2005 American Control Conference
June 8-10, 2005. Portland, OR, USA

Convergence Properties of Continuous-Time Markov Chains with Application to Target Search

Myungsoo Jun and David E. Jeffcoat

Abstract— **This paper considers the search for targets modeled as a discrete state, continuous-time Markov process. Convergence properties are analyzed using the eigenvalues and eigenvectors of a state transition rate matrix without explicitly solving differential equations or calculating matrix exponentials. It also studies the effect of cueing on convergence rate using eigenvalue analysis and optimal control theoretic approach.**

I. INTRODUCTION

In any system-of-systems analysis, consideration of dependencies between systems is imperative. In this paper, we consider a particular type of system interaction, called *cueing*. The interaction could be between similar systems, such as two or more wide area search munitions, or between dissimilar systems, such as a reconnaissance asset and a munition. In this paper, we consider two types of search vehicles that communicate target information to other vehicles, an operation we call cueing. In Shakespeare's day, the word cue meant "a signal (a word, phrase, or bit of stage business) to a performer to begin a specific speech or action." The word is now used more generally for anything serving a comparable purpose. Here we mean any information that provides focus to a search; e.g., information that limits the search area or provides a search heading.

Search theory is one of the oldest areas of operations research [1], with a solid foundation in mathematics, probability and experimental physics. Yet, search theory is clearly of more than academic interest. At times, a search can become an international priority, as in the 1966 search for the hydrogen bomb lost in the Mediterranean Sea near Palomares, Spain involving 34 ships, 2,200 sailors, 130 frogmen and four mini-subs. The search took 75 days, but might have concluded much earlier if cueing had been utilized from the start. A Spanish fisherman had come forward quickly to say he had seen something fall that looked like a bomb, but experts ignored him. Instead, they focused on four possible trajectories calculated by a computer, but for weeks found only airplane pieces. Finally, the fisherman was summoned back. He sent searchers in the right direction, and a sub located the bomb under 2,162 feet of water [2].

Koopman pioneered the application of mathematical process to military search problems during World War II [1]. He discusses the case in which a searcher inadvertently provides information to the target, perhaps allowing the target to employ evasive action [3]. The use of receivers on German U-boats to detect search radar signals in World War II is a classic example. Koopman referred to this type of cueing as "target alerting."

Ablavsky *et al.* [4] proposed a geometric approach for constructing an optimal path to search for a moving target in an arbitrarily shaped region. The path construction method seeks to minimize total flight time and to maximize the likelihood of finding the target. However, the method requires an initial target location, a terrain map that specifies the target's mobility over a region, and a set of predefined shapes for which optimal coverage patterns are known.

The problem of cooperative search for stationary targets using multiple search vehicles was investigated by Baum *et al.* [5]. The search environment is partitioned into cells, and a detection function is defined which relates the probability of detecting a target in a particular cell to the effort expended in that cell (e.g., time spent searching the cell). The classical "effort allocation algorithm" proposed by Stone [1] is used to define an optimal search plan without considering practical constraints on vehicle trajectory. The optimal plan is then modified to construct a physically feasible trajectory. The modified plan is evaluated through simulation by comparing the total search effort required by the modified plan with the effort required by the optimal plan. The evaluation is based on *a priori* probability maps for target locations, with actual target locations fixed and initial search vehicle locations and headings varied over all simulated cases. Cooperation is modeled by allowing search vehicles to share information concerning where each vehicle has searched and whether or not a target was found.

