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Transforming a Nonlinear System to an Extended Brunovsky Canonical Form without Feedback

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Abstract—**Using the differential geometric control theory, we present the necessary and sufficient conditions under which an analytic affine nonlinear system with multiple inputs is equivalent to, via state transformation without feedback, a kind of extended Brunovsky canonical form. The proof is based on a new technique for applying the Frobenius Theorem in a more convenient way by converting a set of quasilinear partial differential equations with the same main portion to a set of homogeneous linear partial differential equations.**

I. Introduction

In the middle of 1990s, the progress on the equivalence of nonlinear system achieved by Celikovsky and Nijmeijer [1] indicates the beginning of the new research of transforming a single input nonlinear system to a high-order normal form. Cheng and Ling [2] presented a necessary and sufficient condition, with a feedback $u=\alpha(\boldsymbol{x})+v$, and Respondek [3], with another feedback $u=\alpha(\boldsymbol{x})+\beta(\boldsymbol{x})v$, of single-input affine systems transformed to *p*-normal form systems.

This work aims to equivalently transform a multi-input nonlinear system to a kind of extended Brunovsky canonical forms (2) via state transformation without feedback. The single-input case of the kind of extended Brunovsky canonical forms is a special class of *p*-normal forms.

II. Main Result

Consider an affine system

$$\Sigma':\quad \dot{\xi}=\boldsymbol{f}(\xi)+\boldsymbol{g}(\xi)\boldsymbol{u} \tag{1}$$

where $\xi=(\xi_1,\cdots,\xi_n)$, $\boldsymbol{u}=(u_1,\cdots u_m)$, $\boldsymbol{f}=(f_1,\cdots,f_n)^T$ is a smooth vector field, $\boldsymbol{g}=(\boldsymbol{g}_1,\cdots,\boldsymbol{g}_m)$, and, for all $1\le i\le m$, $\boldsymbol{g}_i=(g_{i,1},\cdots,g_{i,n})^T$ is again a smooth vector field. The problem considered in this note is when Σ' can be transformed to the following extended Brunovsky canonical form only via coordinate transformation:

$$\Sigma:\dot{x}_{i,1}=x_{i,2}^{p_{i,1}},\cdots,\dot{x}_{i,r_i-1}=x_{i,r_i}^{p_{i,r_i-1}},\dot{x}_{i,r_i}=u_i \tag{2}$$

where $1\le i\le m$, $\sum_{i=1}^{m} r_i=n$, $\boldsymbol{g}_{0,i}=(\underbrace{0,\cdots,0}_{r_1+\cdots+r_i-1},1,0,\cdots,0)$ and $\boldsymbol{f}_0=(x_{1,2}^{p_{1,1}},\cdots,x_{1,r_1}^{p_{1,r_1-1}},0,\cdots\cdots,x_{m,2}^{p_{m,1}},\cdots,x_{m,r_m}^{p_{m,r_m-1}},0)^T$ are smooth vector fields. Denote $\boldsymbol{g}_0=(\boldsymbol{g}_{0,1},\cdots,\boldsymbol{g}_{0,m})$, then (2) can be rewritten as $\dot{\boldsymbol{x}}=\boldsymbol{f}_0(\boldsymbol{x})+\boldsymbol{g}_0(\boldsymbol{x})\boldsymbol{u}$.

Define a set of vector fields for the system Σ'

$$\boldsymbol{X}_1^i=\boldsymbol{g}_i,\cdots,\boldsymbol{X}_{k_i+1}^i=ad_{\boldsymbol{X}_{k_i}^i}^{p_{i,r_i-k_i+1}}\boldsymbol{f}\quad 1\le i\le m,1\le k_i\le r_i \tag{3}$$

Theorem 1: Σ' can be transformed to Σ via coordinate transformation $\boldsymbol{x}=\boldsymbol{T}(\xi)=(T_1(\xi),\cdots,T_n(\xi))$ near $0\in\mathbb{R}^n$ without feedback, if and only if in a neighborhood of $0\in\mathbb{R}^n$, the following conditions satisfy:

(C1) $\boldsymbol{X}_1^1,\cdots,\boldsymbol{X}_{r_1}^1,\cdots,\boldsymbol{X}_1^m,\cdots,\boldsymbol{X}_{r_m}^m$ is full rank

(C2) $[\boldsymbol{X}_{j_1}^{i_1},\boldsymbol{X}_{j_2}^{i_2}]=0\quad 1\le i_1,i_2\le m,1\le j_1\le r_{i_1},1\le j_2\le r_{i_2}$

