the closed-loop bandwidth. An alternative approach is to develop an appropriate reference trajectory that is able to minimize the excitation energy imparted to the system at its natural frequencies.

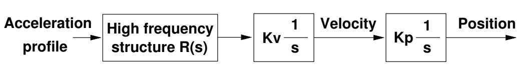

Fig. 1. A typical mechanical flexible system.

Fig. 1 shows a typical mechanical flexible system, where $\frac{1}{s}$ is an integrator, K_v is a velocity constant gain, and K_p is a position constant gain. The high frequency modes can be described as a transfer function $R(s) = \lim_{n\to\infty} \frac{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + 1}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + 1}$ in which an infinite number of lightly damped resonant structures is possible. The goal of vibration suppression trajectory generation is to find a fast input trajectory, under some physical constraint, with minimum possible residual vibration. The position reference input can be generated from a step movement command $s(t) = S\cdot 1(t)$, through a finite support filter, $f(t)$, $0 \le t \le T$, where T is the time duration of the finite support filter. To guarantee that the filtered command reaches the same set point as the step movement command, the integral of $f(t)$ must be equal to 1, i.e., $\int_0^T f(t)dt = 1$. This finite support filter $f(t)$, $0 \le t \le T$, which generates a vibration suppression position reference profile is called a vibration suppression shape filter, or simply a shape filter. In the discrete-time case, if the finite impulse response shape filter is $f[k]$, $0 \le k \le M$, the constraint reduces to $\sum_{k=0}^{M} f[k] = 1$.

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To suppress all the high frequency resonant dynamics in a flexible system, Zhou and Misawa [1], [2] have proposed vibration suppression shape filter and control profile generation based on time-frequency uncertainty and optimal energy concentration functions. In a practical system, a lower resonance frequency mode may exist which is located far from the high frequency resonance modes as shown in Fig. 2. If the low frequency in Fig. 2 is chosen to be a bandwidth in [1], [2] for the vibration suppression profile generation, the time duration of the profile is inefficiently increased. This paper generates a vibration suppression profile for a given specific resonant mode.

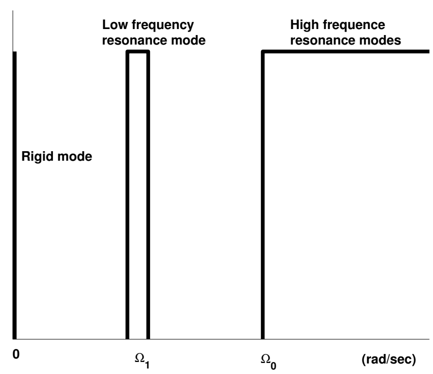

Fig. 2. Illustration of existence of a low resonance frequency mode located far from the high frequency modes in a flexible system.

II. Vibration Suppression Shape Filter Generation for a Specific Resonance Mode

The relationship between the Fourier transform of the control forcing function and the residual vibrations is derived. The undamped case was studied by Yamamura and Ono [3]. The relationship between the Fourier transform of the control forcing function and the residual vibrations for the damped case is studied here.

A. Relationship Between Control Input and Residual Vibration

Consider a unidirection flexible mechanical system with damping in the modal equation description

$$\ddot{x}_0(t) = z_0 u(t),$$
$$\ddot{x}_i(t) + 2\zeta_i\omega_i\dot{x}_i(t) + \omega_i^2 x_i(t) = z_i u(t), \quad i = 1, 2, \cdots,$$

where x_0 is the rigid mode displacement and x_i, $i \geq 1$, is the i^{th} resonant mode displacement. The variable $u(t)$ is a control forcing function, ζ_i is damping ratio of the i^{th} resonant mode ($0 \leq \zeta_i < 1$) and ω_i is the natural angular frequency of the i^{th} resonant mode. The damped natural frequency is $\omega_{di} = \sqrt{1-\zeta_i^2}\omega_i$. Here both ω_i and ω_{di} have units of rad/sec.

