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Robust Performance Analysis of Linear Time-Invariant Parameter-Dependent Systems using Higher-Order Lyapunov Functions

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Abstract— **This paper studies the robust stability/performance analysis of linear time-invariant parameter-dependent systems for which the coefficient matrices in the state-space representations are parameter-dependent in negative as well as positive power series of parameters, and whose parameters are supposed to lie in a given convex region. To analyze the robust stability/performance, we use parameter-dependent Lyapunov functions that are parameter-dependent in negative as well as positive power series, and derive sufficient conditions for them via parametrically affine linear matrix inequalities. Although our formulae have greater numerical complexity than previous works, they encompass formulae based on biquadratic stability, which has been proposed to obtain less conservative conditions, and conventional quadratic stability. We present numerical examples that demonstrate the effectiveness of our proposed methods compared with existing methods.**

I. INTRODUCTION

Much research on the robust stability/performance analysis of linear time-invariant parameter-dependent (LTIPD) systems has been carried out over many years. Conventionally, parameter-independent Lyapunov functions have been used to analyze LTIPD systems ([1] and references therein), but the results are too conservative; the analysis is not tight. For this reason, methods using parameter-dependent Lyapunov functions have been proposed for the robust stability/performance analysis of parameter-dependent systems: parametrically affine Lyapunov functions [2] and quadratically parameter-dependent Lyapunov functions [3], [4], [5]. However, these are restricted to systems with parametrically affine state-space representations. Alternatively, a variety of methods for the stability/performance analysis of systems whose state-space representations are expressed as rationally parameter-dependent matrices have been proposed [6], [7], [8], [9], [10]. Although the effectiveness of these has been demonstrated in the literature theoretically and by numerical examples, they are not entirely satisfactory.

As an example, consider the system depicted in Fig. 1. This system is known as a benchmark problem for robust controller design [11]. Implementing the feedback controller $u = \frac{1.333(s-1.0257)(s+0.1301)}{(s+1.2245)^2+0.8290^2}x_2$ [12] and analyzing the robust stability for variations of k and m_1 (details are given in Section VI), the closed-loop system with $k = 1.25 + \delta_k$ and $m_1 = 1.0+\delta_m$ is confirmed to be robustly stable for all

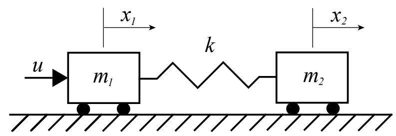

Fig. 1. Benchmark problem

$|\delta_k| \le 0.76252$ and $|\delta_m| < 1.0$ by gridding the uncertainty range. In [10], Ebihara and Hagiwara show the connections with [7] and [9] for stability analysis, and roughly speaking, their analysis method is the best among those, so we now analyze the robust stability using that method. Under the condition that $\delta_m = 0$, the method confirms that the closed-loop system is robustly stable for all $|\delta_k| \le 0.76252$, which is the limit of the variation of δ_k by gridding. However, the method confirms that the closed-loop system is robustly stable for all $|\delta_k| \le 0.762$ and $|\delta_m| \le 0.036182$, and this is much more conservative than the numerical result.

In recent years, a new approach to robust stability analysis has been proposed [13], [14]. Their proposed conditions are necessary and sufficient conditions for robust stability. In particular, Zhang *et al.* show the minimum degree of polynomial-type Lyapunov functions to analyze the stability of parametrically affine systems without introducing any conservatism [14]. However, they have the following drawbacks; we cannot decide the degree of Lyapunov functions preliminarily [13]; that is, if the degree is set to be lower than it is needed, then the analysis may not be satisfactory, and the number of parameters in [14] is unity; this is a big concern to apply their method to actual systems because they usually have several parameters.

In this paper, we address the stability/performance analysis of LTIPD systems and propose a less conservative method of analysis. Our study extends biquadratic stability, proposed by Trofino *et al.* [3], to the case that both the Lyapunov functions and matrices of the state-space representations of LTIPD systems are parameter-dependent in negative as well as positive power series, i.e. $\theta^{-1}, \cdots, \theta^{-N}$ and $\theta^1, \cdots, \theta^M$. The Lyapunov functions used in this study encompass the biquadratic Lyapunov functions proposed in [3] as a special case, and we show the relationship between our proposed analysis condition and a sufficient condition proposed in [3]. Further, we make some observations on numerical complexity of our condition and previously proposed conditions, showing that our method has greater numerical complexity. Finally, using numerical examples, we demonstrate that our methods are less conservative than

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existing methods.

Hereinafter, $\mathsf{He}(X)$ is a shorthand notation for $X+X^T$, $\Xi_2(X_1,X_2,X_3)$ and $\Xi_\infty(X_1,X_2,X_3,X_4,X_5,X_6)$ denote $\begin{bmatrix} X1 & X_2^T \\ X_2 & X_3 \end{bmatrix}$ and $\begin{bmatrix} X1 & X_2 & X_3^T \\ X_2^T & X_5 & X_4^T \\ X_3 & X_4 & X_6 \end{bmatrix}$ respectively, $\otimes$ denotes Kronecker product, and $\mathcal{R}^{n\times m}$ and $\mathcal{S}^{n\times n}$ respectively denote a set of $n\times m$ dimensional real matrices and a set of $n\times n$ dimensional symmetric real matrices. The notations $0_{i,j}$, 0_i, and $\mathbf{0}$ respectively denote an $i\times j$ dimensional zero matrix, an $i\times i$ dimensional zero matrix, and a zero matrix of an appropriate dimension, and I_n denotes an $n\times n$ identity matrix. For a matrix $X\in\mathcal{R}^{l\times m}$ with rank r, $X^\perp\in\mathcal{R}^{(l-r)\times l}$ is defined as a matrix such that $X^\perp X=\mathbf{0}$ and $X^\perp(X^\perp)^T>0$. Furthermore, $\mathsf{diag}(X_1,\cdots,X_k)$ denotes a block diagonal matrix composed of $X_1,\cdots$, and X_k.