(C3) By (C1), we assume

$$\boldsymbol{f}=\sum_{i=1}^{m}\left(\alpha_1^i(\xi)\boldsymbol{X}_1^i+\cdots+\alpha_{r_i}^i(\xi)\boldsymbol{X}_{r_i}^i\right) \tag{4}$$

then for all $1\le i\le m$ and $1\le j\le r_i-1$

$$\phi_j^i=\alpha_j^i(\xi)-x_{i,j+1}^{p_{i,j}},\phi_{r_i}^i=\alpha_{r_i}^i(\xi) \tag{5}$$

are solutions of the equations

$$\begin{aligned}&\sum_{j=1}^{n}f_j(\xi)\frac{\partial\phi}{\partial\xi_j}+\sum_{i=1}^{m}\left(x_2^{p_1}\frac{\partial\phi}{\partial x_{i,1}}+\cdots+x_{n-1}^{p_{n-1}}\frac{\partial\phi}{\partial x_{i,r_i}}\right)=0\\&\sum_{j=1}^{n}g_{i,j}(\xi)\frac{\partial\phi}{\partial\xi_j}+\frac{\partial\phi}{\partial x_{r_1+\cdots+r_i}}=0\quad 1\le i\le m\end{aligned} \tag{6}$$

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where $\phi = \phi(\xi, x)$ is the function of $\mathbb{R}^{2n} \to \mathbb{R}$.

Proof: (Sufficiency): Suppose Σ' and Σ are equivalent to each other via coordinate transformation $x = x(\xi) = (x_1(\xi), \cdots, x_n(\xi))$ without feedback, for all $1 \le i \le m$

$$\begin{aligned} & f_1(\xi)\frac{\partial x_{i,j}}{\partial \xi_1} + \cdots + f_n(\xi)\frac{\partial x_{i,j}}{\partial \xi_n} = x_{i,j+1}^{p_{i,j}} \quad 1 \le j \le r_i - 1 \\ & f_1(\xi)\frac{\partial x_{i,r_i}}{\partial \xi_1} + \cdots + f_n(\xi)\frac{\partial x_{i,r_i}}{\partial \xi_n} = 0; \end{aligned} \tag{7}$$

and for all $1 \le i \le m$

$$\begin{aligned} & g_{i,1}(\xi)\frac{\partial x_{k,j}}{\partial \xi_1} + \cdots + g_{i,n}(\xi)\frac{\partial x_{k,j}}{\partial \xi_n} = 1 \text{ for } i = k \wedge j = r_i \\ & g_{i,1}(\xi)\frac{\partial x_{k,j}}{\partial \xi_1} + \cdots + g_{i,n}(\xi)\frac{\partial x_{k,j}}{\partial \xi_n} = 0 \text{ otherwise} \end{aligned} \tag{8}$$

where $1 \le k \le m$ and $1 \le j \le r_k$.

Notice that all of quasi-linear partial differential equations described as (7) and (8) are equivalent to the set of homogeneous linear partial differential equations (6) [4]. By (6), define $m+1$ $2n$-dimensional smooth vector fields

$$\begin{aligned} & \boldsymbol{F} = \Big(f_1(\xi), \cdots f_n(\xi), x_{1,2}^{p_{1,1}}, \cdots, x_{1,r_1}^{p_{1,r_1-1}}, 0, \\ & \quad \cdots\cdots, x_{m,2}^{p_{m,1}}, \cdots, x_{m,r_m}^{p_{m,r_m-1}}, 0 \Big)^T, \\ & \boldsymbol{G}_i = \Big(g_{i,1}(\xi), \cdots, g_{i,n}(\xi), \underbrace{0, \cdots, 0}_{r_1+\cdots+r_i+1}, 1, 0, \cdots, 0 \Big)^T, \\ & 1 \le i \le m. \end{aligned} \tag{9}$$

Denote the n-dimensional tangent vector field $\boldsymbol{V}_i = (\underbrace{0, \cdots, 0}_{i-1}, 1, 0, \cdots, 0)$; Assume that $\boldsymbol{a} = (a_1, \cdots, a_k)^T$ and $\boldsymbol{b} = (b_1, \cdots, b_l)^T$ are two vector fields, denote $\mathrm{col}(\boldsymbol{a}, \boldsymbol{b}) = (a_1, \cdots, a_k, b_1, \cdots, b_l)^T$. From (C1), (C2) and (C3), define and compute a set of vector fields