If the mechanical system is assumed to be stationary initially, the displacement and velocity of the i^{th} mode of vibration due to the forcing function $u(t)$ are given as:

$$x_i(t) = \frac{z_i}{\omega_{di}} \int_0^t e^{-\zeta_i\omega_i(t-\tau)} \sin(\omega_{di}(t-\tau))u(\tau)d\tau, \; i \geq 1, \tag{1}$$

$$\dot{x}_i(t) = \frac{z_i}{\sqrt{1-\zeta_i^2}} \int_0^t e^{-\zeta_i\omega_i(t-\tau)} \Big(\sqrt{1-\zeta_i^2}\cos(\omega_{di}(t-\tau)) - \zeta_i \sin(\omega_{di}(t-\tau))\Big)u(\tau)d\tau, \; i \geq 1. \tag{2}$$

If the time duration of the forcing function is T_0, i.e., $u(t) = 0, t > T_0$, the Fourier transform of the forcing function is $U(\omega) = \int_0^\infty u(t)e^{-j\omega t}dt = \int_0^{T_0} u(t)e^{-j\omega t}dt$. The displacement and velocity of a resonant mode at the end of the move time T_0 are given by $x_i(T_0), i \geq 1$ and $\dot{x}_i(T_0), i \geq 1$.

From (1) and (2), the following relationship between the end condition of the resonant modes and the Fourier transform of the forcing function is derived.

$$\begin{aligned} &\frac{\dot{x}_i(T_0)}{\omega_i} + \zeta_i x_i(T_0) + j\sqrt{1-\zeta_i^2}x_i(T_0) \\ &= \frac{z_i e^{-\zeta_i\omega_i T_0} e^{j\omega_{di}T_0}}{\omega_i} \int_0^{T_0} e^{\zeta_i\omega_i\tau}u(\tau)e^{-j\omega_{di}\tau}d\tau, \qquad (3) \\ &= \frac{z_i e^{-\zeta_i\omega_i T_0} e^{j\omega_{di}T_0}}{\omega_i} U_e(\omega_{di}), \qquad (4) \end{aligned}$$

where $U_e(\omega_{di}) = \int_0^\infty u_e(t)e^{-j\omega_{di}t}dt = \int_0^{T_0} u_e(t)e^{-j\omega_{di}t}dt$ and $u_e(t) := e^{\zeta_i\omega_i t}u(t)$, $0 \leq t \leq T_0$. Here ":=" denotes "equal to by definition." $U_e(\omega) = \int_0^\infty u_e(t)e^{-j\omega t}dt$ is the Fourier transform of $u_e(t)$.

A relationship between the end conditions of the resonant modes at $t = T_0$ and the Fourier transform of $u_e(t) := e^{\zeta_i\omega_i t}u(t)$, $0 \leq t \leq T_0$, can be derived as $\sqrt{\left[\frac{\dot{x}_i(T_0)}{\omega_i} + \zeta_i x_i(T_0)\right]^2 + (1-\zeta_i^2)x_i^2(T_0)} = \frac{|z_i|e^{-\zeta_i\omega_i T_0}}{\omega_i}|U_e(\omega_{di})|$. Consider the initial amplitude $C_i(T_0)$ of the free vibration which starts at $t = T_0$ [4],

$$C_i(T_0) = \sqrt{\frac{\dot{x}_i^2(T_0)}{\omega_i^2} + x_i^2(T_0)},$$

the following conclusion is drawn on the relationship between the residual vibration and the control input.

Conclusion 2.1: Given a forcing function $u(t)$, $0 \leq t \leq T_0$, the residual vibration of the i^{th} mode immediately after the move time T_0 is eliminated, i.e. $C_i(T_0) = 0$, if and only if the magnitude spectrum of $u_e(t) := e^{\zeta_i\omega_i t}u(t)$, $0 \leq t \leq T_0$, has zero value at the damped natural frequency ω_{di}, i.e. $U_e(\omega_{di}) = U_e(\sqrt{1-\zeta_i^2}\omega_i) = 0$.

The proof of Conclusion 2.1 is given in [5]. The important discovery of Conclusion 2.1 shows that the sufficient and necessary condition to eliminate the residual vibration is that the spectrum of the function $e^{\zeta_i\omega_i t}u(t)$ has zero value at the damped natural frequency $\omega_{di} = \sqrt{1-\zeta_i^2}\omega_i$. Robustness can be improved if higher order derivatives of $U_e(\omega)$ with respect to ω at $\omega = \omega_{di}$ are set to zero, i.e., $\left.\frac{d^k U_e(\omega)}{d\omega^k}\right|_{\omega=\omega_{di}} = 0, \quad k = 1, \cdots, n$. In addition to $U_e(\omega_{di}) = 0$, the n (≥ 1) additional constraints make the values of $|U_e(\omega)|$ around $\omega = \omega_{di}$ be close to the value of $|U_e(\omega_{di})|$ which is zero.