II. PRELIMINARIES

Consider the following LTIPD system:

$$\begin{cases} \dot{x}=A(\theta)x+B(\theta)w \\ z=C(\theta)x+D(\theta)w \end{cases}, \tag{1}$$

where $x\in\mathcal{R}^n$ is the state vector with $x=0$ at $t=0$, $w\in\mathcal{R}^m$ is the disturbance input vector, $z\in\mathcal{R}^l$ is the controlled output vector, and θ_i is a time-invariant parameter that represents plant uncertainties. The matrices in (1) is given as follows:

$$\begin{aligned}
A(\theta)&=\bar\theta_n\bar A\bar\theta_n^T, \quad \bar A\in\mathcal{R}^{n(\sigma+1)\times n(\sigma+1)},\\
B(\theta)&=\bar\theta_n\bar B\bar\theta_m^T, \quad \bar B\in\mathcal{R}^{n(\sigma+1)\times m(\sigma+1)},\\
C(\theta)&=\bar\theta_l\bar C\bar\theta_n^T, \quad \bar C\in\mathcal{R}^{l(\sigma+1)\times n(\sigma+1)},\\
D(\theta)&=\bar\theta_l\bar D\bar\theta_m^T, \quad \bar D\in\mathcal{R}^{l(\sigma+1)\times m(\sigma+1)},\\
\bar\theta_n&=\bar\theta\otimes I_n, \quad \bar\theta_n^T=\bar\theta^T\otimes I_n,\\
\bar\theta_l&=\bar\theta\otimes I_l, \quad \bar\theta_m^T=\bar\theta^T\otimes I_m,\\
\bar\theta&=\big[1\,\big[\theta_1^{-l_1}\cdots\theta_1^{-1}\,\theta_1\cdots\theta_1^{m_1}\big]\\
&\qquad\cdots\big[\theta_k^{-l_k}\cdots\theta_k^{-1}\,\theta_k\cdots\theta_k^{m_k}\big]\big]\in\mathcal{R}^{1\times(\sigma+1)},\\
\sigma&=\textstyle\sum_{i=1}^k(l_i+m_i).
\end{aligned}$$

This expression can represent parameter-dependent matrices of power series from $-2l_i$ to $2m_i$ of the ith parameter. The parameter θ_i is supposed to lie in a given convex region: $\theta\in\mathcal{B}_\theta,\ \forall t\geq 0$, where $\theta=[\theta_1\ \cdots\ \theta_k]^T$.

We define $\tilde\theta$ as follows:

$$\begin{aligned}
\tilde\theta\ &:=\begin{bmatrix} [\theta_1]\cdots[\theta_k] \\ \{\theta_1\} \qquad \mathbf{0} \\ \ddots \\ \mathbf{0} \qquad \{\theta_k\} \end{bmatrix}\in\mathcal{R}^{(\sigma+1)\times\sigma},\\
[\theta_i]\ &=[\underbrace{0\cdots0}_{l_i-1}\ -1\ \theta_i\ \underbrace{0\cdots0}_{m_i-1}],\\
\{\theta_i\}&=\left[\begin{array}{c|c} \{\theta_i\}_{11} & 0_{l_i,m_i} \\ \hline 0_{m_i,l_i} & \{\theta_i\}_{22} \end{array}\right]\in\mathcal{R}^{(l_i+m_i)\times(l_i+m_i)},
\end{aligned}$$

$$\{\theta_i\}_{11}=\begin{bmatrix} \theta_i & & & \mathbf{0} \\ -1 & \ddots & & \\ & \ddots & \theta_i & \\ \mathbf{0} & & -1 & \theta_i \end{bmatrix},\ \{\theta_i\}_{22}=\begin{bmatrix} -1 & \theta_i & & \mathbf{0} \\ & -1 & \ddots & \\ & & \ddots & \theta_i \\ \mathbf{0} & & & -1 \end{bmatrix}.$$

We confirm that $\tilde\theta^\perp=\bar\theta$ and we also easily confirm that $\tilde\theta_n^\perp:=\left(\tilde\theta\otimes I_n\right)^\perp=\bar\theta_n$.

We newly define $\overline{\mathsf{vec}}_{n,m}$ and $\widehat{\mathsf{vec}}_{n,m}$ operators for $X\in\mathcal{R}^{ni\times mj}$, in which $X=\begin{bmatrix} X_{11}\cdots X_{1j} \\ \vdots \ \ddots \ \vdots \\ X_{i1}\cdots X_{ij} \end{bmatrix}$ and $X_{ij}\in\mathcal{R}^{n\times m}$:

$$\begin{aligned}
\overline{\mathsf{vec}}_{n,m}(X)&:=\big[\big[X_{11}^T\ \cdots\ X_{1j}^T\big]\ \cdots\ \big[X_{i1}^T\ \cdots\ X_{ij}^T\big]\big]^T\\
&\quad\in\mathcal{R}^{(nj)i\times m},\\
\widehat{\mathsf{vec}}_{n,m}(X)&:=[[X_{11}\cdots X_{i1}]\cdots[X_{1j}\cdots X_{ij}]]\in\mathcal{R}^{n\times(mi)j}.
\end{aligned}$$

In the case of $n=m$, we define Tr_n for the above X as follows: $\mathrm{Tr}_n(X):=\begin{bmatrix} \mathrm{Tr}(X_{11})\cdots\mathrm{Tr}(X_{1j}) \\ \vdots \ \ddots \ \vdots \\ \mathrm{Tr}(X_{i1})\cdots\mathrm{Tr}(X_{ij}) \end{bmatrix}\in\mathcal{R}^{i\times j}$.

We now show the well-known stability analysis and performance analysis of the LTIPD system (1). In these Lemmas, Lyapunov functions are set as $x^TP(\theta)x$.

