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A Novel Method of Nonlinear System-Simulation with Uncertain Parameters Providing Guaranteed Bounds

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***Abstract*— Uncertain or unknown parameters are often an essential part in several biological or technical applications represented by nonlinear systems. These uncertainties cause numerical and analytical problems in finding guaranteed bounds for the solution of the state space representation for such systems. Unfortunately several industrial applications are demanding exactly these guaranteed bounds in order to fulfil regulations by the state authorities.**

A common and well known method to perform simulations with uncertain parameters is the Monte-Carlo method. This method with its stochastic approach cannot deliver guaranteed bounds. Methods using interval arithmetic provide a lot of overestimation.

Thus we have to develop a novel method to find guaranteed bounds as an initial interval for further methods based on interval arithmetic. In this scope a new method is presented which is using a linear Lyapunov like functions to solve this problem. We achieve guaranteed and finite simulation bounds as a result of our approach. The idea is to find an auxiliary function, which helps to bind the state variables. An example from an industrial application completes the paper.

I. Introduction

The simulation of nonlinear systems is demanded by many technical, biological and chemical systems. Also many industrial applications ask for simulation results of nonlinear systems with uncertain or unknown parameters. This is the main reason, why we have developed a novel method to get guaranteed bounds for the above described systems.

The simulation for such systems cannot be done in one calculation cycle as it is done in the case of known parameters. The time depending set of all possible simulated state variables with all parameter combinations is required. For this purpose we present a new method.

We consider a system of nonlinear ordinary differential equations which represents an uncertain dynamical system in state space representation

$$\dot{x}(t) \quad = \quad f(x(t),u,p) \text{ with } x(0)=x_0, \tag{1}$$

where the vector $x \in \mathbb{R}^n$ and function $f(x(t),u,p)$ represent the time dependent state vector and the nonlinearity, respectively. The input vector is represented by $u \in \mathbb{R}^m$. The vector of uncertain or unknown parameters is given by $p \in \mathbb{R}^q$ which is contained in an interval vector, in other words the uncertain parameters can vary between a lower bound $\underline{p}$ and an upper bound $\overline{p}$ with $p = [\underline{p}, \overline{p}]$.

II. New Simulation Method

The state space representation (1) for the system class under consideration is assumed in the form

$$\dot{x}(t) \quad = \quad f(x(t),u,p) = A\,x(t) + b\,\eta\,(x(t),u) + g \tag{2}$$

where $A \in \mathbb{R}^{n\times n}$ is the system matrix of the linear part, the function $\eta\,(x(t),u), \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}$ represents the nonlinear part, $b \in \mathbb{R}^n$ is a constant vector and the vector g contains all constant parts that do not depend on the state variables, inputs or parameters.

We assume further that the system given by (2) leaves the positive orthant of $\mathbb{R}^n$ invariant. Thus, we have $x(t) \geq 0 \quad \forall \quad t$ if $x(0) \geq 0$ is fulfilled. Our goal is to calculate easily computable upper bounds on the state variables. This is a difficult problem which can in principle be solved through the Monte-Carlo Method [1], [2] with the drawback that the bounds are not guaranteed. A different approach would be to use the theory of differential inequalities [3], [4] which may lead to unstable upper bounds. A recent method is interval arithmetic [5], [6], [8], [9], which also may lead to unstable upper bounds.

In order to overcome these problems we define a linear Lyapunov like function

$$v(t) \quad = \quad c^T\,x(t), \tag{3}$$

with $c^T = (c_1, \cdots, c_n)$ and $c_i \geq 0$. Now we compute the time derivative of (3) along the trajectories of (2).