B. Philosophy Different from Previous Techniques

From the analysis of the relationship between control input and residual vibration of a damped resonant mode, the following conclusion is drawn.

Conclusion 2.2: If there exists a finite support base function $h(t)$, $0 \leq t \leq T_0$, such that $H(\omega_{di}) = H(\sqrt{1-\zeta_i^2}\omega_i) = 0$, then, $h(t)$ may have two possible properties:

1) The function $\frac{h(t)}{e^{\zeta_i\omega_i t}}$ is a control profile candidate to eliminate the residual vibration caused by the resonant mode with the natural frequency ω_i and the damping ratio ζ_i.

2) The function $\frac{h(t)}{e^{\zeta_i\omega_i t}}$ with a constraint $\int_0^{T_0} \frac{h(t)}{e^{\zeta_i\omega_i t}}dt = 1$, is a vibration suppression shape filter that can be used to filter out an arbitrary control profile, and the shaped control profile eliminates the residual vibration caused by the resonant mode with the natural frequency ω_i and the damping ratio ζ_i.

The proof of Conclusion 2.2 is given in [5]. The philosophy of Conclusion 2.2 advocates that a shape filter should be able to be both a traditional shaping filter, as well as a possible command signal in its own right. Let $f_1(t) = \frac{h(t)}{e^{\zeta_i\omega_i t}}$, then the robustness of the properties in Conclusion 2.2 can be improved by the following filter operation,

$$f_n(t) = \int_0^t f_{n-1}(t-\tau)f_1(\tau)d\tau, \quad n \geq 2, \tag{5}$$

and the resultant spectrum of $f_n(t)$ is $F_n(\omega) = F_1^n(\omega)$. Here, the robust control profile or shape filter $f_1(t)$ in (5) is said to have the robustness of order 1. The robust control profile or shape filter $f_n(t)$ generated from the filter operation in (5) is said to have the robustness of order n. In the case that $f(t)$ is a non-continuous impulse function, the robustness improvement method (5) becomes the idea advocated by Singer and Seering [6].

The following analysis shows that the Input Shaping®[1] technique [6] is a special vibration suppression shape filter possessing the second property of Conclusion 2.2. Some disadvantages of the input shaping technique are also demonstrated. The control profile generation technique described in this report simultaneously achieves the two properties in Conclusion 2.2. Also, the control profile generation technique is able to suppress the high frequency unknown vibrations, however the input shaping technique does not have this property, which means that the input shaping technique does have disadvantages if unmodeled high frequency resonance modes exist in a flexible system. These disadvantages are the price of non-smooth input shaping functions.

III. A Special Case (Input Shaping Technique)

In this section, the input shaping technique is proved to be a special case of the functions which only possesses the second property of Conclusion 2.2. Also, the disadvantages of using the input shaping technique are made clear. The input shaping technique [6] was derived using the response of a linear, time-invariant second order system to a sequence of impulses. By setting the amplitude of vibration for a multi-impulse input to be zero, the impulse amplitudes and corresponding impulse starting times can be solved. In [5], it is shown that input shaping technique constructs a small portion of the non-continuous function based shape filter.

A. A Special Case of Conclusion 2.2 Property 2

Let $f(t) = \frac{h(t)}{e^{\zeta_i \omega_i t}}$ and assume it to be an impulse function. First, $f(t)$ is defined to be a two-impulse function, i.e., $f(t) = \begin{cases} A_1 & \text{if } t = t_1, \\ A_2 & \text{if } t = t_2, \\ 0 & \text{otherwise.} \end{cases}$ Since $h(t) = f(t)e^{\zeta_i \omega_i t}$, then to guarantee $H(\omega_{di}) = H(\sqrt{1-\zeta_i^2}\omega_i) = 0$, the following equations must hold:

$$A_1 + A_2 e^{\zeta_i \omega_i t_2} \cos(\omega_{di} t_2) = 0, \tag{6}$$

$$A_2 e^{\zeta_i \omega_i t_2} \sin(\omega_{di} t_2) = 0. \tag{7}$$

Here t_1 is always assumed to be zero to reduce the time duration of the shape filter. With the additional constraint of a shape filter, $A_1 + A_2 = 1$, the resultant function $f(t)$ is given by $f(t) = \begin{cases} \frac{1}{1+K} & \text{if } t = 0, \\ \frac{K}{1+K} & \text{if } t = \Delta T, \\ 0 & \text{otherwise,} \end{cases}$ where $K = e^{-\frac{\zeta_i \pi}{\sqrt{1-\zeta_i^2}}}$ and $\Delta T = \frac{\pi}{\omega_{di}} = \frac{\pi}{\omega_i \sqrt{1-\zeta_i^2}}$. This derived vibration suppression shape filter is exactly the input shaping Zero Vibration (ZV) impulse filter. The base function of this shape filter is

$$h(t) = \begin{cases} 1 & \text{if } t = 0, \\ 1 & \text{if } t = \Delta T, \\ 0 & \text{otherwise,} \end{cases} \tag{8}$$

where $\Delta T = \frac{\pi}{\omega_{di}} = \frac{\pi}{\omega_i \sqrt{1-\zeta_i^2}}$.

To improve robustness, the filter operation in (5) is performed. Let $f_1(t) = f(t)$, then $f_2(t) = \int_0^t f_1(t-\tau) f_1(\tau) d\tau$, and the resultant shape filter with robustness order $n = 2$ is exactly the input shaping Zero Vibration Derivative (ZVD) impulse filter. If $n = 3$, then $f_3(t) = \int_0^t f_2(t-\tau) f_1(\tau) d\tau$, and the resultant shape filter with robustness order $n = 3$ is exactly the input shaping ZVDD impulse filter.

The above impulse filter derivation assumes all the impulses are positive. Negative impulses can also be assumed and the resultant negative shape filter may be shorter than the positive shape filter, however Singhose [7] has pointed out that negative input shapers can cause large unmodeled high frequency vibration. Since the shape filter function in this section is assumed to be an impulse function, it is not a smooth function so the two-impulse or three-impulse functions cannot be used as a velocity profile. The non-smooth shape filter is very sensitive to unmodeled high frequency resonant modes and this detail will be analyzed in the next section. It must be noted that the input shaping technique is only a special case of the non-continuous impulse functions which possess the second property of Conclusion 2.2 and a number of impulse functions possess the second property of Conclusion 2.2.

B. Disadvantages of Using Input Shaping Technique

In this section, the disadvantages of using the input shaping technique are demonstrated. Since the input shaping impulse filter is not smooth, so it is not able to suppress the unmodeled high frequency vibrations in a flexible system if the input shaping impulse filter is designed based on a low frequency resonance mode.

For a second-order harmonic oscillator of the natural frequency ω_i rad/sec and the damping ratio ζ_i, i.e., $\frac{\omega_i^2}{s^2 + 2\zeta_i \omega_i s + \omega_i^2}$, the magnitude of the total response immediately after the N^{th} impulse is given by [6]

$$V_{amp}(\omega_i, \zeta_i) = e^{-\zeta_i \omega_i t_N} \frac{\omega_i}{\sqrt{1-\zeta_i^2}} \sqrt{(AC(\omega_i, \zeta_i))^2 + (AS(\omega_i, \zeta_i))^2}, \tag{9}$$

where $AC(\omega_i, \zeta_i) = \sum_{k=1}^{N} A_k e^{\zeta_i \omega_i t_k} \cos\left(\omega_i \sqrt{1-\zeta_i^2} t_k\right)$ and $AS(\omega_i, \zeta_i) = \sum_{k=1}^{N} A_k e^{\zeta_i \omega_i t_k} \sin\left(\omega_i \sqrt{1-\zeta_i^2} t_k\right)$. A_k and t_k are the amplitude and time location at which the impulse occurs, N is the total number of impulses, and t_N is the time of the last impulse. The sensitivity of the impulse shape filter can be displayed graphically by a sensitivity curve: a plot of residual vibration amplitude versus

[1]Input Shaping® is a registered trademark of Convolve, Inc. in the United States. When this technique is referred to in this report, the terms "input shaping" or "input shaper" are used.