Lemma 1 (Stability): The system (1) is robustly stable for all $\theta\in\mathcal{B}_\theta$ if and only if there exists $P(\theta)>0$ such that

$$\mathsf{He}\{P(\theta)A(\theta)\}<0,\ \forall\theta\in\mathcal{B}_\theta. \tag{2}$$

Lemma 2 (H_∞ performance): The system (1) is robustly stable and its H_∞ performance is bounded by γ_∞ for all $\theta\in\mathcal{B}_\theta$ if and only if there exists $P(\theta)>0$ such that

$$\begin{aligned}
\Xi_\infty\big[&\mathsf{He}\{P(\theta)A(\theta)\},P(\theta)B(\theta),C(\theta),D(\theta),\\
&-\gamma_\infty I_m,-\gamma_\infty I_l\big]<0,\ \forall\theta\in\mathcal{B}_\theta.
\end{aligned} \tag{3}$$

Hereinafter, we assume $D(\theta)=\mathbf{0}$ when considering H_2 performance.

Lemma 3 (H_2 performance): The system (1) is robustly stable and its H_2 performance is bounded by γ_2 for all $\theta\in\mathcal{B}_\theta$ if and only if there exists $P(\theta)>0$ and $N(\theta)>0$ such that

$$\Xi_2\{N(\theta),P(\theta)B(\theta),P(\theta)\}>0,\ \forall\theta\in\mathcal{B}_\theta, \tag{4}$$

$$\Xi_2[\mathsf{He}\{P(\theta)A(\theta)\},C(\theta),-I_l]<0,\ \forall\theta\in\mathcal{B}_\theta, \tag{5}$$

$$\gamma_2^2>\mathsf{Tr}\{N(\theta)\},\ \forall\theta\in\mathcal{B}_\theta. \tag{6}$$

III. PROPOSED METHODS

We first define multi-quadratic stability and performance analysis based on multi-quadratic stability, and then show tractable sufficient conditions for those.

A. Multi-Quadratic Stability

We set $P(\theta)$ in Lemmas 1, 2, and 3 as follows:

$$P(\theta)=\bar\theta_n\bar P\bar\theta_n^T, \tag{7}$$

where $\bar P\in\mathcal{S}^{n(\sigma+1)\times n(\sigma+1)}$. This expression can represent parameter-dependent $P(\theta)$ of power series from $-2l_i$ to $2m_i$ of the ith parameter. Using this $P(\theta)$, we define multi-quadratic stability.

Definition 1 (Multi-Quadratic Stability): The system (1) is said to be multi-quadratically stable if there exists $P(\theta)>0$ defined in (7) such that (2) holds.

Based on this definition of multi-quadratic stability, we derive a new condition for the stability analysis of (1).

Lemma 4: The system (1) is multi-quadratically stable for all $\theta \in \mathcal{B}_\theta$ if and only if there exist a symmetric matrix $\bar{P}$ and matrices $F(\theta) \in \mathcal{R}^{n\sigma \times n(\sigma+1)}$ and $M(\theta) \in \mathcal{R}^{n\sigma \times n(\sigma+1)}$ such that

$$\bar{P} + \mathsf{He}\left\{\tilde{\theta}_n F(\theta)\right\} > 0,\ \forall\theta \in \mathcal{B}_\theta, \tag{8}$$

$$\mathsf{He}\left(\bar{P}\bar{\theta}_n^T \theta_n \bar{A}\right) + \mathsf{He}\left\{\tilde{\theta}_n M(\theta)\right\} < 0,\ \forall\theta \in \mathcal{B}_\theta. \tag{9}$$

Proof: We first confirm that the following two conditions are equivalent for a given $\bar{P}$. This fact is easily derived from the Elimination lemma [15].

(i) $\bar{\theta}_n \bar{P} \bar{\theta}_n^T > 0,\ \forall\theta \in \mathcal{B}_\theta$

(ii) $\exists F(\theta)$ such that $\bar{P} + \mathsf{He}\left\{\tilde{\theta}_n F(\theta)\right\} > 0,\ \forall\theta \in \mathcal{B}_\theta$

Using the above equivalence, the existence of a positive $P(\theta)$ that satisfies (2) in Lemma 1 is equivalent to the existence of $\bar{P}$, $F(\theta)$, and $M(\theta)$ such that (8) and (9) hold. This completes the proof. ∎

We obtain the following lemma that characterizes H_∞ performance based on multi-quadratic stability from Lemma 2 after applying the Elimination lemma to (3).

Lemma 5: The system (1) is multi-quadratically stable and its H_∞ performance is bounded by γ_∞ for all $\theta \in \mathcal{B}_\theta$ if and only if there exist a symmetric matrix $\bar{P}$ and matrices $F(\theta) \in \mathcal{R}^{n\sigma \times n(\sigma+1)}$ and $M(\theta) \in \mathcal{R}^{(n\sigma+m\sigma)\times\{n(\sigma+1)+m(\sigma+1)+l\}}$ such that (8) and

$$\begin{aligned}\Xi_\infty \Big\{\mathsf{He}\left(\bar{P}\bar{\theta}_n^T \theta_n \bar{A}\right), \bar{P}\bar{\theta}_n^T \theta_n \bar{B}, \bar{\theta}_l \bar{C}, \bar{\theta}_l \bar{D},& \\ -\mathsf{diag}\left(\gamma_\infty I_m, 0_{m\sigma}\right), -\gamma_\infty I_l\Big\}& \\ + \mathsf{He}\left\{\begin{bmatrix}\tilde{\theta}_n & \mathbf{0} \\ \mathbf{0} & \tilde{\theta}_m \\ 0_{l,n\sigma} & 0_{l,m\sigma}\end{bmatrix} M(\theta)\right\} < 0,\ \forall\theta \in \mathcal{B}_\theta.&\end{aligned} \tag{10}$$

We now show a lemma that characterizes H_2 performance based on multi-quadratic stability. We set $N(\theta)$ in Lemma 3 as $\bar{\theta}_m \bar{N}(\theta) \bar{\theta}_m^T$, where $\bar{N}(\theta) \in \mathcal{S}^{m(\sigma+1)\times m(\sigma+1)}$. We then obtain the following inequality instead of (6).

$$\bar{\theta}\left[\mathsf{diag}\left(\gamma_2^2, 0_\sigma\right) - \mathrm{Tr}_m\left\{\bar{N}(\theta)\right\}\right]\bar{\theta}^T > 0,\ \forall\theta \in \mathcal{B}_\theta$$

After applying the Elimination lemma to the above inequality, (4), and (5), we obtain the following lemma on H_2 performance analysis from Lemma 3.

