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Stability of Damped Membranes and Plates with Distributed Inputs

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Abstract— **This paper proves the stability of boundary and distributed damped membranes and Kirchhoff plates under distributed inputs. Distributed viscous or Kelvin-Voigt damping ensures weakly bounded response to bounded transverse loading for pinned membranes and clamped plates. Damping on part of the boundary can also weakly stabilize the forced response, provided the damped and undamped boundary normals satisfy certain conditions. For example, damping on half and one side of the boundary is sufficient for circular and rectangular domains, respectively.**

I. INTRODUCTION

Many engineering applications have distributed parameter models governed by partial differential equations. Often forcing of unknown but bounded magnitude disturbs the system and the boundedness of the response needs to be determined. Without damping, flexible structures are not stable due to resonances corresponding to natural frequencies in the system. Bounded sinusoidal inputs at these frequencies cause unbounded response.

One approach to determine the stability of distributed parameter flexible systems is to discretize using Galerkin, FEM, or finite difference approximations [1]. The system reduces to a set of finite, second order differential equations with mass, damping, and stiffness matrices. These systems are exponentially (and bounded input - bounded output) stable if the stiffness matrix is positive definite (no rigid body modes) and the damping matrix satisfies complete or pervasive damping conditions [2]. Unfortunately, these conditions only apply to the discretized model, not the full order distributed system.

Recently, important progress has been made in the stability analysis of two dimensional distributed parameter systems. Cavalcanti and Oquendo [3] show exponential and polynomial decay for a partially viscoelastic nonlinear wave equation subject to nonlinear and localized frictional damping. Cheng [4] proves the continuity of the input/output map for boundary control systems through the system transfer function. Komornik [5] and Lagnese [6] use the multiplier method to prove the boundary stabilization of membranes and plates. Guesmia [7] provides decay estimates when integral inequalities can not be applied due to the lack of dissipativity. Zhao and Rahn [8] apply the energy multiplier method to damped strings and beams, proving bounded response to distributed inputs.

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This paper extends the approach in [8] to two dimensional membranes and plates using a variety of integral inequalities [9] - [11]. Distributed viscous and material damping and boundary damping are shown to stabilize the response to bounded distributed inputs.

II. MATHEMATICAL PRELIMINARIES

Damping results from viscoelastic material behavior, frictional interaction between contacting surfaces, or movement through a dissipative fluid. Distributed (viscous and material) and boundary (viscous) damping are analyzed in this paper. Viscous damping forces are proportional to transverse velocity. Kelvin-Voigt damping is due to material viscoelasticity and proportional to material strain rate.

The following equalities and inequalities are used extensively in this paper and are presented without proof (see [9] - [11] for details). Throughout the paper, we assume that a 2D open, bounded, connected, Lipschitz domain with boundary Γ is defined.

A. Equalities

The Divergence Theorem applies to vector fields $V = P(x_1, x_2)\,\mathbf{i} + Q(x_1, x_2)\,\mathbf{j}$ as follows

$$\int_\Omega \left(\frac{\partial P}{\partial x_1} + \frac{\partial Q}{\partial x_2}\right) dx = \int_\Gamma (P dx_2 - Q dx_1). \tag{1}$$

The normal derivative of $\omega(\mathbf{x}, t)$ is defined as

$$\frac{\partial \omega}{\partial n} = \nabla\omega \cdot \mathbf{n} \quad on\ \Gamma, \tag{2}$$

where $\mathbf{n}$ is the unit-normal vector to Γ pointing toward the exterior of Ω.

The following integral equalities apply to $\omega \in H^1(\Omega)$ and $\upsilon \in H^2(\Omega)$

$$\int_\Omega \triangle\upsilon\,\omega\, dx = \int_\Gamma \frac{\partial \upsilon}{\partial n}\omega d\Gamma - \int_\Omega \nabla\upsilon \cdot \nabla\omega\, dx, \tag{3}$$

$$\int_\Omega \mathbf{r} \cdot \nabla\omega\, dx = \int_\Gamma (\mathbf{r}\cdot\mathbf{n})\omega d\Gamma - \int_\Omega (\nabla\cdot\mathbf{r})\,\omega dx. \tag{4}$$

The divergence of products can be calculated as follows:

$$\nabla\cdot(\omega\mathbf{a}) = \omega\nabla\cdot\mathbf{a} + (\nabla\omega)\cdot\mathbf{a}, \tag{5}$$

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a}) + (\nabla\cdot\mathbf{a})\,\mathbf{b} + (\nabla\cdot\mathbf{b})\,\mathbf{a}, \tag{6}$$

where $\mathbf{r}$, $\mathbf{a}$, $\mathbf{b}$ are two dimensional vectors.