$$\begin{aligned} & \boldsymbol{Y}_1^i = \boldsymbol{G}_i = \mathrm{col}(\boldsymbol{X}_1^i, \boldsymbol{V}_{r_1+\cdots+r_i}), \\ & \boldsymbol{Y}_2^i = ad_{\boldsymbol{Y}_1^i}^{p_{i,r_i-1}} \boldsymbol{F} = \mathrm{col}(ad_{\boldsymbol{X}_1^i}^{p_{i,r_i-1}} \boldsymbol{f}, ad_{\boldsymbol{V}_{r_1+\cdots+r_n}}^{p_{i,r_i-1}} \boldsymbol{f}_0) \\ & = \mathrm{col}(\boldsymbol{X}_2^{p_{i,r_i-1}}, p_{i,r_i-1}! \boldsymbol{V}_{r_1+\cdots+r_i-1}), \\ & \cdots \\ & \boldsymbol{Y}_{r_i}^i = ad_{\boldsymbol{Y}_{n-1}^i}^{p_{i,1}} \boldsymbol{F} = \mathrm{col}(ad_{\boldsymbol{X}_{n-1}^i}^{p_{i,1}} \boldsymbol{f}, ad_{\boldsymbol{V}_{r_1+\cdots+r_n}}^{p_{i,1}} \boldsymbol{f}_0) \\ & = \mathrm{col}(\boldsymbol{X}_n^{p_{i,1}}, p_{i,1}! \cdots p_{i,r_i-1}! \boldsymbol{V}_{r_1+\cdots+r_{i-1}+1}), \\ & 1 \le i \le m. \end{aligned} \tag{10}$$

Obviously, for all integers $0 \le i, j \le n-1$, $[\boldsymbol{Y}_i, \boldsymbol{Y}_j] = 0$ are satisfied. By Frobenius Theorem, first-order linear partial differential equations

$$< d\phi, \boldsymbol{Y}_1^i >= 0, \cdots, < d\phi, \boldsymbol{Y}_{r_i}^i >= 0 \quad 1 \le i \le m \tag{11}$$

have n linearly independent solutions. In fact (6) is

$$< d\phi, \boldsymbol{F} >= 0, \ < d\phi, \boldsymbol{G}_i >=< d\phi, \boldsymbol{Y}_1^i >= 0 \ \ 1 \le i \le m \tag{12}$$

Because of (C1), $\{\boldsymbol{X}_0, \boldsymbol{X}_1, \cdots, \boldsymbol{X}_{n-1}\}$ is full rank, then for $1 \le i \le m$, there uniquely exists a set of smooth functions $\alpha_1^i(\xi), \cdots, \alpha_{r_1}^i(\xi)$, which satisfies:

$$\boldsymbol{f} = \sum_{i=1}^{m} \left(\alpha_1^i(\xi) \boldsymbol{X}_1^i + \cdots + \alpha_{r_i}^i(\xi) \boldsymbol{X}_{r_i}^i \right) \tag{13}$$

and meanwhile

$$\boldsymbol{f}_0 = \sum_{i=1}^{m} \left(x_{i,2}^{p_{i,1}} \boldsymbol{V}_1 + \cdots + x_{i,r_i}^{p_{i,r_i-1}} \boldsymbol{V}_{r_1-1} + 0 \cdot \boldsymbol{V}_{r_1} \right) \tag{14}$$

Then there must be

$$\alpha_j^i(\xi) - x_{i,j+1}^{p_{i,j}} = 0, \alpha_{r_i}^i(\xi) = 0 \tag{15}$$

for all $1 \le i \le m$ and $1 \le j \le r_i - 1$.

By (C3) and theories of partial differential equation, (15) is the first integral which satisfies (7) and (8). Then we can conclude that (7) and (8) have n linearly independent solutions if they satisfy (C1-C3).

(Necessity): Since Lie bracket satisfies $\boldsymbol{T}_*[\boldsymbol{X}, \boldsymbol{Y}] = [\boldsymbol{T}_*\boldsymbol{X}, \boldsymbol{T}_*\boldsymbol{Y}]$, where $\boldsymbol{T}_*$ is the tangent mapping induced by coordinate transform $\boldsymbol{T}$.

So the necessities of (C1) and (C2) only need to be proved in Σ, and they are obviously satisfied in Σ. Suppose (C3) is not satisfied and $\mathrm{span}\{\boldsymbol{Y}_1^i, \cdots, \boldsymbol{Y}_n^i \mid 1 \le i \le m\}$ is not the involution closure of $\mathrm{span}\{\boldsymbol{F}, \boldsymbol{G}_i \mid 1 \le i \le m\}$, then the dimension of the solution space of (6) is less than n, which indicates that the n linearly independent solutions can not be found. Hence (C3) is necessary.

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