This results in

$$\begin{aligned} \dot{v}(t) &= c^T\,\dot{x}(t) \\ &= c^T(A\,x(t) + b\,\eta\,(x(t),u) + g) \\ &= c^T\,A\,x(t) + c^T\,b\,\eta\,(x(t),u) + c^T\,g. \end{aligned} \tag{4}$$

If we now assume

$$c^T\,b \quad = \quad 0 \text{ and} \tag{5}$$

$$\text{all components } (c^T A)_i \quad < \quad 0 \text{ with } i = 1, \cdots, n, \tag{6}$$

we compute

$$\dot{v}(t) \quad = \quad (c^T A)\,x(t) + c^T g, \tag{7}$$

thus the nonlinear function η is eliminated. Due to the fact that $x(t) > 0\ \forall\ t$ and the assumption (6) on $c^T A$ we can bound the right hand side of (7). This results in

$$
\begin{aligned}
\dot{v}(t) &= c^T g - \left[\frac{-c^T A x(t)}{v(t)}\right] \cdot v(t) \\
\dot{v}(t) &= c^T g - \left[\frac{-c^T A x(t)}{c^T x(t)}\right] \cdot v(t) \\
\dot{v}(t) &= c^T g - \gamma v(t) \qquad (8) \\
\text{with } \gamma &= \frac{-c^T A x(t)}{c^T x(t)}. \qquad (9)
\end{aligned}
$$

Our aim is to bound $\dot{v}(t)$, thus we have to minimize γ, according to

$$
\begin{aligned}
\gamma &= \min_{x>0, x\neq 0} \left(-\frac{c^T A x}{c^T x}\right), \text{ which is equivalent to} \\
\gamma &= \min_{x>0, c^T x=1} \left(-c^T A x\right). \qquad (10)
\end{aligned}
$$

The linear optimization problem (10) has the solution

$$
\gamma = \min\left(\frac{-(c^T A)_1}{c_1}, \frac{-(c^T A)_2}{c_2}, \cdots, \frac{-(c^T A)_n}{c_n}\right). \quad (11)
$$

Now we are in the position to find an upper bound for the linear Lyapunov like function itself. Therefore we have to compute a solution for the differential inequality

$$
\dot{v}(t) \le c^T g - \gamma \cdot v(t). \qquad (12)
$$

Using the Gronwall Lemma we calculate an upper bound as

$$
v(t) \le v(0) e^{-\gamma t} + c^T g \int_0^t e^{-\gamma(t-\tau)} d\tau \qquad (13)
$$

which results in

$$
v(t) \le v(0) e^{-\gamma t} + \frac{1}{\gamma} c^T g \left(1 - e^{-\gamma t}\right) = \tilde{v}(t) . \qquad (14)
$$

Thus, $\tilde{v}(t)$ is an upper bound for the linear Lyapunov like function $v(t)$. As a result of the guaranteed bound (14) for the linear Lyapunov like function the state variables are also bounded. For each state variable the bound is given by

$$
0 \le x_i \le \frac{\tilde{v}_i(t)}{c_i} \quad \forall \quad i = 1, \cdots, n . \qquad (15)
$$

This equation represents the guaranteed bounds for the state variables of the given nonlinear system.

III. Optimizing the Bounds

The essential part of the problem is done. Guaranteed bounds for nonlinear systems with unknown or uncertain parameters are found. But we can do more. We use c to optimize the bounds for time to infinity ($t \to \infty$). According to (15) we find guaranteed bounds for the state variables as $0 \le x_i(t) \le \frac{\tilde{v}_i(t)}{c_i}$. So we can formulate the mimization of each state variable x_i for infinite time durations as an optimization problem

$$
\begin{aligned}
&\min_c \left(\lim_{t\to\infty} \left(\frac{\tilde{v}_i(t)}{c_i}\right)\right) \text{ subject to} \qquad (16) \\
c^T b &= 0, \\
c &> 0, \\
(c^T A)_i &< 0 \text{ for } i = 1, \cdots, n \text{ and} \\
\gamma &= \min_i \left(\frac{-(c^T A)_i}{c_i}\right). \qquad (17)
\end{aligned}
$$

The constrained optimization formulated in (16) leads exactly to a vector c for optimal bounds of the state variables for at $t \to \infty$. Moreover $\tilde{v}_i(t)$ is to be simplified as

$$
\lim_{t\to\infty} \tilde{v}_i(t) = \lim_{t\to\infty} \left(v(0) \underbrace{e^{-\gamma t}}_{\to 0} + \frac{1}{\gamma} c^T g - \frac{1}{\gamma} c^T g \underbrace{e^{-\gamma t}}_{\to 0}\right) = \frac{c^T g}{\gamma}.
$$