B. Inequalities

$$(\mathbf{a}\cdot\mathbf{b}) \le \delta\,|\mathbf{a}|^2 + \frac{1}{\delta}\,|\mathbf{b}|^2. \tag{7}$$

The Poincaré inequality

$$\int_\Omega \omega^2 dx \le m_1 \int_\Omega |\nabla\omega|^2\,dx \tag{8}$$

holds $\forall\omega \in H^2(\Omega)$ with $\omega = 0$ on Γ for some constant $m_1 > 0$. The Sobolev inequality

$$\int_{\Gamma_1} \omega^2 dx \le m_2 \int_\Omega |\nabla\omega|^2\,dx,\ \forall\omega \in H^1(\Omega) \tag{9}$$

where m_2 is a positive constant, $\Gamma = \Gamma_0 \cup \Gamma_1$, and $\omega = 0$ on Γ_0.

III. Damped Membranes

For the damped membrane model shown in Fig. 1, we assume that the membrane is inextensible and perfectly flexible, the in-plane stress P is constant, and bounded distributed forcing $f(\mathbf{x},t)$ is applied in the domain Ω. First, a membrane with distributed viscous and material damping is considered. Then, we consider a membrane without damping in the field equation and with a damped boundary condition on Γ_1 with the remaining boundary Γ_0 pinned.

A. Distributed Damped Membranes

The field equation, boundary conditions, and initial conditions of the damped membrane are

$$\begin{aligned}
\rho\ddot{\omega} + b\dot{\omega} - D\Delta\dot{\omega} - P\Delta\omega &= f \quad in\ \Omega\times R_+, &(10)\\
\omega(\mathbf{x},t) &= 0 \quad on\ \Gamma\times R_+, &(11)\\
\omega(\mathbf{x},0) &= \omega_0\ on\ \Omega, &(12)\\
\dot{\omega}(\mathbf{x},0) &= \dot{\omega}_0\ on\ \Omega, &(13)
\end{aligned}$$

where dots indicate time differentiation, ρ is the mass/area, b is viscous damping, D is Kelvin-Voigt damping, Γ is the boundary, Ω is the open, bounded, connected, Lipschitz, 2D domain, and $\mathbf{n}$ is the unit-normal vector to Γ pointing toward the exterior of Ω. We assume the models presented in this paper are well-posed and possess a unique solution for all initial conditions and bounded inputs.

Theorem 1: The response of the damped membrane governed by (10) - (13) is weakly bounded if either b or D is nonzero and $f(\mathbf{x},t)$ is bounded $\forall\mathbf{x}\in\Omega$ and $t>0$ ($f\in L_\infty$).

Proof: The energy of the membrane

$$E = \frac{1}{2}\int_\Omega (\rho\dot{\omega}^2 + P\,|\nabla\omega|^2)dx \ge 0 \tag{14}$$

has a time rate of change which can be upper bounded by

$$\begin{aligned}
\dot{E} &= \int_\Omega [\dot{\omega}\,(f - b\dot{\omega} + D\Delta\dot{\omega} + P\Delta\omega)\\
&\quad + P\nabla\omega\cdot\nabla\dot{\omega}]\,dx\\
&\le \delta_1\int_\Omega \dot{\omega}^2 dx + \frac{1}{\delta_1}\int_\Omega f^2 dx - b\int_\Omega \dot{\omega}^2 dx\\
&\quad + D\int_\Gamma \frac{\partial\dot{\omega}}{\partial n}\dot{\omega}\,d\Gamma - D\int_\Omega |\nabla\dot{\omega}|^2\,dx\\
&\quad + P\int_\Gamma \frac{\partial\omega}{\partial n}\dot{\omega}\,d\Gamma - P\int_\Omega (\nabla\omega\cdot\nabla\dot{\omega})\,dx\\
&\quad + P\int_\Omega (\nabla\omega\cdot\nabla\dot{\omega})\,dx\\
&\le -\left(b + \frac{D}{2m_1} - \delta_1\right)\int_\Omega \dot{\omega}^2 dx\\
&\quad - \frac{D}{2}\int_\Omega |\nabla\dot{\omega}|^2\,dx + \frac{1}{\delta_1}\int_\Omega f^2 dx, \qquad (15)
\end{aligned}$$