Lemma 6: The system (1) is multi-quadratically stable and its H_2 performance is bounded by γ_2 for all $\theta \in \mathcal{B}_\theta$ if and only if there exist symmetric matrices $\bar{P}$ and $\bar{N}(\theta)$, and matrices $F(\theta) \in \mathcal{R}^{(m\sigma+n\sigma)\times\{m(\sigma+1)+n(\sigma+1)\}}$, $M(\theta) \in \mathcal{R}^{n\sigma\times\{n(\sigma+1)+l\}}$ and $H(\theta) \in \mathcal{R}^{\sigma\times(\sigma+1)}$ such that

$$\Xi_2\left\{\bar{N}(\theta), \bar{P}\bar{\theta}_n^T \bar{\theta}_n \bar{B}, \bar{P}\right\} + \mathsf{He}\left\{\begin{bmatrix}\tilde{\theta}_m & \mathbf{0} \\ \mathbf{0} & \tilde{\theta}_n\end{bmatrix} F(\theta)\right\} > 0,\ \forall\theta \in \mathcal{B}_\theta, \tag{11}$$

$$\begin{aligned}\Xi_2\left\{\mathsf{He}\left(\bar{P}\bar{\theta}_n^T \bar{\theta}_n \bar{A}\right), \bar{\theta}_l \bar{C}, -I_l\right\}& \\ + \mathsf{He}\left\{\begin{bmatrix}\tilde{\theta}_n \\ 0_{l,n\sigma}\end{bmatrix} M(\theta)\right\} < 0,\ \forall\theta \in \mathcal{B}_\theta,&\end{aligned} \tag{12}$$

$$\mathsf{diag}\left(\gamma_2^2, 0_\sigma\right) - \mathrm{Tr}_m\left\{\bar{N}(\theta)\right\} + \mathsf{He}\left\{\tilde{\theta} H(\theta)\right\} > 0,\ \forall\theta \in \mathcal{B}_\theta. \tag{13}$$

The proofs are omitted here as they are similar to the proof of Lemma 4.

We now verify the tractability of the LMIs in these lemmas; that is, whether or not the LMIs are parametrically affine. For (8) and (13), if $\bar{N}(\theta)$ and the newly introduced parameter-dependent matrices $F(\theta)$ and $H(\theta)$ are set to be parameter-independent, then these inequalities become parametrically affine and we need only solve them at vertices of $\mathcal{B}_\theta$ instead of at all points of $\mathcal{B}_\theta$. However, other LMIs are not parametrically affine even if the newly introduced matrices $F(\theta)$ and $M(\theta)$ are set to be parameter-independent, so we must solve these LMIs at all points of $\mathcal{B}_\theta$; that is, solve infinitely many LMIs. We derive tractable sufficient conditions for those LMIs in the next subsection.

B. Tractable Sufficient Conditions

In the previous subsection, we derive new necessary and sufficient conditions for the stability and performance analysis of an LTIPD system (1). However, we had to solve infinitely many LMIs because the derived LMIs are not parametrically affine. To tackle this problem, we apply the Elimination lemma again to those LMIs. Here, we show the details for (9).

First, we set $M(\theta)$ in (9) as follows:

$$M(\theta) = \left(I_\sigma \otimes \bar{\theta}_n\right)\hat{M}_1(\theta)\left(I_{\sigma+1} \otimes \bar{\theta}_n^T\right),$$

where $\hat{M}_1(\theta) \in \mathcal{R}^{n\sigma(\sigma+1)\times n(\sigma+1)^2}$. This expression means that the every element of $M(\theta)$ is a function of $\bar{\theta}$, $\bar{\theta}^T$, and the corresponding block matrix of $\hat{M}_1(\theta)$. Using this expression, the term $\tilde{\theta}_n M(\theta)$ in (9) becomes $\tilde{\theta}_n M(\theta) = \left(I_{\sigma+1} \otimes \bar{\theta}_n\right)\tilde{\theta}_{n(\sigma+1)}\hat{M}_1(\theta)\left(I_{\sigma+1} \otimes \bar{\theta}_n^T\right)$. Using the operator $\widehat{\mathsf{vec}}$, we obtain an alternative representation for $\bar{\theta}_n \bar{A}$ as $\bar{\theta}_n \bar{A} = \widehat{\mathsf{vec}}_{n,n}(\bar{A})\left(I_{\sigma+1} \otimes \bar{\theta}_n^T\right)$. Further, using the operator $\overline{\mathsf{vec}}$ we obtain an alternative representation for $\bar{P}\bar{\theta}_n^T$ as $\bar{P}\bar{\theta}_n^T = \left(I_{\sigma+1} \otimes \bar{\theta}_n\right)\overline{\mathsf{vec}}_{n,n}(\bar{P})$. These representations are the key idea in our proposed method.

With these preliminaries, we now show our main result on stability analysis.

Theorem 1: The system (1) is multi-quadratically stable for all $\theta \in \mathcal{B}_\theta$ if and only if there exist a symmetric matrix $\bar{P} \in \mathcal{S}^{n(\sigma+1)\times n(\sigma+1)}$ and matrices $F(\theta) \in \mathcal{R}^{n\sigma\times n(\sigma+1)}$, $\hat{M}_1(\theta) \in \mathcal{R}^{n\sigma(\sigma+1)\times n(\sigma+1)^2}$ and $M_2(\theta) \in \mathcal{R}^{n\sigma(\sigma+1)\times n(\sigma+1)^2}$ such that (8) and

$$\begin{aligned}\mathsf{He}\left\{\overline{\mathsf{vec}}_{n,n}(\bar{P})\widehat{\mathsf{vec}}_{n,n}(\bar{A})\right\} + \mathsf{He}\left\{\tilde{\theta}_{n(\sigma+1)}\hat{M}_1(\theta)\right\}& \\ + \mathsf{He}\left\{\left(I_{\sigma+1} \otimes \tilde{\theta}_n\right)M_2(\theta)\right\} < 0,\ \forall\theta \in \mathcal{B}_\theta.&\end{aligned} \tag{14}$$

Proof: Note that $\left(I_{\sigma+1} \otimes \tilde{\theta}_n\right)^\perp = I_{\sigma+1} \otimes \bar{\theta}_n$. Then, we obtain (14) from (9) after applying the Elimination lemma, similarly to Lemma 4. This completes the proof. ∎

We obtain the following theorems for performance analysis, similarly to Theorem 1. Their proofs are omitted here as they are similar to the proof of Theorem 1.