Using this the optimization problem is rewritten as

$$
\begin{aligned}
&\min_c \left(\frac{c^T g}{\gamma c_i}\right) \text{ subject to} \qquad (18) \\
c^T b &= 0, \\
c &> 0, \\
(c^T A)_i &< 0 \text{ for } i = 1, \cdots, n \text{ and} \\
\gamma &= \min_i \left(\frac{-(c^T A)_i}{c_i}\right).
\end{aligned}
$$

The last constraint $\gamma = \min_i \left(\frac{-(c^T A)_i}{c_i}\right)$ is non differentiable and we rewrite the problem as an extended optimization problem. This results in

$$
\begin{aligned}
&\min_{c,\hat{\gamma}} \left(\frac{c^T g}{\hat{\gamma} c_i}\right) \text{ subject to} \qquad (19) \\
c^T b &= 0, \\
c &> 0, \\
\text{all components of } (c^T A)_i &< 0 \text{ for } i = 1, \cdots, n \text{ and} \\
\hat{\gamma} &\le \left(\frac{-(c^T A)_i}{c_i}\right) \text{ for } i = 1, \cdots, n.
\end{aligned}
$$

For each state variable x_i a single constrained optimization has to be done and provides optimal c and $\hat{\gamma}$. The results lead to optimal bounds of the specific state variable. Moreover these bounds can be used as a starting point for other methods e.g. interval arithmetic. These methods profit a lot from optimal bounds, because overestimation is a serious problem in simulations of nonlinear systems with uncertain parameters. In the following we will present an example, and show how the presented method is applied to a real industrial application.

IV. Waste Water Treatment Example

A. Introduction

Sewage is treated in order to remove and decompose substances that are harmful to the natural environment. The substances are either directly decomposed or first separated and then decomposed, by using or promoting natural purification action. In other words, a treatment plant is built on separation and decomposition technologies. There is an large number of micro organisms inhabiting the natural environment that decompose organic substances. Biological treatment creates and maintains a habitat for these microbes. It promotes the proliferation of as many microbes as possible in a restricted space in order to decompose organic substances. The environmental capacity of a river is based on the number of microbes that live in it.

B. Active Sludge Model

Now we have do define a model for the above introduced waste water treatment. The main part of treatment plants is the final sedimentation. In the following we use a reduced system of differential equations [Fig.1] of the Active Sludge Model (ASM) [12]. The ASM describes the biological and chemical reactions in waste water treatment plants final sedimentation.

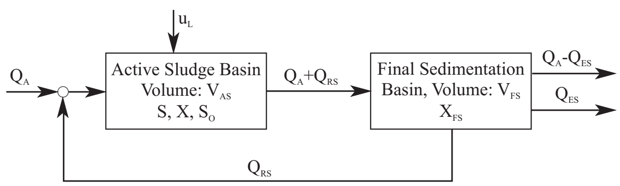

Fig. 1. Reduced model of a waste water treatment system.

The system of differential equations [12] is defined as

$$\dot{S}(t) = \frac{Q_A}{V_{AS}}(S_A - S(t)) - \mu(t)\frac{1}{Y_H}X(t) \tag{20}$$

$$\dot{X}(t) = -\frac{Q_A}{V_{AS}}X(t) + \frac{Q_{RS}}{V_{AS}}(X_{FS}(t) - X(t)) + (\mu(t) - b_H)X(t) \tag{21}$$

$$\dot{S}_O(t) = \frac{Q_A}{V_{AS}}(S_{OA} - S_O(t)) - \frac{1-Y_H}{Y_H}\mu(t)X(t) + \left(1 - \frac{S_O(t)}{S_{O,sat}}\right)u_L \tag{22}$$

$$\dot{X}_{FS}(t) = \frac{Q_A + Q_{RS}}{V_{FS}}X(t) - \frac{Q_{ES} + Q_{RS}}{V_{FS}}X_{FS}(t) \tag{23}$$

with $\mu(t) = \hat{\mu}_H \frac{S(t)}{S(t)+K_S} \frac{S_O(t)}{S_O(t)+K_{OS}}$ as an abbreviation for the Monod kinetic (reduced growth due to nutrient limitations). In the model the state variables are the bacteria concentrations X (active sludge) and X_{FS} (final sedimentation), as well as the active sludge substrate S and oxygen concentrations S_O. S_A and S_{OA} represent the influent substrate and oxygen concentration. In this example the constant terms are the flow rates Q_A, Q_{RS} and Q_{ES}, the basin volumes V_{AS} and V_{FS}, the oxygen saturation $S_{O,sat}$ and the aeration u_L. The uncertain parameters are the heterotrophic yield Y_H, the half-saturation coefficients K_S and K_{OS}, as well as the heterotrophic decay rate b_H and the maximum specific growth rate $\hat{\mu}_H$.