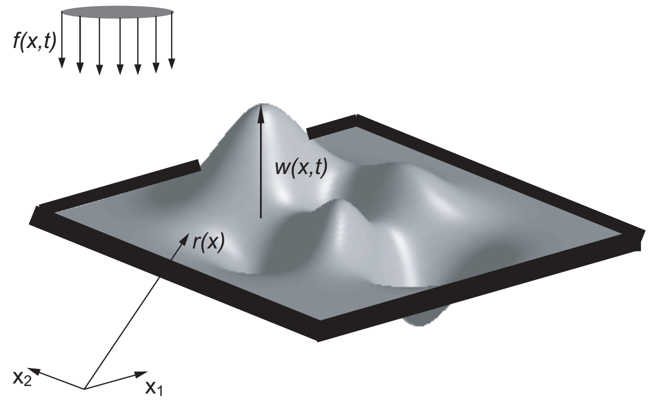

Fig. 1. Schematic diagram of a distributed damped membrane with distributed disturbances.

where (3), (7), and (8) are used. Inequality (15) lacks the $\int_\Omega |\nabla\omega|^2$ term that appears in E. We therefore define a new functional by adding the crossing term $C(t)$

$$V(t) = E(t) + \beta C(t), \tag{16}$$

where

$$C(t) = \rho\int_\Omega \dot{\omega}\omega dx. \tag{17}$$

The functional $V(t)$ is positive because

$$\begin{aligned}
|C(t)| &\le \frac{1}{2}\rho\int_\Omega \left(\dot{\omega}^2 + \omega^2\right)dx\\
&\le \frac{1}{2}\rho\int_\Omega \left(\dot{\omega}^2 + m_1\,|\nabla\omega|^2\right)dx\\
&\le \frac{\rho\max(1,m_1)}{\min(\rho,P)}\frac{1}{2}\int_\Omega \left(\rho\dot{\omega}^2 + P\,|\nabla\omega|^2\right)dx\\
&= \eta E, \qquad (18)
\end{aligned}$$

using inequalities (7) and (8), where

$$\eta = \frac{\rho\max(1,m_1)}{\min(\rho,P)}. \tag{19}$$

This means that

$$0 \le \lambda_1 E(t) \le V(t) \le \lambda_2 E(t), \tag{20}$$

where

$$\begin{aligned}\lambda_1 &= 1-\beta\eta > 0,\\ \lambda_2 &= 1+\beta\eta > 1,\end{aligned}$$

for sufficiently small β. Differentiation of the crossing term produces

$$\begin{aligned}\dot{C} &= \int_\Omega \rho\ddot{\omega}\,\omega dx + \int_\Omega \rho\dot{\omega}^2\,dx\\ &= \int_\Omega (f - b\dot{\omega} + D\Delta\dot{\omega} + P\Delta\omega)\,\omega dx + \rho\int_\Omega \dot{\omega}^2\,dx\\ &= \rho\int_\Omega \dot{\omega}^2\,dx + \dot{C}_1 + \dot{C}_2 + \dot{C}_3 + \dot{C}_4. \qquad (21)\end{aligned}$$

The terms in (21) simplify as follows

$$\begin{aligned}\dot{C}_1 &= \int_\Omega f\omega dx\\ &\le \delta_2\int_\Omega \omega^2 dx + \frac{1}{\delta_2}\int_\Omega f^2 dx\\ &\le \delta_2 m_1\int_\Omega |\nabla\omega|^2\,dx + \frac{1}{\delta_2}\int_\Omega f^2 dx, \qquad (22)\end{aligned}$$

$$\begin{aligned}\dot{C}_2 &= -\int_\Omega b\dot{\omega}\omega dx\\ &\le b\delta_3\int_\Omega \omega^2 dx + \frac{b}{\delta_3}\int_\Omega \dot{\omega}^2 dx\\ &\le bm_1\delta_3\int_\Omega |\nabla\omega|^2\,dx + \frac{b}{\delta_3}\int_\Omega \dot{\omega}^2 dx, \qquad (23)\end{aligned}$$

$$\begin{aligned}\dot{C}_3 &= \int_\Omega D\omega\Delta\dot{\omega} dx\\ &= D\int_\Gamma \frac{\partial\dot{\omega}}{\partial n}\omega d\Gamma - D\int_\Omega (\nabla\omega\cdot\nabla\dot{\omega})\,dx\\ &\le D\delta_4\int_\Omega |\nabla\omega|^2\,dx + \frac{D}{\delta_4}\int_\Omega |\nabla\dot{\omega}|^2\,dx, \qquad (24)\end{aligned}$$