frequency error. Let $q = \omega_{actual}/\omega_{model}$, (9) becomes

$$V_{amp}(q\omega_i, \zeta_i) = e^{-\zeta_i q\omega_i t_N} \frac{q\omega_i}{\sqrt{1-\zeta^2}} \sqrt{\left(AC(q\omega_i, \zeta_i)\right)^2 + \left(AS(q\omega_i, \zeta_i)\right)^2}, \quad (10)$$

where $\omega_i = \omega_{model}$, $\omega_{actual} = q\omega_i$, $AC(q\omega_i, \zeta_i) = \sum_{k=1}^{N} A_k e^{\zeta_i q\omega_i t_k} \cos\left(q\omega_i \sqrt{1-\zeta_i^2} t_k\right)$, and $AS(q\omega_i, \zeta_i) = \sum_{k=1}^{N} A_k e^{\zeta_i q\omega_i t_k} \sin\left(q\omega_i \sqrt{1-\zeta_i^2} t_k\right)$. Since for any finite impulse shape filter $f(t)$, $0 \le t \le T_0$, the integral of $f(t)$ is $\int_0^{T_0} f(t)dt = 1$ and the rigid body movement amplitude can be assumed as $\int_0^{T_0} f(t)dt = 1$. The residual vibration level can be defined as a percentage of the rigid body motion amplitude, i.e., $\frac{V_{amp}(q\omega_i,\zeta_i)}{\int_0^{T_0} f(t)dt} = V_{amp}(q\omega_i, \zeta_i)$. For the impulse shape filter case, $\int_0^{T_0} f(t)dt = \sum_{k=1}^{N} A_k = 1$.

Fig. 3 shows the sensitivity curve of the input shaping ZVD impulse filter with $\omega_{model} = 1$ rad/sec and different damping $\zeta_i = 0, 0.05, 0.2$. This curve shows that the residual vibration is amplified at the unmodeled high frequency. For example, if $\omega_{actual} = 2$ rad/sec and the damping ratio $\zeta_i = 0$, then the ZVD input shaping impulse filter based on $\omega_{model} = 1$ rad/sec and $\zeta_i = 0$ will result a residual vibration amplification of 200% for the second-order oscillator described by $\frac{\omega_i^2}{s^2+2\zeta_i\omega_i s+\omega_i^2}$. The residual vibration amplitude is extremely large compared with the rigid body motion amplitude of 1.

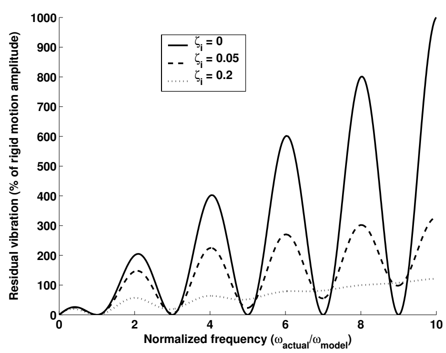

Fig. 3. ZVD input shaping residual vibration level versus actual natural frequency.

Since the impulse shape filter is not smooth, it is not able to suppress the unmodeled high frequency vibrations in a flexible system. Moreover, it cannot be used as a velocity profile. In the following analysis, a smooth shape filter will be developed. The smoothness of the shape filter suppresses the high frequency vibration.

The definition in (10) is different from the sensitivity concept of Singer [6] and Singhose [7]. In their definition, the sensitivity curve is expressed as the magnitude of the total response immediately after the N^{th} impulse divided by the magnitude of the response with unit impulse occurring at time 0. Since the magnitude of the response with unit impulse occurring at time 0 is given by $\frac{\omega_i}{\sqrt{1-\zeta_i^2}}$, their definition of residual vibration level is $\frac{V_{amp}(q\omega_i,\zeta_i)}{\frac{\omega_i}{\sqrt{1-\zeta_i^2}}\sum_{k=1}^{N} A_k}$ or simply

$$e^{-\zeta_i q\omega_i t_N} \sqrt{\left(AC(q\omega_i, \zeta_i)\right)^2 + \left(AS(q\omega_i, \zeta_i)\right)^2}, \quad (11)$$

because $V_{amp}(q\omega_i, \zeta_i)$ exactly has the term $\frac{\omega_i}{\sqrt{1-\zeta_i^2}}$ and $\sum_{k=1}^{N} A_k = 1$.

It is known that the magnitude of the response with unit impulse occurring at time 0, which is $\frac{\omega_i}{\sqrt{1-\zeta_i^2}}$, linearly increases with respect to actual undamped natural frequency ω_i if ζ_i is assumed to be a constant. Therefore, the definition (11) does not express the absolute residual vibration magnitude. For example, the definition (11) shows that the residual vibration level is 100% when the actual undamped natural frequency is 0. However, the true magnitude of the residual vibration given in (10) is 0 when the actual undamped natural frequency is 0. Fig. 4 shows the sensitivity plot of the input shaping ZVD impulse filter with $\omega_{model} = 1$ rad/sec using Singhose's definition (11).