To compute reliable upper bounds for the bacteria concentrations as well as for the substrate concentration the differential equations (20), (21) and (22) are used. the system is positive and thus all state variables are ≥ 0.

C. New Simulation Method Appliance

Using the notation of equation (2) the matrices and vectors follow from the model (20), (21) and (23) as

$$x_a(t) = \begin{pmatrix} S(t) & X(t) & X_{FS}(t) \end{pmatrix}^T \tag{24}$$

$$A_a = \begin{pmatrix} -\frac{Q_A}{V_{AS}} & 0 & 0 \\ 0 & -\frac{Q_A+Q_{RS}}{V_{AS}} - b_H & \frac{Q_{RS}}{V_{AS}} \\ 0 & \frac{Q_A+Q_{RS}}{V_{FS}} & -\frac{Q_{ES}+Q_{RS}}{V_{FS}} \end{pmatrix} \tag{25}$$

$$b_a = \begin{pmatrix} -\frac{1}{Y_H} & 1 & 0 \end{pmatrix}^T \tag{26}$$

$$g_a = \begin{pmatrix} \frac{Q_A}{V_{AS}}S_A & 0 & 0 \end{pmatrix}^T. \tag{27}$$

The nonlinear function is defined as

$$\eta(t) = \hat{\mu}_H \frac{S(t)}{S(t)+K_S} \frac{S_O(t)}{S_O(t)+K_{OS}} X(t) = \mu(t)X(t). \tag{28}$$

We are applying the constrained optimization

$$\min_{c_a,\hat{\gamma}_a} \left(\frac{c_a^T g_a}{\hat{\gamma}_a c_{ai}}\right) \text{ subject to} \tag{29}$$

$$\begin{aligned} c_a^T b &= 0, \\ c_a &> 0, \\ (c_a^T A)_i &< 0 \text{ for } i = 1, \cdots, 3 \text{ and} \\ \hat{\gamma}_a &\leq \left(\frac{-(c_a^T A)_i}{c_{a_i}}\right) \text{ for } i = 1, \cdots, 3. \end{aligned}$$

The constrained optimization leads to a certain vector c_a with its optimal $\hat{\gamma}_a$, which deliver optimal bounds for a state variable simulation. In fact the results of the constrained optimization are the same for the state variables of the active sludge bacteria concentration X and bacteria concentration in final sedimentation X_{FS}. In this case the vector c_a is resulting in

$$c_a^T = \begin{pmatrix} V_{AS}Y_H & V_{AS} & V_{FS} \end{pmatrix}.$$

The vector c_a for an optimal bound of the active sludge substrate state variable is calculated using the `fmincon` function from the Matlab Optimization Toolbox. In this case we will get a numerical solution and the values are

$$c_a^T = \begin{pmatrix} 0.14344 & 0.21409 & 0.64246 \end{pmatrix}.$$

In section 2 we proved the existence of a bounded $\tilde{v}_a(t)$ which can be used to determine the range of the state variables. According to (14) $\tilde{v}_a(t)$ is derived as

$$\tilde{v}_a(t) = v(0)e^{-\gamma_a t} + \frac{1}{\gamma_a} c_a^T g \left(1 - e^{-\gamma_a t}\right).$$

According to the constrained optimization (16) the optimal value for γ_a results depending on the state variables to

$$\begin{aligned}\gamma_a &= 0.003960 \text{ for } X \text{ and } X_{FS},\\ \gamma_a &= 0.012046 \text{ for } S.\end{aligned}$$