$$\begin{aligned}\dot{C}_4 &= \int_\Omega P\omega\Delta\omega dx\\ &= \int_\Gamma P\frac{\partial\omega}{\partial n}\omega d\Gamma - P\int_\Omega |\nabla\omega|^2\,dx\\ &= -P\int_\Omega |\nabla\omega|^2\,dx, \qquad (25)\end{aligned}$$

using the boundary condition (11) and (3), (7), and (8). Substitution of (22) - (25) into (21) yields

$$\begin{aligned}\dot{C} \le\ & \frac{1}{\delta_2}\int_\Omega f^2 dx + \left(\rho + \frac{b}{\delta_3}\right)\int_\Omega \dot{\omega}^2\,dx\\ & - [P - (\delta_2 + b\delta_3)\,m_1 - D\delta_4]\int_\Omega |\nabla\omega|^2\,dx\\ & + \frac{D}{\delta_4}\int_\Omega |\nabla\dot{\omega}|^2\,dx. \qquad (26)\end{aligned}$$

Substitution of the derivative of crossing term (26) into (14) produces

$$\begin{aligned}\dot{V} \le\ & -\left[b + \frac{D}{2m_1} - \delta_1 - \beta\left(\rho + \frac{b}{\delta_3}\right)\right]\int_\Omega \dot{\omega}^2 dx\\ & -\beta\,[P - (\delta_2 + b\delta_3)\,m_1 - D\delta_4]\int_\Omega |\nabla\omega|^2\,dx\\ & -D\left(\frac{1}{2} - \frac{\beta}{\delta_4}\right)\int_\Omega |\nabla\dot{\omega}|^2\,dx\\ & +\left(\frac{1}{\delta_1} + \frac{\beta}{\delta_2}\right)\int_\Omega f^2 dx\\ \le\ & -\lambda_3 E + \varepsilon, \qquad (27)\end{aligned}$$

where, for sufficiently small β, δ_1, δ_2, δ_3, and δ_4,

$$\frac{1}{2} \ge \frac{\beta}{\delta_4}, \qquad (28)$$

$$\varepsilon_1 = b + \frac{D}{2m_1} - \delta_1 - \beta\left(\rho + \frac{b}{\delta_3}\right) > 0, \qquad (29)$$

$$\varepsilon_2 = \beta\,[P - (\delta_2 + b\delta_3)\,m_1 - D\delta_4] > 0, \qquad (30)$$

$$\varepsilon = \left(\frac{1}{\delta_1} + \frac{\beta}{\delta_2}\right)\max_{t\in[0,\infty)}\int_\Omega f^2 dx < \infty, \qquad (31)$$

$$\lambda_3 = \frac{\min(\varepsilon_1,\varepsilon_2)}{\max(\rho, P)} > 0 \qquad (32)$$

for bounded f. Using (20), we obtain

$$\dot{V} \le -\lambda V + \varepsilon, \qquad (33)$$

where $\lambda = \lambda_3/\lambda_2$, with the solution

$$V(t) \le V(0)e^{-\lambda t} + \frac{\varepsilon}{\lambda} \in L_\infty \qquad (34)$$

and

$$E(t) \le \frac{1}{\lambda_1}V(t) \in L_\infty. \qquad (35)$$

□

Thus, the system is weakly stable with respect to the energy norm.

B. Boundary Damped Membranes

We remove the distributed damping in (10) to obtain the two dimensional wave equation with partially damped boundary conditions shown in Fig. 2. The governing equations are

$$\begin{aligned}\rho\ddot{\omega} - P\Delta\omega &= f \quad in\ \Omega\times R_+, \qquad (36)\\ \omega &= 0\ on\ \Gamma_0\times R_+, \qquad (37)\\ P\frac{\partial\omega}{\partial n} + c\dot{\omega} &= 0\ on\ \Gamma_1\times R_+, \qquad (38)\end{aligned}$$

where $\Gamma = \Gamma_0 \cup \Gamma_1$, c is boundary damping coefficient, and the initial conditions are given in (12) and (13). We assume the boundary normals satisfy

$$\begin{aligned}\mathbf{r}\cdot\mathbf{n} &\le 0\ \ on\ \ \Gamma_0, \qquad (39)\\ \mathbf{r}\cdot\mathbf{n} &> 0\ \ on\ \ \Gamma_1, \qquad (40)\end{aligned}$$

where $\mathbf{r} = \mathbf{x} - \mathbf{x}_0$ and $\mathbf{x}_0$ is the origin [5] [6].