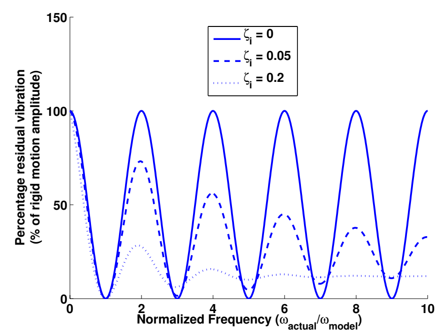

Fig. 4. ZVD input shaping percentage residual vibration level versus actual natural frequency.

IV. Another Special Case (Optimal Arbitrary Time-Delay Filter (OATF))

In this section, the optimal arbitrary time-delay filter (OATF) described by Magee and Book [8] is proved to be a special case of the functions which only possesses the second property of Conclusion 2.2. The OATF technique was first presented in Singh and Vadali [9].

OATF technique chooses a cost function involving both the error signal and the time rate of change in the error signal. In [5], it is shown that OATF technique, like input shaping technique, only constructs a small portion of the non-continuous impulse function based shape filter. The OATF that minimizes the elastic response of a single mode of vibration is given by [8]

$$f(t) = \begin{cases} 1 & \text{if } t = 0, \\ -2\cos(\omega_{di} T_1) e^{-\zeta_i \omega_i T_1} & \text{if } t = T_1, \\ e^{-2\zeta_i \omega_i T_1} & \text{if } t = 2T_1, \\ 0 & \text{otherwise,} \end{cases} \quad (12)$$

where T_1 is an arbitrary time-delay value, ω_i rad/sec is the undamped natural frequency, ζ_i is the damping ratio, and $\omega_{di} = \sqrt{1-\zeta_i^2}\omega_i$ is the damped natural frequency.

It is very simple to know that the function $h(t) = f(t)e^{\zeta_i \omega_i t}$ from the three-impulse function $f(t)$ in (12) is a base function such that the Fourier transform of $h(t)$, $H(\omega)$ is zero at $\omega = \omega_{di}$, i.e., $H(\omega_{di}) = 0$. First, $h(t) = f(t)e^{\zeta_i \omega_i t}$ is given by

$$h(t) = \begin{cases} 1 & \text{if } t = 0, \\ -2\cos(\omega_{di} T_1) & \text{if } t = T_1, \\ 1 & \text{if } t = 2T_1, \\ 0 & \text{otherwise,} \end{cases} \tag{13}$$

and the Fourier transform of three-impulse function $h(t)$, i.e. $H(\omega)$ at $\omega = \omega_{di}$ is given by $H(\omega_{di}) = 1 - 2\cos(\omega_{di}T_1)e^{-j\omega_{di}T_1} + e^{-2j\omega_{di}T_1} = 1 - (e^{j\omega_{di}T_1} + e^{-j\omega_{di}T_1})e^{-j\omega_{di}T_1} + e^{-2j\omega_{di}T_1} = 0$. Hence the OATF $f(t)$ in (12) can be generated from the above three-impulse base function $h(t)$ in (13) by the operation from the second property of Conclusion 2.2. Note that the area of shape filter $f(t)$ in (12) is not normalized to be 1.

V. Shape Filter Generation Using Rectangle Window or Zeroth Order Polynomial

A. Continuous-Time Rectangle Based Shape Filter

Assume the vibration suppression shape filter is generated from a rectangle window or zeroth order polynomial given by $h(t) = \begin{cases} \frac{1}{T}, & \text{if } 0 \le t \le T, \\ 0, & \text{otherwise.} \end{cases}$ The Fourier transform of $h(t)$ is given by $H(\omega) = \int_0^T \frac{1}{T} e^{-j\omega t} dt = \frac{1-e^{-j\omega T}}{j\omega T}$, and the magnitude spectrum is given by $|H(\omega)| = \left|\frac{1-e^{-j\omega T}}{j\omega T}\right| = \left|\frac{\sin(\omega T/2)}{(\omega T)/2}\right|$. If $H(\omega_{di}) = H(\sqrt{1-\zeta_i^2}\omega_i) = 0$, then $T = \frac{2\pi}{\omega_{di}}$, which is the same time duration as the input shaping ZVD impulse filter. So a smooth shape filter can be generated as

$$\begin{aligned} f(t) &= \frac{h(t)/e^{\zeta_i \omega_i t}}{\int_0^\infty h(t)/e^{\zeta_i \omega_i t} dt} = \frac{h(t)/e^{\zeta_i \omega_i t}}{(1-e^{-\zeta_i \omega_i T})/(\zeta_i \omega_i T)}, \\ &= \begin{cases} \frac{\zeta_i \omega_i}{1-e^{-\zeta_i \omega_i T}} e^{-\zeta_i \omega_i t} & \text{if } 0 \le t \le T, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{14}$$