Using $\tilde{v}_a(t)$ for determining the upper bound the active sludge bacteria concentration is

$$\begin{aligned}& 0 \le X(t) \le \frac{\tilde{v}_a(t)}{c_{a_2}}\\ \Rightarrow\, & 0 \le X(t) \le \frac{v_a(0)e^{-\gamma_a t}}{c_{a_2}} + \frac{c_a^T g_a(1-e^{-\gamma_a t})}{c_{a_2}\gamma_a} \qquad (30)\\ \Rightarrow\, & 0 \le X(t) \le \frac{v_a(0)e^{-\gamma_a t}}{V_{AS}} + \frac{Y_A Q_A S_A(1-e^{-\gamma_a t})}{V_{AS}\gamma_a}. \qquad (31)\end{aligned}$$

Bounds for substrate and return-sludge bacteria concentration result in

$$\begin{aligned}& 0 \le S(t) \le \frac{\tilde{v}_a(t)}{c_{a_1}}\\ \Rightarrow\, & 0 \le S(t) \le \frac{v_a(0)e^{-\gamma_a t}}{c_{a_1}} + \frac{c_a^T g_a(1-e^{-\gamma_a t})}{c_{a_1}\gamma_a} \qquad (32)\end{aligned}$$

and

$$\begin{aligned}& 0 \le X_{FS}(t) \le \frac{\tilde{v}_a(t)}{c_{a_3}}\\ \Rightarrow\, & 0 \le X_{FS}(t) \le \frac{v_a(0)e^{-\gamma_a t}}{c_{a_3}} + \frac{c_a^T g_a(1-e^{-\gamma_a t})}{c_{a_3}\gamma_a}\\ \Rightarrow\, & 0 \le X_{FS}(t) \le \frac{v_a(0)e^{-\gamma_a t}}{V_{FS}} + \frac{Y_A Q_A S_A(1-e^{-\gamma_a t})}{V_{FS}\gamma_a}. \qquad (33)\end{aligned}$$

In order to calculate bounds for the active sludge oxygen concentration S_O we have to use the differential equations for S_O (21), X (22) and X_{FS} (23). Thus, the state space vector and the matrix A_b as well as the vectors b_b and g_b are given by

$$\begin{aligned}x_b(t) &= \begin{pmatrix} S_O(t) & X(t) & X_{FS}(t)\end{pmatrix}^T \qquad (34)\\ A_b &= \begin{pmatrix} -\frac{Q_A}{V_{AS}} - \frac{u_L}{S_{O,sat}} & 0 & 0\\ 0 & \frac{Q_A-Q_{RS}}{V_{AS}} - b_H & \frac{Q_{RS}}{V_{AS}}\\ 0 & \frac{Q_A+Q_{RS}}{V_{FS}} & -\frac{Q_{ES}+Q_{RS}}{V_{FS}}\end{pmatrix} \qquad (35)\\ b_b &= \begin{pmatrix} -\frac{1}{\psi} & 1 & 0\end{pmatrix}^T \qquad (36)\\ g_b &= \begin{pmatrix} \frac{Q_A}{V_{AS}} S_A + u_L & 0 & 0\end{pmatrix}^T \qquad (37)\end{aligned}$$

where $\psi = \frac{Y_H}{1-Y_H}$ is used as an abbreviation. Like in the first part of this example the nonlinear function is defined as

$$\eta(t) = \hat{\mu}_H \frac{S(t)}{S(t)+K_S}\frac{S_O(t)}{S_O(t)+K_{OS}}X(t) = \mu(t)X(t). \qquad (38)$$

We achieve the c_b from the above describes constrained optimization, which results in

$$c_b^T = \begin{pmatrix} \psi \cdot V_{AS} & V_{AS} & V_{FS}\end{pmatrix}. \qquad (39)$$

From now we use the abbreviations

$$\begin{aligned}\alpha &= \psi\left[Q_A S_{OA} + V_{AS} u_L\right]\\ \beta &= -\psi\left[Q_A + V_{AS}\frac{u_L}{S_{O,satt}}\right].\end{aligned}$$

According to (14) $\tilde{v}_b(t)$ is derived as

$$\tilde{v}_b(t) = v(0)e^{-\gamma_b t} + \frac{1}{\gamma_b} c_b^T g\left(1-e^{-\gamma_b t}\right)$$

with the linear optimization result

$$\gamma_b = 0.003960.$$

Hence v_b is constrained by (14) with an upper bound $\tilde{v}_b$ which results in

$$v_b(t) \le v_b(0)e^{-\gamma_b t} + \frac{\alpha}{\gamma_b}(1-e^{-\gamma_b t}) = \tilde{v}_b(t). \qquad (40)$$