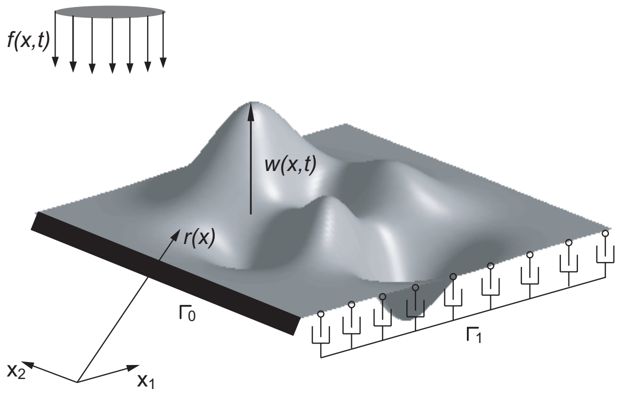

Fig. 2. Schematic diagram of a boundary damped membrane with distributed disturbances.

Theorem 2: The response of the boundary damped membrane governed by (36) - (38) is weakly bounded if $c > 0$, $f \in L_\infty$, and the boundary normal conditions (39) and (40) are satisfied.

Proof: The energy given in (14) has a time derivative

$$\begin{aligned}\dot{E} &= \int_\Omega (\dot{\omega} f + P\dot{\omega}\Delta\omega + P\nabla\omega\cdot\nabla\dot{\omega})dx \\ &\leq -c\int_{\Gamma_1}\dot{\omega}^2 d\Gamma + \delta_1\int_\Omega \dot{\omega}^2 dx + \frac{1}{\delta_1}\int_\Omega f^2 dx, \quad (41)\end{aligned}$$

using (3), (7), and (38).

The boundary damper does not match the distributed input, providing neither a negative kinetic nor potential energy term in Ω. A positive functional is defined as in (16) with a different crossing term

$$C(t) = C_1 + C_2, \quad (42)$$

where $C_1 = 2\int_\Omega \rho\dot{\omega}\ (\mathbf{r}\cdot\nabla\omega)\,dx$ and $C_2 = \int_\Omega \rho\dot{\omega}\,\omega dx$.

We bound the crossing term (42) with respect to the system energy as follows:

$$\begin{aligned}|C(t)| &\leq 2\rho R\int_\Omega |\dot{\omega}|\,|\nabla\omega|\,dx \\ &\quad +\frac{1}{2}\rho\int_\Omega \left(\dot{\omega}^2+\omega^2\right)dx \\ &\leq \frac{2\rho R}{2}\int_\Omega\left(\dot{\omega}^2+|\nabla\omega|^2\right)dx \\ &\quad +\frac{1}{2}\rho\int_\Omega\left(\dot{\omega}^2+m_1|\nabla\omega|^2\right)dx \\ &\leq \eta E, \quad (43)\end{aligned}$$

using (7) and (8), where

$$R = \sup_{\Gamma_1}\|\mathbf{r}(\mathbf{x})\|, \quad (44)$$

$$\eta = \frac{\rho\max\left[(2R+1),\ (2R+m_1)\right]}{\min(\rho, P)}. \quad (45)$$

The time derivative of the crossing term (42) depends on

$$\begin{aligned}\dot{C}_1 &= 2\int_\Omega (f+P\Delta\omega)\ (\mathbf{r}\cdot\nabla\omega)\,dx \\ &\quad +2\rho\int_\Omega \dot{\omega}\ (\mathbf{r}\cdot\nabla\dot{\omega})\,dx \\ &\leq 2R\int_\Omega |f|\,|\nabla\omega|\,dx + 2P\int_\Omega \Delta\omega\ (\mathbf{r}\cdot\nabla\omega)\,dx \\ &\quad +2\rho\int_\Omega \dot{\omega}\ (\mathbf{r}\cdot\nabla\dot{\omega})\,dx \\ &\leq 2R\delta_2\int_\Omega |\nabla\omega|^2 dx + \frac{2R}{\delta_2}\int_\Omega f^2 dx \\ &\quad +\dot{C}_3+\dot{C}_4, \quad (46)\end{aligned}$$

where inequality (7) is used.