Curtis *et al.* [6] proposed a coordination strategy to simultaneously accomplish area search and task assignment. The problem consists of multiple homogeneous search agents and multiple fixed targets with unknown initial conditions. Each target has both a position in the search space (an undirected graph) and a value. Search agents have partial knowledge of the position and value of known targets and the routes of other agents. Dynamic constraints are modeled by linear constraints on the time required for an agent to reach a point in the search region. The objective function measures the total value of unengaged targets; this objective is used both to measure the value of searching a region and the value of engaging a target. Individual and team objectives are traded whenever an individual agent replans its own route or chooses to engage a target using an

M. Jun is with Graduate Engineering and Research Center, University of Florida, Shalimar, FL 32579, USA. mjun@gerc.eng.ufl.edu.
He is supported by AFOSR Grant F49620-03-1-0170.
D. E. Jeffcoat is with Air Force Research Laboratory, Munitions Directorate Eglin Air Force Base, FL, USA. david.jeffcoat@eglin.af.mil.
He is supported by AFOSR Grant LRIR 99MN01COR.

objective function common to all agents, but based on that agent's partial knowledge of past actions of other agents.

Slater [7] examined the benefits of cooperation for two search-and-engage vehicles searching for targets in a region containing both real and false targets. Vehicles cooperate by providing to each other the location of any target attacked. The unengaged vehicle will move to the attacked target, complete a damage assessment, and attack the target if the first vehicle did not destroy it. If the target is already destroyed, the second vehicle will continue its own search. Real and false targets are assumed randomly distributed according to a uniform distribution over the search area, with the probability of encountering targets following a Poisson distribution with the Poisson parameter based on the target density. Slater compared the number of targets destroyed with and without cooperation for various model parameters, such as target density and probability of correct target identification.

This paper investigates the effect of cooperation on the search performance analytically. We use a Markov chain analysis to examine cueing as a coupling mechanism between searchers. A Markov chain approach to target detection can be found in [1], which deals with the optimal allocation of effort to detect a target. A prior distribution of the target's location is assumed known to the searcher. The states correspond to cells that contain a target at a discrete time with a specified probability. In this research, the states correspond to detection states for individual search-and-engage vehicles. Jeffcoat [10] examined the case of two cooperative searchers. He determined the effect of cueing on the probability of target detection via a Markov chain analysis and uses matrix exponentiation to obtain numerical solutions. He showed the effect of cueing on convergence for a two vehicle example. This paper analyzes the effect of cueing on convergence more generally by examining convergence properties of continuous-time Markov chains. The approach used in this paper is analogous to convergence properties of discrete-time Markov chains in [11]. It utilizes the eigenvalues and eigenvectors of a state transition rate matrix without calculating computationally intensive matrix exponentials or solving a set of differential equations.

II. PROBLEM DESCRIPTION

Consider a search-only vehicle with a detection rate of λ targets detected per unit of time, and a search-and-engage vehicle with a detection rate of θ detections per time unit. Assume that cueing increases the search-and-engage vehicle's detection capability by a factor of k. That is, let the cued detection rate for an individual search-and-engage vehicle be given by $k\theta$ per time unit. We assume that once a search-only or search-and-engage asset detects a target, it immediately cues one of the search-and-engage vehicles. This cue could take the form of a target coordinate, a search heading, or any other information that improves detection rate. We wish to examine the impact of these coupled system capabilities (λ, θ, and k) on the time required to engage targets.

The detection rate can be considered a measure of some system characteristic (e.g., sensor capability) or of the conflict scenario (e.g., target density). The cueing factor, k, represents the value of the information communicated. The higher the value of k, the more valuable the information, where value might represent the precision of a target coordinate or the timeliness of the information communicated. We assume that there are sufficient targets in the battle space so that all search-and-engage vehicles have the opportunity to find and engage at least one target. As a measure of effectiveness (MOE), we use the time required to achieve a threshold probability that all search-and-engage vehicles have engaged a target.

III. STATE MODEL

This section briefly describes continuous-time Markov chains and modeling of vehicle states using continuous-time Markov chains.