There from the upper bound for the oxygen concentration is

$$\begin{aligned}S_O(t) &= \frac{1}{\psi V_{AS}}\left\{\psi V_{AS} S_O(t)\right\} \qquad (41)\\ &\le \frac{1}{\psi V_{AS}}\underbrace{\left\{\psi V_{AS} S_O(t) + V_{AS} X(t) + V_{FS} X_{FS}(t)\right\}}_{=v_{b_1}(t)}\\ \Rightarrow\, 0 \le S_O(t) &\le \frac{\tilde{v}_{b_1}(t)}{\psi V_{AS}}\\ &= \frac{v_b(0)}{\psi V_{AS}}e^{-\gamma_b t} + \left(\frac{Q_A S_{OA}}{\gamma_b V_{AS}} + \frac{u_L}{\gamma_b}\right)(1-e^{-\gamma_b t}). \qquad (42)\end{aligned}$$

The already bounded concentrations for the bacteria in the active sludge (X) and the final sedimentation (X_{FS}) can also be bounded with the second linear Lyapunov like function. This leads to

$$\begin{aligned}0 \le X(t) &\le \frac{\tilde{v}_b(t)}{V_{AS}}\\ &= \frac{v_b(0)}{V_{AS}}e^{-\gamma_b t} + \frac{\psi}{\gamma_b}\left(\frac{Q_A S_{OA}}{V_{AS}} + u_L\right)(1-e^{-\gamma_b t}) \qquad (43)\end{aligned}$$

and

$$\begin{aligned}0 \le X_{FS}(t) &\le \frac{\tilde{v}_b(t)}{V_{FS}}\\ &= \frac{v_b(0)}{V_{FS}}e^{-\gamma_b t} + \frac{\psi}{\gamma_b}\left(\frac{Q_A S_{OA}}{V_{FS}} + \frac{V_{AS} u_L}{V_{FS}}\right)(1-e^{-\gamma_b t}). \qquad (44)\end{aligned}$$

These bounds for the active sludge and final sedimentation bacteria concentration are not as rigid as the prior determined bounds (31) and (33). Thus the bounds found using the first linear Lyapunov like function are used for the simulations.

V. Simulation Results

In this section the Monte-Carlo method [1], [2] is compared with our new method using linear Lyapunov like functions. We achieve good results for quite long time intervals such as several 1000 hours.

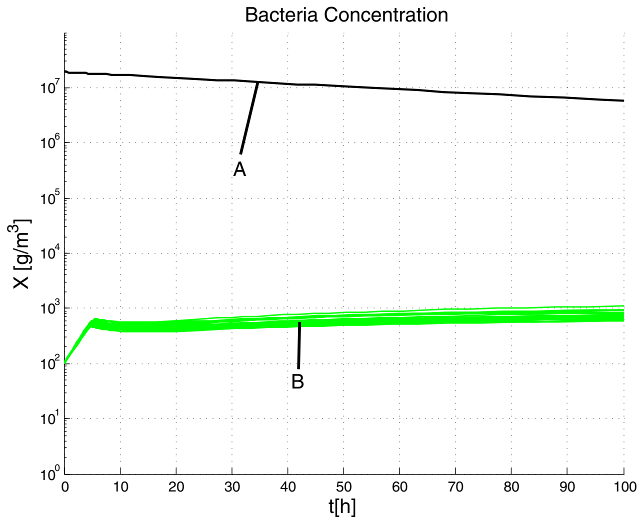

Fig. 2. Simulations of the bacteria concentration with with logarithmic scale
A: according to linear Lyapunov like functions
B: provided by the Monte-Carlo method

For these simulations the uncertain parameters of the nonlinear system are bounded as follows. The heterotrophic yield is set to $Y_H = [0.38, 0.75] \frac{gCSB}{gCSB}$, the half-saturation coefficients are $K_S = [5, 225] \frac{gCSB}{m^3}$ and $K_{OS} = [0.01, 0.2] \frac{gO_2}{m^3}$. Moreover the heterotrophic maximum growth rate is set to $\mu_H = [0.6, 13.2] \frac{1}{d}$. As one can see in Fig. 2 and Fig. 3 the upper bound provided by linear Lyapunov like functions is much higher than any of the Monte-Carlo simulation. This comes from the constrained optimization, where we ask for the best upper bound at $t \longrightarrow \infty$.