The terms in (46) simplify as follows

$$\dot{C}_3 = 2P\int_\Omega \Delta\omega\ (\mathbf{r}\cdot\nabla\omega)\,dx$$

$$\begin{aligned}&= 2P\int_\Gamma (\mathbf{r}\cdot\mathbf{n})\,|\nabla\omega|^2 d\Gamma \\ &\quad -2P\int_\Omega |\nabla\omega|^2 dx - P\int_\Omega \mathbf{r}\cdot\nabla\left(|\nabla\omega|^2\right)dx \\ &\leq \frac{Rc^2}{P}\int_{\Gamma_1}\dot{\omega}^2 d\Gamma, \quad (47)\end{aligned}$$

using the boundary conditions (37) and (38), (2) - (4), and (7). Based on (2) and the boundary condition (39), $2P\int_{\Gamma_0}(\mathbf{r}\cdot\mathbf{n})\,|\nabla\omega|^2 d\Gamma \leq 0$ is dropped from (47).

$$\begin{aligned}\dot{C}_4 &= 2\rho\int_\Omega \dot{\omega}\ (\mathbf{r}\cdot\nabla\dot{\omega})\,dx \\ &\leq 2\rho R\int_{\Gamma_1}\dot{\omega}^2 d\Gamma - 4\rho\int_\Omega \dot{\omega}^2 dx - \dot{C}_4, \quad (48)\end{aligned}$$

using the boundary conditions, (2), and (7). Solving (48),

$$\dot{C}_4 \leq \rho R\int_{\Gamma_1}\dot{\omega}^2 d\Gamma - 2\rho\int_\Omega \dot{\omega}^2 dx. \quad (49)$$

The time derivative of crossing term C_2

$$\begin{aligned}\dot{C}_2 &= \int_\Omega (f+P\Delta\omega)\,\omega\,dx + \rho\int_\Omega \dot{\omega}^2 dx \\ &\leq -\left[P-\delta_3 m_1 - \frac{1}{2}cm_2\right]\int_\Omega |\nabla\omega|^2 dx \\ &\quad +\rho\int_\Omega \dot{\omega}^2 dx + \frac{1}{\delta_3}\int_\Omega f^2 dx \\ &\quad +\frac{1}{2}c\int_{\Gamma_1}\dot{\omega}^2 d\Gamma, \quad (50)\end{aligned}$$

using (3) and (7) - (9). Substitution of (42), (46), (47), and

(49) into (16) yields

$$\begin{aligned}\dot{V} \leq & -\left\{c-\beta\left[R\left(\frac{c^2}{P}+\rho\right)+\frac{1}{2}c\right]\right\}\int_{\Gamma_1}\dot{\omega}^2 d\Gamma \\ & -\beta\left[P-2R\delta_2-\delta_3 m_1-\frac{1}{2}cm_2\right]\int_\Omega |\nabla\omega|^2 dx \\ & -(\beta\rho-\delta_1)\int_\Omega \dot{\omega}^2 dx \\ & +\left[\frac{1}{\delta_1}+\beta\left(\frac{2R}{\delta_2}+\frac{1}{\delta_3}\right)\right]\int_\Omega f^2 dx, \qquad (51)\end{aligned}$$

where for sufficiently small β, δ_1, δ_2, and δ_3,

$$c \geq \beta\left[R\left(\frac{c^2}{P}+\rho\right)+\frac{1}{2}c\right], \qquad (52)$$

$$\varepsilon_1 = \beta\rho-\delta_1 > 0, \qquad (53)$$

$$\varepsilon_2 = \beta\left[P-2R\delta_2-\delta_3 m_1-\frac{1}{2}cm_2\right] > 0, \qquad (54)$$

$$\varepsilon = \left(\frac{1}{\delta_1}+\frac{2\beta R}{\delta_2}+\frac{\beta}{\delta_3}\right)\max_{t\in[0,\infty)}\int_\Omega f^2 dx, \qquad (55)$$

$$\lambda_3 = \frac{\min(\varepsilon_1,\varepsilon_2)}{\max(\rho,P)} > 0. \qquad (56)$$

Therefore, (35) holds and the response is weakly bounded.

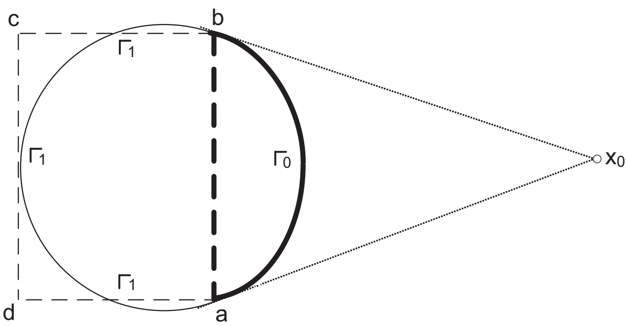

Fig. 3. Circular (solid) and rectangular (dashed) domain showing damped (thin) and undamped (thick) boundaries.