A. Continuous-Time Markov Chains

A continuous-time Markov chain is a Markov process $\mathbf{x}(t)$ consisting of a family of staircase functions with continuities at the random points $\mathbf{t}_n$ [12], that is, transitions between the discrete states occur at any time. We denote by

$$p_i(t) = \mathbb{P}\{\mathbf{x}(t) = a_i\}$$

the state probabilities of $\mathbf{x}(t)$ and by

$$\pi_{ij}(t_1, t_2) = \mathbb{P}\{\mathbf{x}(t_2) = a_j | x(t_1) = a_i\}$$

its transient probabilities. These functions satisfy the followings:

$$\sum_j \pi_{ij}(t_1, t_2) = 1$$
$$\sum_i p_i(t_1)\pi_{ij}(t_1, t_2) = p_j(t_2).$$

A Markov process $\mathbf{x}(t)$ is homogeneous if its transition probabilities depend on the difference $\tau = t_2 - t_1$

$$\pi_{ij}(\tau) = \mathbb{P}\{\mathbf{x}(t + \tau) = a_j | x(\tau) = a_i\}.$$

A Markov matrix $\Pi(t)$, or a state transition matrix, is defined by a matrix whose ij element is $\pi_{ij}(t)$. We denote by

$$Q = \begin{bmatrix} q_{11} & \cdots & q_{1n} \\ \vdots & \cdots & \vdots \\ q_{n1} & \cdots & q_{nn} \end{bmatrix}$$

the state transition rate matrix whose elements $q_{ij} = \dot{\pi}_{ij}(0^+)$ are the derivatives from the right of the elements $\pi_{ij}(\tau)$. Note that $\sum_j q_{ij} = 0$ because $\sum_j \pi_{ij}(\tau) = 1$. Also we can say $q_{ii} = -\sum_{j \neq i} q_{ij} \leq 0$ and $q_{ij} \geq 0$, $i \neq j$.

Since $p(t+\tau) = [p_1(t+\tau) \ \cdots \ p_n(t+\tau)] = p(t)\Pi(t)$, we obtain

$$\dot{p}(t) = p(t)Q$$

and its formal solution is

$$p(t) = p_0 e^{Qt}$$

where $p_0 = p(0)$.

B. Modeling of Vehicle States

We model the possible vehicle states using discrete states, with transitions between states possible at any real-valued time greater than zero. For example, the search-only vehicle has two possible states — `search` and `detect and cue`. Note that the search-only vehicle transitions from the `search` state to the `detect and cue` state at rate λ, and that the transition back to the `search` state is instantaneous, denoted by an infinite transition rate. That is, once the search vehicle cues a search-and-engage vehicle, the search vehicle immediately resumes its search for additional targets. The system model is a continuous-time Markov chain because transitions between the discrete states can occur at any time.

Search-and-engage vehicles have three possible states, `search uncued`, `search cued`, and `detect and engage`. The transition rate from the uncued to the cued state depends on the detection rate of the search-only vehicle. We assume that the search-only vehicle will cue only one search-and-engage vehicle for each target detected, and that the cues are equally distributed if all search-and-engage vehicles are available. Under these assumptions, the transition rate from `search uncued` to `search cued` is λ/n if n search-and-engage vehicles are available, but increases to λ if only one search-and-engage vehicle is available to accept a cue. This assumption implies that the search vehicle is aware of the current state of both search-and-engage vehicles, and the search vehicle can transmit information to a single vehicle. Even if a transmission is broadcast on a common frequency, we assume that the transmitted data can be *tagged* for use only by an individual search-and-engage vehicle.

Recall that search-and-engage vehicles have the ability to search independently, so that a vehicle could transition directly from the `search uncued` to the `detect and engage` state. We also assume that each search-and-engage vehicle is a single shot asset, so that `detect and engage` is a terminal state. Since the state transitions of search-and-engage vehicles are unidirectional, e.g., there is no transition to `search uncued` from each state and no transition to `search cued` from `detect and engage`, the state transition rate matrix has triangular form.