When $\zeta_i = 0$, the shape filter $f(t)$ is simply equal to $h(t)$. To improve robustness, the filter operation in (5) can be performed.

Robustness can also be further improved by the filter operation in (5). The price of the robustness improvement is that the time duration of the shape filter is increased. Fig. 5 shows the resultant shape filter function $f_1(t)$ in the time domain and the magnitude spectrum $|F_1(\omega)|$ in the frequency domain, with $\omega_i = 1$ rad/sec and $\zeta_i = 0.05$. Fig. 6 shows the resultant shape filter function $f_2(t)$ and the magnitude spectrum $|F_2(\omega)|$, with $\omega_i = 1$ rad/sec and $\zeta_i = 0.05$. It can be seen that $f_2(t)$ may be used as a robust velocity profile for the rigid body mode.

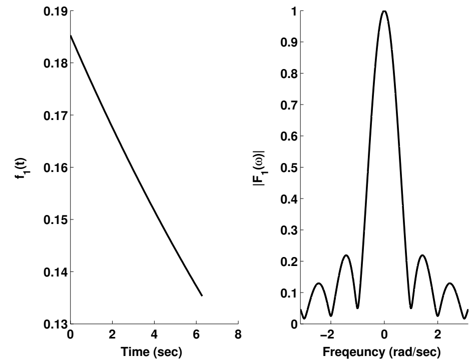

Fig. 5. Rectangle based shape filter $f_1(t)$ with $\omega_i = 1, \zeta_i = 0.05$.

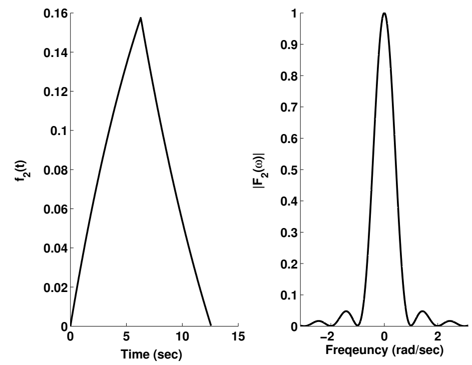

Fig. 6. Rectangle based shape filter $f_2(t)$ with $\omega_i = 1, \zeta_i = 0.05$.

B. Discrete-Time Rectangle Based Shape Filter Generation

Now the discrete-time rectangle based shape filter is derived. If the sampling period is T_s seconds and the total discrete-time sequence has $M+1$ impulses, the rectangle function is $h[k] = \begin{cases} 1, & \text{if } 0 \le k \le M, \\ 0, & \text{otherwise.} \end{cases}$ The discrete-time Fourier transform of $h[k]$ is given by $H(\omega) = \sum_{k=0}^{M} h[k]e^{-j\omega k} = e^{-j\omega M}\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)}$, and the magnitude spectrum of $h[k]$ is given by $|H(\omega)| = \left|\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)}\right|$.

Here the unit of ω in the discrete-time Fourier transform is radian. If $H(\omega_{di}T_s) = H(\sqrt{1-\zeta_i^2}\ \omega_i T_s) = 0$, then $\omega_{di}T_s = \frac{2\pi}{M+1}$ and $M = \frac{2\pi}{\omega_{di}T_s} - 1$. If M is a positive integer, a smooth shape filter can be generated as

$$f[k] = \frac{h[k]/e^{\zeta_i \omega_i k T_s}}{\sum_{m=0}^{M} h[m]/e^{\zeta_i \omega_i m T_s}}, \tag{15}$$

$$= \begin{cases} Be^{-\zeta_i \omega_i k T_s} & \text{if } 0 \le k \le M, \\ 0 & \text{otherwise,} \end{cases} \tag{16}$$

where $B = \frac{1-e^{-\zeta_i \omega_i T_s}}{1-e^{-\zeta_i \omega_i (M+1) T_s}}$. When $\zeta_i = 0$, the shape filter $f[k]$ is simply equal to $h[k]/(M+1)$. To improve robustness, the filter operation in (5) can be performed.