Theorem 2: The system (1) is multi-quadratically stable and its H_∞ performance is bounded by γ_∞ for all $\theta \in \mathcal{B}_\theta$ if and only if there exist a symmetric matrix $\bar{P} \in \mathcal{S}^{n(\sigma+1)\times n(\sigma+1)}$ and matrices $F(\theta) \in \mathcal{R}^{n\sigma\times n(\sigma+1)}$, $\hat{M}_1(\theta) \in \mathcal{R}^{\{n\sigma(\sigma+1)+m\sigma(\sigma+1)\}\times\{n(\sigma+1)^2+m(\sigma+1)^2+l\}}$ and $M_2(\theta) \in \mathcal{R}^{\{n\sigma(\sigma+1)+m\sigma(\sigma+1)\}\times\{n(\sigma+1)^2+m(\sigma+1)^2+l\}}$ such that (8) and

$$\begin{aligned}&\Xi_\infty\left[\mathsf{He}\left\{\overline{\mathsf{vec}}_{n,n}(\bar{P})\widehat{\mathsf{vec}}_{n,n}(\bar{A})\right\},\overline{\mathsf{vec}}_{n,n}(\bar{P})\widehat{\mathsf{vec}}_{n,m}(\bar{B}),\right.\\&\quad\widehat{\mathsf{vec}}_{l,n}(\bar{C}),\widehat{\mathsf{vec}}_{l,m}(\bar{D}),-\mathsf{diag}\left(\gamma_\infty I_m,0_{m\sigma},0_{m(\sigma+1)\sigma}\right),\\&\quad\left.-\gamma_\infty I_l\right]+\mathsf{He}\left\{X_1\hat{M}_1(\theta)\right\}+\mathsf{He}\left\{X_2M_2(\theta)\right\}<0,\\&\qquad\qquad\forall\theta\in\mathcal{B}_\theta,\end{aligned}\tag{15}$$

where X_1 and X_2 are defined as follows:

$$X_1=\begin{bmatrix}\tilde{\theta}_{n(\sigma+1)} & \mathbf{0}\\ \mathbf{0} & \tilde{\theta}_{m(\sigma+1)}\\ 0_{l,n\sigma(\sigma+1)} & 0_{l,m\sigma(\sigma+1)}\end{bmatrix},$$
$$X_2=\begin{bmatrix}I_{\sigma+1}\otimes\tilde{\theta}_n & \mathbf{0}\\ \mathbf{0} & I_{\sigma+1}\otimes\tilde{\theta}_m\\ 0_{l,n\sigma(\sigma+1)} & 0_{l,m\sigma(\sigma+1)}\end{bmatrix}.$$

Theorem 3: The system (1) is multi-quadratically stable and its H_2 performance is bounded by γ_2 for all $\theta \in \mathcal{B}_\theta$ if and only if there exist symmetric matrices $\bar{P} \in \mathcal{S}^{n(\sigma+1)\times n(\sigma+1)}$ and $\bar{N}(\theta) \in \mathcal{S}^{m(\sigma+1)\times m(\sigma+1)}$, and matrices $\hat{F}_1(\theta) \in \mathcal{R}^{\{m\sigma(\sigma+1)+n\sigma(\sigma+1)\}\times\{m(\sigma+1)^2+n(\sigma+1)^2\}}$, $F_2(\theta) \in \mathcal{R}^{\{m\sigma(\sigma+1)+n\sigma(\sigma+1)\}\times\{m(\sigma+1)^2+n(\sigma+1)^2\}}$, $\hat{M}_1(\theta) \in \mathcal{R}^{n\sigma(\sigma+1)\times\{n(\sigma+1)^2+l\}}$, $M_2(\theta) \in \mathcal{R}^{n\sigma(\sigma+1)\times\{n(\sigma+1)^2+l\}}$ and $H(\theta) \in \mathcal{R}^{\sigma\times(\sigma+1)}$ such that (13) and

$$\begin{aligned}&\Xi_2\left[\mathsf{diag}\left\{\bar{N}(\theta),0_{m\sigma(\sigma+1)}\right\},\overline{\mathsf{vec}}_{n,n}(\bar{P})\widehat{\mathsf{vec}}_{n,m}(\bar{B}),\right.\\&\qquad\qquad\left.\mathsf{diag}\left(\bar{P},0_{n\sigma(\sigma+1)}\right)\right]\\&\quad+\mathsf{He}\left\{X_3\hat{F}_1(\theta)\right\}+\mathsf{He}\left\{X_4F_2(\theta)\right\}<0,\ \forall\theta\in\mathcal{B}_\theta,\end{aligned}\tag{16}$$

$$\begin{aligned}&\Xi_2\left[\mathsf{He}\left\{\overline{\mathsf{vec}}_{n,n}(\bar{P})\widehat{\mathsf{vec}}_{n,n}(\bar{A})\right\},\widehat{\mathsf{vec}}_{l,n}(\bar{C}),-I_l\right]\\&\quad+\mathsf{He}\left\{X_5\hat{M}_1(\theta)\right\}+\mathsf{He}\left\{X_6M_2(\theta)\right\}<0,\ \forall\theta\in\mathcal{B}_\theta,\end{aligned}\tag{17}$$

where X_3, X_4, X_5, and X_6 are defined as follows:

$$X_3=\begin{bmatrix}\tilde{\theta}_{m(\sigma+1)} & \mathbf{0}\\ \mathbf{0} & \tilde{\theta}_{n(\sigma+1)}\end{bmatrix},\ X_4=\begin{bmatrix}I_{\sigma+1}\otimes\tilde{\theta}_m & \mathbf{0}\\ \mathbf{0} & I_{\sigma+1}\otimes\tilde{\theta}_n\end{bmatrix},$$
$$X_5=\begin{bmatrix}\tilde{\theta}_{n(\sigma+1)}\\ 0_{l,n\sigma(\sigma+1)}\end{bmatrix},\ X_6=\begin{bmatrix}I_{\sigma+1}\otimes\tilde{\theta}_n\\ 0_{l,n\sigma(\sigma+1)}\end{bmatrix}.$$