EXAMPLE 1
Let us consider two search-and-engage vehicles. Table I shows the states of two vehicles, in which "U" denotes `search uncued`, "C" denotes `search cued`, and "D" denotes `detect and engage`. The state diagram of the system is shown in Figure 1. Then we can formulate the transition rate matrix $Q \in \mathbb{R}^{9\times 9}$ with an upper triangular form. □

TABLE I
STATE SPACE FOR TWO SEARCH-AND-ENGAGE (S&E) VEHICLES.

State	S&E 1	S&E 2	Next State
1	U	U	2,3,4,7
2	U	C	3,5,8
3	U	D	6,9
4	C	U	5,6,7
5	C	C	6,8
6	C	D	9
7	D	U	8,9
8	D	C	9
9	D	D	

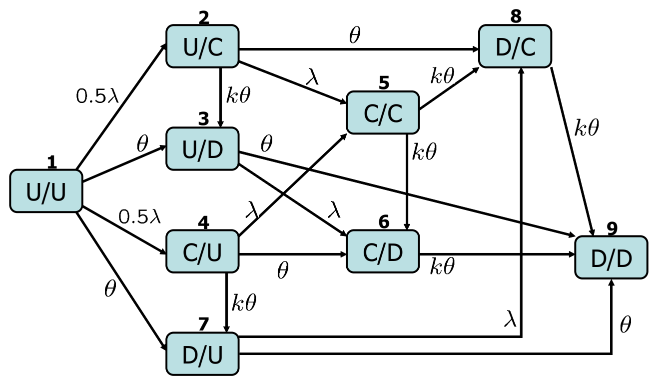

Fig. 1. System state diagram for two search-and-engage vehicles

IV. MAIN RESULTS

In this section, we provide convergence properties of continuous-time Markov chains without explicitly solving differential equations or calculating matrix exponentials. We derive a bound on settling time for the state probabilities. We also examine the characteristics of the transition rate matrix associated with continuous-time Markov chains. Finally, we apply these results to analyze the effect of cueing in a search application. The search problem is modeled using a linear quadratic regulator (LQR) formulation.

A. Convergence Rate of Continuous-Time Markov Chains

Proposition 1 *Any transition rate matrix Q has an eigenvalue equal to* 0.

Proof: Define the vector $v = [1 \ \ 1 \ \ \cdots \ \ 1]^T$, then it is easily verified that $Qv = 0$ from the fact that $\sum_j q_{ij} = 0$ for all $1 \leq i \leq n$. ■

The next theorem provides an upper bound on the difference between the transient state value and its steady state value.

Theorem 1 *Suppose that $Q \in \mathbb{C}^{n\times n}$ can be transformed to the Jordan form $\hat{Q} = diag\,(J_1, \cdots, J_m)$ where the size* *of the Jordan block J_i is j_i, $i = 1, \cdots, m$ and has non-positive eigenvalues $\lambda_1 = 0, \lambda_2, \cdots, \lambda_m$ corresponding to Jordan block $J_1, \cdots, J_m$, respectively. Assume that $j_1 = 1$. Then,*

$$\|p(t) - \bar{p}\| \le \sum_{i=2}^{m} \sum_{k=1}^{j_i} \sum_{\ell=1}^{j_i-k+1} \|a_\ell v_{k+\ell-1}\| e^{\lambda_i t} \frac{t^{k-1}}{(k-1)!} \tag{1}$$

where $\bar{p} = \lim_{t\to\infty} p(t)$, $v_i, v_{i+1}, \cdots, v_{i+j_i-1}$ are generalized left eigenvectors corresponding to λ_i, $i = 1, \cdots, m$, and a_k, $k = 1, \cdots, n$, are the complex coefficients satisfying $p_0 = \sum_{k=1}^{n} a_k v_k$.