C. Comparison of Rectangle Based Shape Filter and ZVD Input Shaping®

In this section, a comparison between the discrete-time rectangle based shape filter $f_1[k]$ and the ZVD input shaper is performed. The sampling period T_s is chosen to be $\frac{\pi}{100\omega_{di}}$. First, observe that the continuous-time rectangle based shape filter $f_1(t)$ in (14) has the same time duration as the ZVD input shaper which is $\frac{2\pi}{\omega_{di}}$. In the discrete-time case, the time duration of the rectangle based shape filter $f_1[k] = f[k]$ in (16) is $MT_s = \left(\frac{2\pi}{\omega_{di}T_s} - 1\right) T_s = \frac{2\pi}{\omega_{di}} - T_s$. This results that the time duration of discrete-time rectangle based shape filter $f_1[k]$ is always one sample period T_s less than the time duration of the ZVD input shaper.

The residual vibration level (10) can be plotted for the rectangle based shape filter $f_1[k]$. Fig. 7 shows the sensitivity curve of the rectangle based shape filter $f_1[k]$ with $\omega_{model} = 1$ rad/sec and different damping $\zeta_i = 0, 0.05, 0.2$. Although the sensitivity curve of the rectangle based shape filter at the model natural frequency $\omega = \omega_{model} = 1$ rad/sec is not as flat as that of the ZVD input shaper, the high frequency unmodeled dynamics are suppressed by the smoothness of the rectangle based shape filter. Fig. 8 shows the sensitivity plot of the rectangle based shape filter with $\omega_{model} = 1$ rad/sec using Singhose's definition (11). It must be noted that both definition (10) and definition (11) assume the direct input to the second-order oscillator are impulses. Sensitivity analysis including the baseline command such as a unit step command will be reported separately.

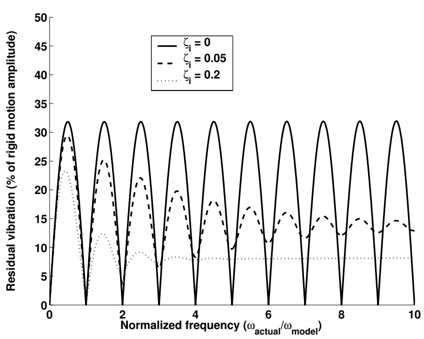

Fig. 7. Rectangle based shape filter $f_1[k]$ sensitivity plot versus actual natural frequency.

D. Discrete-Time Shape Filter Generation with an Arbitrary Sampling Period

The previous analysis assumes the discrete-time sequence has an exact integer number of impulses. In practice, the calculation result of $M = \frac{2\pi}{\omega_{di}T_s} - 1$ may not be an integer, but a floating point number. Discrete-time shape filter generation with an arbitrary sampling period is studied in [10]. It is also clear from (8) that the continuous-time ZV input shaper is also a rectangle based shape filter $f_1[k]$ with the sampling period $\Delta T = \frac{\pi}{\omega_{di}}$. Sensitivity analysis with the sampling period will be reported separately.

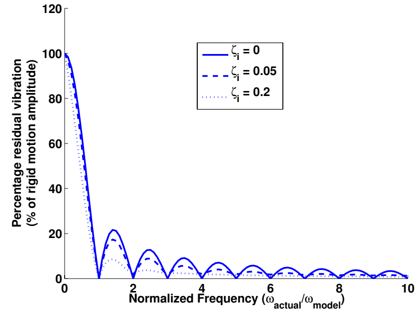

Fig. 8. Rectangle based shape filter $f_1[k]$ percentage residual vibration level versus actual natural frequency.

VI. Conclusions

A vibration suppression shape filter is generated from a continuous function. The robustness can be arbitrarily improved and the robustness brings about a smoother profile. The shape filter can also be used as a velocity profile in the case of zero initial and final values. The methods in this paper were tested on hard disk drive position control at the Oklahoma State University Advanced Controls Laboratory. The experimental results of both input shaping and rectangle based shape filter are reported in [11]. The methods in this paper are patented (pending). Commercial use of these methods requires written permission from the Oklahoma State University.

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