If $\bar{N}(\theta)$ and the newly introduced matrices $F(\theta)$, $\hat{F}_1(\theta)$, $F_2(\theta)$, $\hat{M}_1(\theta)$, $M_2(\theta)$ and $H(\theta)$ in Theorems 1, 2, and 3 are set to be parameter-independent, then all LMIs in these theorems become parametrically affine LMIs; that is, we need only solve these LMIs at vertices of $\mathcal{B}_\theta$ to verify them, and we now have the following sufficient conditions.

Theorem 4: The system (1) is multi-quadratically stable for all $\theta \in \mathcal{B}_\theta$ if there exist a symmetric matrix $\bar{P}$ and matrices $F(\theta) = F$, $\hat{M}_1(\theta) = \hat{M}_1$ and $M_2(\theta) = M_2$ such that (8) and (14) hold for all vertices of $\mathcal{B}_\theta$.

Theorem 5: The system (1) is multi-quadratically stable and its H_∞ performance is bounded by γ_∞ for all $\theta \in \mathcal{B}_\theta$ if there exist a symmetric matrix $\bar{P}$ and matrices $F(\theta) = F$, $\hat{M}_1(\theta) = \hat{M}_1$ and $M_2(\theta) = M_2$ such that (8) and (15) hold for all vertices of $\mathcal{B}_\theta$.

Theorem 6: The system (1) is multi-quadratically stable and its H_2 performance is bounded by γ_2 for all $\theta \in \mathcal{B}_\theta$ if there exist symmetric matrices $\bar{P}$ and $\bar{N}(\theta) = \bar{N}$, and matrices $\hat{F}_1(\theta) = \hat{F}_1$, $F_2(\theta) = F_2$, $\hat{M}_1(\theta) = \hat{M}_1$, $M_2(\theta) = M_2$ and $H(\theta) = H$ such that (13), (16), and (17) hold for all vertices of $\mathcal{B}_\theta$.

Remark 1: One way to convexify (6) is to restrict $N(\theta)$ to be parametrically affine, similarly to [4]. However, it leads to conservatism. In our formulation, $N(\theta)$ is set to be parameter-dependent in negative as well as positive power series of parameters, and this yields less conservative formulation than one with being set $N(\theta)$ to be parametrically affine.

IV. RELATIONSHIPS WITH EXISTING RESULTS

In this section, we show the relationships between Theorem 4, a sufficient condition for biquadratic stability proposed by Trofino and de Souza [3], and conventional quadratic stability. We consider the following parametrically affine system in this section.

$$\begin{aligned}\dot{x}&=A(\theta)x\\ A(\theta)&=[A_0\ A_1\ \cdots\ A_k]\,\bar{\theta}_n^T\\ \bar{\theta}&=[1\,\theta_1\cdots\theta_k]\in\mathcal{R}^{1\times(k+1)}\end{aligned}\tag{18}$$

An LTIPD system (18) can be represented as one of (1); i.e. $l_1 = \cdots = l_k = 0$ and $m_1 = \cdots = m_k = 1$ in (1). We have an alternative representation for $A(\theta)$ in (18) as $A(\theta) = \bar{\theta}_n\bar{A}\bar{\theta}_n$, where $\bar{A} = \begin{bmatrix}A_0\,A_1\cdots A_k\\ 0_{nk,n(k+1)}\end{bmatrix}$. In addition, from the definition of $\tilde{\theta}$, we have the following representation of $\tilde{\theta}$; $\tilde{\theta} = \begin{bmatrix}\theta_1\ \cdots\ \theta_k\\ -I_k\end{bmatrix}$.

Biquadratic stability is defined as follows.

Definition 2 (Biquadratic Stability [3]): The system (18) is said to be biquadratically stable if there exists a symmetric matrix $P(\theta) = \bar{\theta}_n\bar{P}\bar{\theta}_n^T > 0$ such that (2) holds.

Trofino and de Souza propose the following sufficient condition for biquadratic stability after applying the Elimination lemma to an inequality; $\bar{\theta}_n\bar{P}\bar{\theta}_n^T > 0$ and (2), and setting newly introduced matrices to be parameter-independent.

Lemma 7: [3] The system (18) is biquadratically stable for all $\theta \in \mathcal{B}_\theta$ if there exist a symmetric matrix $\bar{P} \in \mathcal{S}^{n(k+1)\times n(k+1)}$ and matrices $F \in \mathcal{R}^{nk\times n(k+1)}$ and $M \in \mathcal{R}^{nk\times n(k+1)}$ such that (19) and (20) hold for all vertices of $\mathcal{B}_\theta$.

$$\bar{P}+\mathsf{He}\left(\tilde{\theta}_nF\right)>0\tag{19}$$

$$\mathsf{He}\left(\bar{P}\bar{\theta}_n^T\left[A_0\ A_1\ \cdots\ A_k\right]\right)+\mathsf{He}\left(\tilde{\theta}_n M\right)<0 \qquad (20)$$

We will claim that if there exists a symmetric matrix $\bar{P}$ and matrices F and M such that (19) and (20) hold, then there always exist a symmetric matrix $\bar{P}$ and matrices F, $\hat{M}_1$, and M_2 such that (8) and (14) in Theorem 4 hold.