Proof: Let us define $T = [v_1^T \; v_2^T \; \cdots \; v_n^T]^T$. Using that $p(t) = p_0 e^{Qt}$ and $TQ = \hat{Q}T$, we have

$$\begin{aligned} p(t) &= \sum_{k=1}^{n} a_k v_k e^{Qt} \\ &= \sum_{k=1}^{n} \left(a_k v_k \left(I + Qt + \frac{Q^2}{2!} t^2 + \cdots \right) \right) \\ &= \sum_{k=1}^{n} \left(a_k v_k T^{-1} \left(I + \hat{Q}t + \frac{\hat{Q}^2}{2!} t^2 + \cdots \right) T \right) \\ &= \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} e^{\hat{Q}t} T. \end{aligned}$$

Observing that

$$\begin{aligned} e^{J_i t} &= e^{\lambda_i t} \begin{bmatrix} 1 & t & t^2/2! & \cdots & t^{j_i-1}/(j_i-1)! \\ 0 & 1 & t & \cdots & t^{j_i-2}/(j_i-2)! \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \\ &\triangleq e^{\lambda_i t} M_i, \end{aligned}$$

we have

$$\begin{aligned} p(t) &= \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} \mathrm{diag}\left(e^{\lambda_1 t} M_1, \cdots, e^{\lambda_m t} M_m \right) T \\ &= \sum_{i=1}^{m} e^{\lambda_i t} \sum_{k=1}^{j_i} \sum_{\ell=1}^{j_i-k+1} a_\ell v_{k+\ell-1} \frac{t^{k-1}}{(k-1)!} \end{aligned}$$

Since $\lambda_i < 0$, $i = 2, \cdots, m$, we have $e^{\lambda_i t} t^{k-1}/(k-1)! \to 0$ for a finite k as $t \to \infty$, thus, $\bar{p} = a_1 v_1$. Therefore, we have

$$\begin{aligned} \|p(t) - \bar{p}\| &= \left\| \sum_{i=2}^{m} e^{\lambda_i t} \sum_{k=1}^{j_i} \sum_{\ell=1}^{j_i-k+1} a_\ell v_{k+\ell-1} \frac{t^{k-1}}{(k-1)!} \right\| \\ &\le \sum_{i=2}^{m} \sum_{k=1}^{j_i} \sum_{\ell=1}^{j_i-k+1} \|a_\ell v_{k+\ell-1}\| e^{\lambda_i t} \frac{t^{k-1}}{(k-1)!} \end{aligned}$$

■

If the matrix Q is diagonalizable, Theorem 1 reduces to the following corollary.

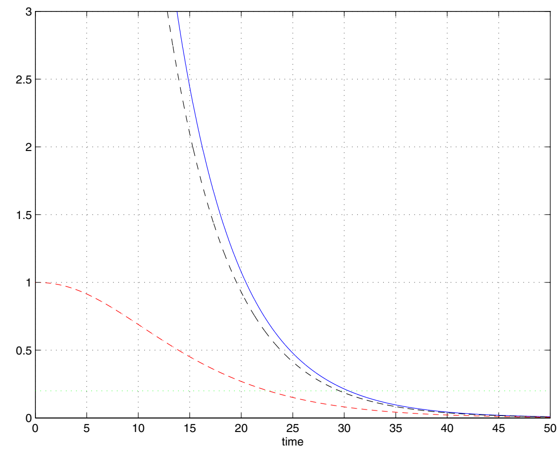

Fig. 2. Convergence

Corollary 1 *Suppose that $Q \in \mathbb{C}^{n\times n}$ is diagonalizable and has non-positive eigenvalues, then*

$$\|p(t) - \bar{p}\| \le \sum_{k=2}^{n} \|a_k v_k\| e^{\lambda_k t} \tag{2}$$

where $\bar{p} = \lim_{t\to\infty} p(t)$, $v_1, \cdots, v_n$ are a basis of left eigenvectors corresponding to $\lambda_1 = 0, \lambda_2, \cdots, \lambda_n$ respectively, and a_k, $k = 1, \cdots, n$, are the complex coefficients satisfying $p_0 = \sum_{k=1}^{n} a_k v_k$.