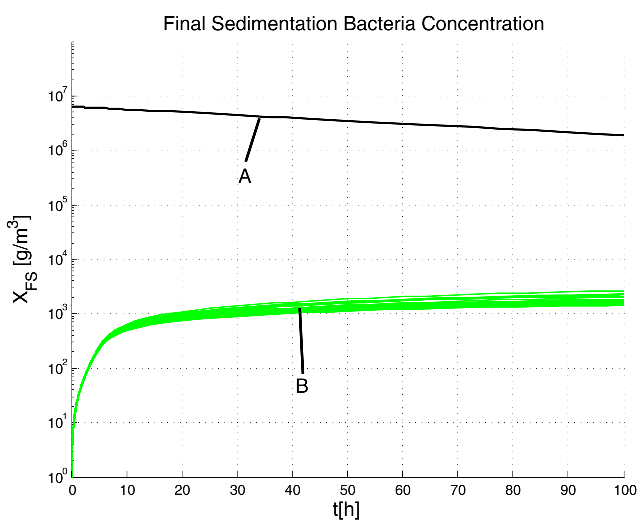

Fig. 3. Simulations of the final sedimentation bacteria concentration with logarithmic scale
A: according to linear Lyapunov like functions
B: provided by the Monte-Carlo method

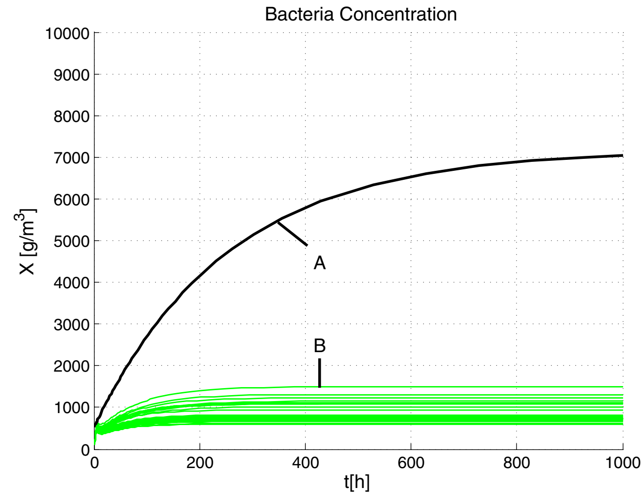

Fig. 4. Simulations of the bacteria concentration with linear scale
A: according to linear Lyapunov like functions
B: provided by the Monte-Carlo method

Fig. 4 and Fig. 5 show the simulation results for the bacteria concentration with a linear ordinate scale, which is represented by the space variable X. The simulation results, which we achieve Fig. 4, come from the simulation with vector c and $\hat{\gamma}$ that are the result of a constrained optimization for the sludge concentration S. In opposite to Fig. 4 the simulation result shown in Fig. 5 comes from a constrained optimization for the bacteria concentration X. It is possible to get in short time periods tighter bounds for a state variable, if the result of a constrained optimization for another state variable is used. Nevertheless the best bounds for $t \longrightarrow \infty$ are achieved with the described method. This is shown in Fig. 6 and Fig. 7 in logarithmic scale.