We now assume that there exist $\bar{P}$, F, and M such that (19) and (20) hold for all vertices of $\mathcal{B}_\theta$. Inequality (19) is the same as (8) in Theorem 4 for the parametrically affine LTIPD system (18). Therefore, if there exist a symmetric matrix $\bar{P}$ and a matrix F such that (19) holds, then (8) always holds with the same $\bar{P}$ and F.

We now check inequality (14). We set $\bar{P}$ and M in Lemma 7 as follows:

$$\bar{P}=\begin{bmatrix}P_0 & P_1 & \cdots & P_k\\ P_1^T & P_{1,1} & \cdots & P_{1,k}\\ \vdots & \vdots & \ddots & \vdots\\ P_k^T & P_{1,k}^T & \cdots & P_{k,k}\end{bmatrix},\ M=\begin{bmatrix}M_{1,1} & \cdots & M_{1,k+1}\\ \vdots & \ddots & \vdots\\ M_{k,1} & \cdots & M_{k,k+1}\end{bmatrix},$$

where $P_i\in\mathcal{R}^{n\times n}$, $P_{i,j}\in\mathcal{S}^{n\times n}(i=j)$, $\mathcal{R}^{n\times n}(i\neq j)$, and $M_{i,j}\in\mathcal{R}^{n\times n}$. Further, let $\mathsf{He}(S)$ denote the left-hand term of (20); S denotes $\bar{P}\bar{\theta}_n^T\left[A_0\ A_1\ \cdots\ A_k\right]+\tilde{\theta}_n M$, and let $S_{i,j},\ i,j=1,\cdots,k+1$ denote the (i,j)th n-dimensional square matrix of S. Using these notations, we set $\hat{M}_1$ and M_2 in Theorem 4 as follows with a sufficiently small positive number ε:

$$\hat{M}_1=\begin{bmatrix}\hat{M}_{1_{1,1}} & \cdots & \hat{M}_{1_{1,k+1}}\\ \vdots & \ddots & \vdots\\ \hat{M}_{1_{k,1}} & \cdots & \hat{M}_{1_{k,k+1}}\end{bmatrix},\ \hat{M}_{1_{i,j}}=\begin{bmatrix}M_{i,j} & \mathbf{0}\\ \mathbf{0} & 0_{nk}\end{bmatrix},$$
$$M_2=\begin{bmatrix}\begin{bmatrix}P_1A_0 & \\ \vdots & \varepsilon I_{nk}\\ P_kA_0 & \end{bmatrix} & \cdots & \begin{bmatrix}P_1A_k & \\ \vdots & 0_{nk}\\ P_kA_k & \end{bmatrix}\\ \vdots & \ddots & \vdots\\ \begin{bmatrix}P_{1,k}^TA_0 & \\ \vdots & 0_{nk}\\ P_{k,k}A_0 & \end{bmatrix} & \cdots & \begin{bmatrix}P_{1,k}^TA_k & \\ \vdots & \varepsilon I_{nk}\\ P_{k,k}A_k & \end{bmatrix}\end{bmatrix}. \qquad (21)$$

With these definitions, we obtain the following inequality from (14) in Theorem 4.

$$\mathsf{He}\left(\begin{bmatrix}\tilde{S}_{1,1} & \cdots & \tilde{S}_{1,k+1}\\ \vdots & \ddots & \vdots\\ \tilde{S}_{k+1,1} & \cdots & \tilde{S}_{k+1,k+1}\end{bmatrix}\right)<0,$$
$$\tilde{S}_{i,j}=\begin{bmatrix}S_{i,j} & \mathbf{0}\\ \mathbf{0} & 0_{nk}\end{bmatrix},\ i,j=1,\cdots,k+1,\ i\neq j,$$
$$\tilde{S}_{i,i}=\begin{bmatrix}S_{i,i} & \varepsilon\left[\theta_1 I_n\ \cdots\ \theta_k I_n\right]\\ \mathbf{0} & -\varepsilon I_{nk}\end{bmatrix},\ i=1,\cdots,k+1$$

We apply congruence transformations with the following matrices, $i=1,\cdots,k-1$, to the above inequality:

$$\mathsf{diag}\left(I_{ni},\ Z_i,\ I_{n(k+1)^2-n\{(k+1)i+1\}}\right),$$

$$Z_i=\begin{bmatrix}0_n & 0_n & \cdots & 0_n & I_n\\ 0_n & I_n & & \mathbf{0} & 0_n\\ \vdots & & \ddots & & \vdots\\ 0_n & \mathbf{0} & & I_n & 0_n\\ I_n & 0_n & \cdots & 0_n & 0_n\end{bmatrix}\in\mathcal{R}^{n(ki+1)\times n(ki+1)}.$$

This transformation means the exchange of the $i+1$th n-dimensional square row/column matrices for the $i(k+1)+1$th n-dimensional square row/column matrices. We then obtain the following inequality:

$$\mathsf{He}\left(\begin{bmatrix}S & T_1 & \cdots & T_k\\ 0_{n(k+1)} & -\varepsilon I_{n(k+1)} & & \mathbf{0}\\ \vdots & & \ddots & \vdots\\ 0_{n(k+1)} & \mathbf{0} & \cdots & -\varepsilon I_{n(k+1)}\end{bmatrix}\right)<0,$$

$$T_i=\begin{bmatrix}\varepsilon\theta_i I_n & \mathbf{0}\\ \mathbf{0} & \begin{bmatrix}0_{n(i-1),nk}\\ \varepsilon\left[\theta_1 I_n\cdots\theta_k I_n\right]\\ 0_{n(k-i),nk}\end{bmatrix}\end{bmatrix},\ i=1,\ \cdots,\ k.$$

This inequality is equivalent to the following two conditions after applying the Schur complement.

$$-\varepsilon<0,\ \mathsf{He}(S)+\frac{\varepsilon}{2}\left(\theta_1^2+\cdots+\theta_k^2\right)I_{n(k+1)}<0$$

If we set ε to be sufficiently small, then the above conditions hold because θ_i is bounded; inequality (14) in Theorem 4 always holds for the same $\bar{P}$ in Lemma 7 and $\hat{M}_1$ and M_2 defined in (21).