Proof: It can be easily proved by setting $m = n$ and $j_i = 1$, $i = 1, \cdots, n$ in Eq. (1). ■

The next example illustrates how eigenvalues and eigenvectors can be used to estimate an upper bound of convergence of Markov chains.

EXAMPLE 2
Consider the case with two search-and-engage vehicles described in the previous section with $\lambda = 0.1$, $\theta = 0.05$, $k = 4$ and $p_0 = [1 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0]$. The transition rate matrix Q is shown in Eq. (5). Computing eigenvalues and generalized eigenvectors, we have the Jordan form $\hat{Q} = TQT^{-1}$. See Eq. (6) and Eq. (7). They were computed by using MATLAB function `jordan`.

Note that there is only one Jordan block with size 2. From Theorem 1, we observe that $\|p(t) - \bar{p}\| \le 0.2$ when $t \ge 30.42$. If we consider the ninth state only, we have $|p_9(t) - 1| \le 0.2$ when $t \ge 29.53$. See Fig. 2. The solid (top) line denotes the bound of $\|p(t) - \bar{p}\|$, that is, $\sum_{i=2}^{m} \sum_{k=1}^{j_i} \sum_{\ell=1}^{j_i-k+1} \|a_\ell v_{k+\ell-1}\| e^{\lambda_i t} t^{k-1}/(k-1)!$, the dash-dotted (middle) line the bound of $|p_9(t) - 1|$, and the dashed (bottom) line the true trajectory of $1 - p_9(t)$. □

B. Analysis of Properties of $p(t)$

This section examines the relationship between the state transition rate matrix Q and convergence properties of $p(t)$.

$$Q = \begin{bmatrix} -0.2 & 0.05 & 0.05 & 0.05 & 0 & 0 & 0.05 & 0 & 0 \\ 0 & -0.35 & 0.2 & 0 & 0.1 & 0 & 0 & 0.05 & 0 \\ 0 & 0 & -0.15 & 0 & 0 & 0.1 & 0 & 0 & 0.05 \\ 0 & 0 & 0 & -0.35 & 0.1 & 0.05 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.4 & 0.2 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.2 & 0 & 0 & 0.2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.15 & 0.1 & 0.05 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.2 & 0.2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{5}$$

$$T = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -0.0313 & 0.0313 & -0.0312 & -0.1250 & 0.1562 & 0.0312 & 0.1562 & -0.1875 \\ 0 & 0 & -0.0446 & 0 & 0 & -0.0893 & -0.0446 & -0.0893 & 0.2679 \\ 0 & 0 & 0 & 0 & -0.2381 & 0.2381 & 0 & 0.2381 & -0.2381 \\ 0.3571 & 0.1190 & -0.8333 & 0.1190 & 0.1190 & -0.9524 & -0.8333 & -0.9524 & 2.8571 \\ 0 & 0 & 0 & 0 & 0 & -0.0536 & 0 & -0.0536 & 0.1071 \\ 0 & 0 & 0 & 0 & 0 & 1.0536 & 0 & -17.6131 & 16.5595 \\ 0 & 1.8182 & -1.8182 & 7.9890 & 19.6144 & -27.6034 & -7.9890 & -21.4327 & 29.4217 \\ 0 & 0 & -0.2308 & 0 & 0 & -0.4617 & 1.8261 & 3.6522 & -4.7858 \end{bmatrix}, \tag{6}$$

$$\hat{Q} = \text{diag}\left(0, -0.35, -0.15, -0.4, \begin{bmatrix} -0.2 & 1 \\ 0 & -0.2 \end{bmatrix}, -0.2, -0.35, -0.15\right) \tag{7}$$

The following proposition states under which conditions $p(t)$ has one absorbing state.