Here the same simulations are done as in Fig. 4 and

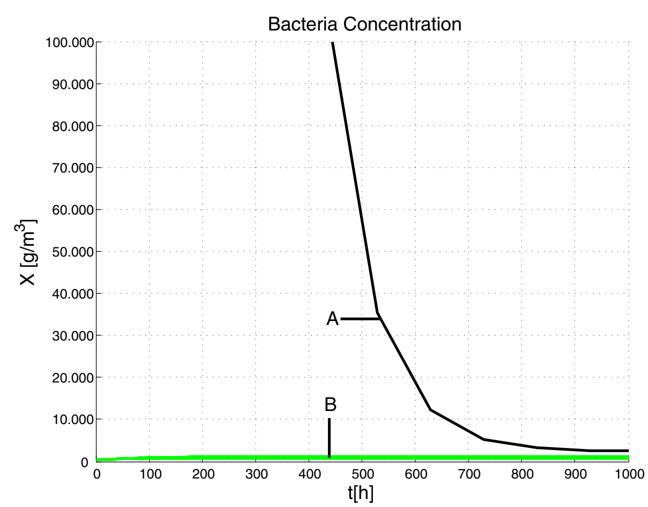

Fig. 5. Simulations of the bacteria concentration with linear scale
A: according to linear Lyapunov like functions
B: provided by the Monte-Carlo method

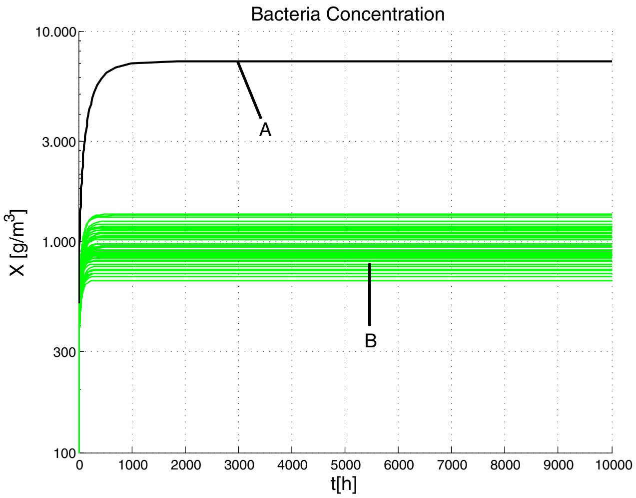

Fig. 6. Simulations of the bacteria concentration with logarithmic scale
A: according to linear Lyapunov like functions
B: provided by the Monte-Carlo method

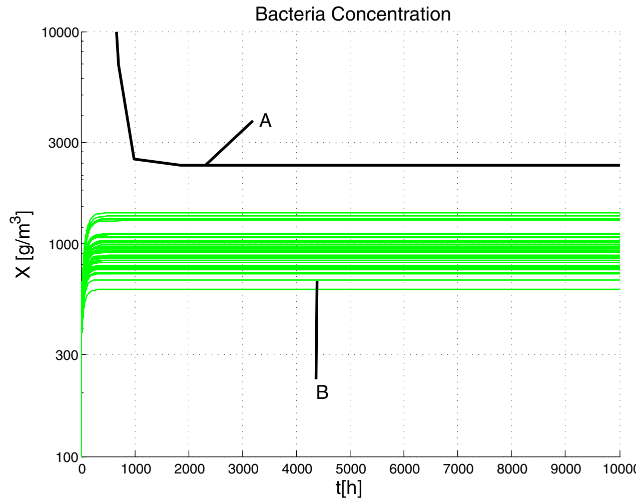

Fig. 7. Simulations of the bacteria concentration with logarithmic scale
A: according to linear Lyapunov like functions
B: provided by the Monte-Carlo method

Fig. 5, in other words the simulation for Fig. 6 is done with the result of a constrained optimization for the sludge concentration S. Ihe best bound for bacteria concentration X is shown in Fig. 7.

VI. Conclusion

In this paper we propose a new algorithm for a guaranteed simulation of nonlinear systems with uncertain parameters, which uses linear Lyapunov like functions. To achieve optimal bounds for $t \longrightarrow \infty$ we apply a constrained optimization. A short overview on the algorithm and the mathematical application has been given. We have implemented the algorithms in Matlab [11] to perform the simulations, calculating the optimal vector c by using the Optimization Toolbox and to generate the figures. Using this algorithm guaranteed simulation of nonlinear systems with uncertain parameters is computed. It is demonstrated for a nonlinear waste water treatment system. Future research will concentrate on the improvement of the algorithm. For future work it is possible to use other constraints for the optimization, thus the upper bound in shorter time intervals are smaller. Another approach is to perform several steps with modified Lyapunov like functions in order to decrease the size of the resulting set of guaranteed simulations.

VII. Acknowledgement

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