□

The partially damped boundary normal conditions (39) and (40) require damping on part of the boundary ($\Gamma_1 \neq \varnothing$). We are free to choose the origin $\mathbf{x}_0$ to determine the minimal Γ_1 for stability. If $\mathbf{x}_0$ is located at the center of a star shaped domain, then the entire boundary has $\mathbf{r}\cdot\mathbf{n} > 0$ so $\Gamma_1 = \Gamma$ and the entire boundary must be damped. Locating $\mathbf{x}_0$ outside Ω, however leads to $\Gamma_0 \neq \varnothing$ and part of the boundary need not be damped. Fig. 3 shows example circular (solid) and rectangular (dashed) domains with $\mathbf{x}_0 \notin \Omega$. In both cases, damping is not required on $\overline{ab}$. In the limit as $\mathbf{x}_0 \to \infty$, half of the circular domain is damped. For the rectangular domain as $\mathbf{x}_0 \to \infty$, $\mathbf{r}\cdot\mathbf{n} < 0$ on $\overline{ab}$ and $\mathbf{r}\cdot\mathbf{n} = \mathbf{0}$ on $\overline{bc}$ and $\overline{da}$, so $\mathbf{r}\cdot\mathbf{n} \leq 0$ on $\overline{da}\cup\overline{ab}\cup\overline{bc}$. Thus, only one side $\overline{cd} = \Gamma_1$ requires damping.

IV. Distributed Plates

In this section, we investigate the stability of distributed and boundary damped Kirchhoff plates with distributed excitation. We assume the plates are inextensible and homogeneous with uniform cross-section.

A. Distributed Damped Plates

The field equation of the distributed damped plate includes distributed viscous and material damping and forcing:

$$\rho\ddot{\omega}+b\dot{\omega}+D\Delta^2\dot{\omega}+D_E\Delta^2\omega = f \quad in\ \Omega\times R_+, \qquad (57)$$

with boundary condition

$$\omega = 0\ on\ \Gamma\times R_+, \qquad (58)$$

$$\frac{\partial\omega}{\partial n} = 0\ on\ \Gamma\times R_+, \qquad (59)$$

where D_E is the plate flexural rigidity. The initial conditions are given in (12) and (13).

Theorem 3: The response of the damped plate governed by (57) - (59) is weakly bounded if either b or D is nonzero and $f \in L_\infty$.

Proof: The energy of the plate is

$$\begin{aligned}E = & \frac{1}{2}\int_\Omega\left\{2(1-\mu)\left[\left(\frac{\partial^2\omega}{\partial x_1\partial x_2}\right)^2-\frac{\partial^2\omega}{\partial x_1^2}\frac{\partial^2\omega}{\partial x_2^2}\right]\right. \\ & \left.+\rho\dot{\omega}^2+D_E(\Delta\omega)^2\right\}dx, \qquad (60)\end{aligned}$$

where μ is Poisson's ratio. The Gaussian curvature $2(1-\mu)\left[\left(\frac{\partial^2\omega}{\partial x_1\partial x_2}\right)^2-\frac{\partial^2\omega}{\partial x_1^2}\frac{\partial^2\omega}{\partial x_2^2}\right]$ complicates the energy. For a clamped plate with either a rectangular domain or a smooth boundary, however, the Gaussian curvature integral is zero [12].

Elimination of the Gaussian curvature integral and differentiation of (60) produces

$$\begin{aligned}\dot{E} = & \int_\Omega\left[\dot{\omega}\left(f-b\dot{\omega}-D\Delta^2\dot{\omega}-D_E\Delta^2\omega\right)\right. \\ & \left.+D_E\Delta\omega\Delta\dot{\omega}\right]dx \\ \leq & -\left(b+\frac{D}{2m_1^2}-\delta_1\right)\int_\Omega\dot{\omega}^2 dx \\ & +\frac{1}{\delta_1}\int_\Omega f^2 dx-\frac{D}{2}\int_\Omega(\Delta\dot{\omega})^2 dx, \qquad (61)\end{aligned}$$

where (2), (3), (7), and (8) are used.