Proposition 2 *Suppose that Q has an eigenvalue 0 with algebraic multiplicity 1 and the other eigenvalues are negative with arbitrary multiplicities. Then, if Q is a triangular matrix, $p(t)$ has only one absorbing state, viz., $p(t)$ converges to one state with probability 1 as $t \to \infty$.*

Proof: Since each off-diagonal element of Q is nonnegative, the row with diagonal element 0 has all zero elements, that is, if $q_{kk} = 0$, $q_{ki} = 0$, $\forall\ i = 1, \cdots, n$. Without loss of generality, we assume $k = n$. Then, the left eigenvector associated to the eigenvalue 0 is $v_1 = [0\ \ 0\ \ \cdots\ \ 0\ \ 1]$. Since $\bar{p} = a_1 v_1$, $\bar{p}$ has only one non-zero element. ■

The next proposition provides conditions for the non-absorbing cases.

Proposition 3 *Suppose that Q has an eigenvalue 0 with algebraic multiplicity $m > 1$ and the other eigenvalues are negative with arbitrary multiplicities. If Q is a triangular matrix and the size of Jordan block J_i, $i = 1, \cdots, m$, is one, then $p(t)$ does not have an absorbing state and $\bar{p}$ has m non-zero elements.*

Proof: Following the proof of Proposition 2, $\bar{p} = \sum_{i=1}^{m} a_i v_i$ and $\bar{p}$ has m non-zero elements. ■

C. Analysis of Cueing Effect

Since we have assumed that cueing increases the search-and-engage vehicle's detection capability by a factor of k, we can expect a large value of k increases convergence rate. First, let us consider the case when Q is triangular. In this case, it is easy to obtain the eigenvalues of Q since they are the diagonal elements of Q. It can be noticed from Theorem 1 that the state probabilities converge faster when the eigenvalues are more negative. This is illustrated in the following example.

EXAMPLE 3
Consider the case with two search-and-engage vehicles described in the previous section. The eigenvalues dependent on k are $-(k+1)\theta - \lambda$, $-2k\theta$ and $-k\theta$. As k increases, they become more negative since $\theta > 0$. The trajectories of $p_9(t)$ are plotted in Figure 3 for $k = 1$ (solid line) , $k = 4$ (dashed line), $k = 10$ (dash-dot line), and $k = 30$ (dotted line). We observe that $p_9(t)$ converges faster as the value of k increases. □

Proposition 4 *Suppose that all eigenvalues of Q are non-positive and $\lambda_1(k), \cdots, \lambda_m(k)$ are the eigenvalues of $Q(k)$ dependent upon k. The Markov chain with transition rate matrix $Q(k_1)$ converges faster than one with $Q(k_2)$ if $\lambda_1(k_1) < \lambda_1(k_2), \cdots, \lambda_m(k_1) < \lambda_1(k_2)$.*

Proof: Omitted. ■

Let us consider non-triangular cases. Since we assume that cueing increases an individual searcher's detection

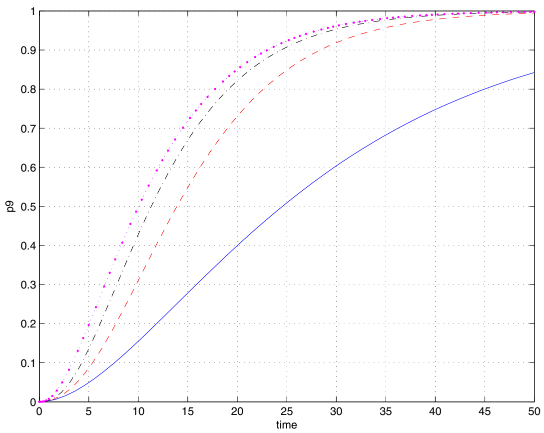

Fig. 3. Effect of Cueing

capability by a factor of k, it can be noticed that the state transition rate matrix Q is affine with respect to k, i.e., $Q = kQ_1 + Q_2$. If we define $x(t) = p(t)^T$, we have

$$\dot{x}(t) = Q^T x(t) = Q_2^T x(t) + kQ_1^T x(t).$$

The system can be transformed into a state feedback configuration

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \\ u(t) &= -kx(t) \end{cases}$$

where $A = Q_2^T$ and $B = -Q_1^T$. This problem can be considered as a linear-quadratic regulator (LQR) problem with cost function

$$V = \int_0^T \left(x(\tau)^T W x(\tau) + u(\tau)^T R u(\tau) \right) d\tau + x(T)^T M x(T) \quad (3)$$