From this discussion, if a parametrically affine LTIPD system (18) satisfies Lemma 7, then the system always satisfies Theorem 4. Thus our proposed methods encompass analysis methods based on biquadratic stability as a special case. As Trofino and de Souza prove, if parametrically affine LTIPD systems (18) satisfy quadratic stability, then they satisfy Lemma 7 [3]. Therefore if the systems satisfy quadratic stability, then they always satisfy Theorem 4; that is, our proposed methods also encompass analysis methods based on quadratic stability.

V. NUMERICAL COMPLEXITY

In this section, we compare the numerical complexity of our proposed method and previously proposed methods for stability analysis.

We assume that the size of the state vector is n, the number of parameters is k, and the LTIPD system (1) has parametrically affine matrices in the state-space representation; that is, we now consider a parametrically affine system (18). Under this assumption, quadratic stability and biquadratic stability are both applicable and we compare the numerical complexity of three methods; quadratic stability (QS), Lemma 7, and Theorem 4. Table I shows the number of decision variables in LMIs. QS has $1+2^k$ LMIs and other methods have 2×2^k LMIs to be solved. The table shows that our method has the greatest number of decision variables. Although our proposed method has much greater numerical complexity than existing results, we will show that it is effective in analyzing the stability and performance of LTIPD systems in the next section.

TABLE I

NUMERICAL COMPLEXITY FOR PARAMETRICALLY AFFINE SYSTEMS

Method	The number of decision variables
QS	$n(n+1)/2$
Lemma 7	$(nk+n)(nk+n+1)/2+n^2k(k+1)+n^2k(k+1)$
Theorem 4	$(nk+n)(nk+n+1)/2+n^2k(k+1)+2n^2k(k+1)^3$

VI. NUMERICAL EXAMPLES

First, we demonstrate the effectiveness of our proposed method with an example introduced in Section I. This is a slightly revised form of the example in [10]. The plant is a closed-loop system of the following plant and controller.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k/m_1 & k/m_1 & 0 & 0 \\ k & -k & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/m_1 \\ 0 \end{bmatrix} u$$

$$u = \frac{1.333(s-1.0257)(s+0.1301)}{(s+1.2245)^2+0.8290^2} x_2$$

We first maximize γ by the bisection algorithm such that the closed-loop system with $m_1 = 1.0$ and $k = 1.25+\delta_k$ is robustly stable for all $|\delta_k| \leq \gamma$. Both Theorem 4 with setting $l_1 = 0$ and $m_1 = 1$ and the method in [10] confirm that the system is robustly stable for all $|\delta_k| \leq 0.76252$, which is the limit obtained by gridding. We next maximize γ by the bisection algorithm such that the closed-loop system with $m_1 = 1.0+\delta_m$ and $k = 1.25+\delta_k$, $|\delta_k| \leq 0.762$ is robustly stable for all $|\delta_m| \leq \gamma$. Theorem 4 with setting $l_1 = 1$, $m_1 = 0$, $l_2 = 0$, and $m_2 = 1$ confirms that the system is robustly stable for all $|\delta_m| \leq 0.97186$, which is very close to the limit $|\delta_m| < 1.0$ obtained by gridding. The method in [10] confirms that the system is robustly stable for all $|\delta_m| \leq 0.036182$. No methods in [8] can confirm that the system is robustly stable. These results show that our method is less conservative than those previous works.

Second, we introduce an example in [13].

$$A(\theta) = \begin{bmatrix} -12 & -7 & 7 \\ -11 & -13 & -5 \\ -2 & 9 & -8 \end{bmatrix} + \alpha I_3 + \theta_1 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 3 & 0 \end{bmatrix} + \theta_2 \begin{bmatrix} 1 & 2 & 0 \\ -3 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

For the above $A(\theta)$, we first maximize α by the bisection algorithm such that the system is robustly stable for all $|\theta_i| \leq 1.0$, $i = 1, 2$. Theorem 4 with setting $l_1 = 0$, $m_1 = 1$, $l_2 = 0$, and $m_2 = 1$ gives the maximum α as 5.2432, which is the same value as obtained by gridding and the method in [13]. We next maximize α by the bisection algorithm such that the system with $\theta_1 = \theta_2$ is robustly stable for all $|\theta_1| \leq 1.0$. Theorem 4 with setting $l_1 = 0$ and $m_1 = 1$ gives the maximum α as 5.4177, which is the same value as obtained by gridding and the method in [13]. Thus, the conservatism of our method vanishes for this example.

Finally, we analyze the robust H_2 performance of a system in [4]; $\{A_0 + \theta A_1, B_0 + \theta B_1, I_2, \mathbf{0}\}$, where $[A_0|A_1]$ and $[B_0|B_1]$ are respectively given as follows:

$$\left[\begin{array}{cc|cc} -1.65 & -9.5 & -1.3 & -20 \\ 0 & -7 & 2 & -10 \end{array}\right], \quad \left[\begin{array}{cc|cc} 2.1 & -3.75 & 2.2 & 0.5 \\ -4 & -3.5 & -6 & -5 \end{array}\right].$$

We set the range of θ to be $|\theta| \leq 0.6$. The method in [4] gives the minimum of γ_2^2 as 35.004. Theorem 6 with setting $l_1 = 0$ and $m_1 = 1$, and setting $l_1 = 0$ and $m_1 = 2$ respectively gives the minimum of γ_2^2 as 33.846 and 33.827. We obtain the minimum of γ_2^2 as 33.827 by gridding. Thus, our method is less conservative than existing results and the conservatism of our method vanishes for this example.

VII. CONCLUSIONS

This paper proposes sufficient conditions for stability, H_∞, and H_2 performance analyses of LTIPD systems for which the state-space representations are parameter-dependent in negative as well as positive power series of parameters. In this study, we use parameter-dependent Lyapunov functions that are parameter-dependent in negative as well as positive power series of parameters. Although our proposed methods introduce additional matrices and require more computational effort to solve than previous works, numerical examples show that our methods are less conservative than previous works.

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