Both viscous and material damping match the disturbance input, producing a negative kinetic energy term in $\dot{E}$. The energy cannot be used to prove stability, however, because the time derivative lacks the $-\int_\Omega(\Delta\omega)^2 dx$ term that is found in E . We therefore add the crossing term in (17) to form positive functional (16). The crossing term can be bounded by (18), where

$$\eta = \frac{\rho\max\left(1,m_1^2\right)}{\min(\rho,D_E)}. \qquad (62)$$

The time derivative of the crossing term (17) has the form of (26) with

$$\dot{C}_3 \leq D\delta_4 \int_\Omega (\Delta\omega)^2\, dx + \frac{D}{\delta_4}\int_\Omega (\Delta\dot{\omega})^2\, dx \tag{63}$$

and

$$\dot{C}_4 = -D_E \int_\Omega (\Delta\omega)^2\, dx. \tag{64}$$

Substitution of (22), (23), (63), and (64) into (26) produces

$$\begin{aligned}\dot{V} \leq\ & -\left[b + \frac{D}{2m_1^2} - \delta_1 - \beta\left(\rho + \frac{b}{\delta_3}\right)\right]\int_\Omega \dot{\omega}^2 dx \\ & -\beta\left[D_E - (\delta_2 + b\delta_3)\, m_1^2 - D\delta_4\right]\int_\Omega (\Delta\omega)^2\, dx \\ & -D\left(\frac{1}{2} - \frac{\beta}{\delta_4}\right)\int_\Omega (\Delta\dot{\omega})^2\, dx \\ & +\left(\frac{1}{\delta_1} + \frac{\beta}{\delta_2}\right)\int_\Omega f^2 dx \\ \leq\ & -\lambda_3 E + \varepsilon, \end{aligned} \tag{65}$$

where, for sufficiently small β, δ_1, δ_2, δ_3, and δ_4,

$$\frac{1}{2} \geq \frac{\beta}{\delta_4}, \tag{66}$$

$$\varepsilon_1 = b + \frac{D}{2m_1^2} - \delta_1 - \beta\left(\rho + \frac{b}{\delta_3}\right) > 0, \tag{67}$$

$$\varepsilon_2 = \beta\left[D_E - (\delta_2 + b\delta_3)\, m_1^2 - D\delta_4\right] > 0, \tag{68}$$

$$\varepsilon = \left(\frac{1}{\delta_1} + \frac{\beta}{\delta_2}\right)\max_{t\in[0,\infty)}\int_\Omega f^2 dx < \infty, \tag{69}$$

$$\lambda_3 = \frac{\min(\varepsilon_1, \varepsilon_2)}{\max(\rho, D_E)} > 0. \tag{70}$$

Therefore, (35) holds and the system is weakly stable. □

B. Boundary Damped Plates

For the boundary clamped plate model, the viscous and material damping are removed from the field equation and clamped boundary condition is changed to a damper on Γ_1. The field equation and boundary conditions are

$$\rho\ddot{\omega} + D_E\Delta^2\omega = f \quad in\ \Omega \times R_+, \tag{71}$$

$$\omega = 0\ on\ \Gamma_0 \times R_+, \tag{72}$$

$$\frac{\partial}{\partial n}\omega = 0\ on\ \Gamma_0 \times R_+, \tag{73}$$

$$\Delta\omega = 0\ on\ \Gamma_1 \times R_+, \tag{74}$$

$$D_E\frac{\partial}{\partial n}\Delta\omega - c\dot{\omega} = 0\ on\ \Gamma_1 \times R_+, \tag{75}$$

and the initial conditions are given in (12) and (13)

Theorem 4: The response of the boundary damped plate governed by (71) - (75) is bounded if $c > 0$, $f \in L_\infty$, and the normal boundary conditions (39) and (40) are satisfied.

V. CONCLUSION

This paper shows that distributed and boundary damping can ensure bounded response for pinned membranes and clamped plates under distributed excitation. Either external, viscous damping or internal, material damping ensures weak stability with respect to the energy norm. The distributed input can include spatial and time variations provided it is L_2 spatially and L_∞ temporally bounded, respectively. Thus, time-bounded point forces are allowed because they have a bounded L_2 spatial norm. Boundary damping must satisfy the normal boundary conditions (39) and (40) to ensure stability. Circular and rectangular domains satisfy these conditions with damping on half and one side, respectively. For each of the cases studied, $\varepsilon = 0$ if $f = 0$ so without inputs these systems are weakly exponentially stable.

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