It is well-known (see [13]) that if (A, B) is controllable the controller minimizing the above cost function is given by

$$k = R^{-1} B^T P(t) \quad (4)$$

where $P(t)$ is the solution of the following matrix Riccati equation

$$-\dot{P} = A^T P + PA + W - PBR^{-1}B^T P$$
$$P(T) = M.$$

In order to observe the effect of cueing, define $W = 0$ and $M = \rho I$. If we make the value ρ large, the cost function V penalizes the state x at time T and makes $x(T)$ small. When we consider a constant k, we can observe a large k penalizes the state x at time T more, thus, makes fast response of the system.

V. CONCLUSION

We have considered a cooperative search for targets with communication of cues, or target information, between searchers. The search is modeled as a discrete state, continuous time Markov process. We have investigated convergence properties of continuous-time Markov chains and provided a bound on settling time for the state probabilities. The bound is a function of eigenvalues and eigenvectors of the state transition rate matrix and provides information on convergence rate or response time without explicitly solving a set of differential equations or calculating a matrix exponential. Finally, we have applied these results to analyze the effect of cueing in a search application. We have also provided a linear quadratic regulator (LQR) formulation for the search problem.

REFERENCES

[1] L. Stone, *Theory of Optimal Search*, 2nd ed. Military Applications Section, Operations Research Society of America, 1989.

[2] D. Woolls, "A chronicle of four lost nukes," Houston Chronicle, July 12, 2003.

[3] B. Koopman, *Search and Screening: General Principles with Historical Applications*. Pergamon Press, 1980.

[4] V. Ablavsky and M. Snorrason, "Optimal search for a moving target: a geometric approach," in *Proc. of the AIAA GNC Conference*, Denver, CO, August 2000.

[5] M. L. Baum and K. M. Passino, "A search-theoretic approach to cooperative control for uninhibited air vehicles," in *Proc. of the AIAA GNC Conference*, Monterey, CA, August 2002.

[6] J. W. Curtis and R. Murphey, "Simultaneous area search and task assignment for a team of cooperative agents," in *Proc. of the AIAA GNC Conference*, Austin, TX, August 2003.

[7] G. L. Slater, "Cooperation between UAVs in a search and destroy mission," in *Proc. of the AIAA GNC Conference*, Austin, TX, August 2003.

[8] B. Koopman, "The theory of search II: Target detction," *Operations Research*, vol. 4, pp. 503–531, 1956.

[9] A. Washburn, "Search for a moving target: Upper bound on detection probability," in *Search Theory and Applications*, B. Haley and L. Stone, Eds. New York, NY: Plenum Press, 1980, pp. 231–237.

[10] D. E. Jeffcoat, "Coupled detection rates: An introduction," in *Theory and Algorithms for Cooperative Systems*, D. Grundel, R. Murphey, and P. Pardalos, Eds. New Jersey: World Scietific Publishing, 2004, pp. 157–167.

[11] J. S. Rosenthal, "Convergence rates of Markov chains," *SIAM Review*, vol. 37, pp. 387–405, 1995.

[12] A. Popoulis and S. U. Pillai, *Probability, Random Variables, and Stochastic Processes*, 4th ed. McGraw-Hill, 2002.

[13] B. D. O. Anderson and J. B. Moore, *Optimal Control: Linear Quadratic Methods*. Englewood Cliffs, NJ: Prentice-Hall, 1989.