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State-Space Formulas for Gain-Scheduled ℓ_1-Optimal Controllers

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***Abstract*— Design of gain-scheduled controllers has attracted wide attention in $\mathcal{H}_\infty$ control. Within the framework of ℓ_1-optimal control, a novel method for gain-scheduled output-feedback control has been proposed recently using a special control structure. This paper extends this method to more general linear parameter-varying systems. Furthermore an explicit way of constructing the resulting gain-scheduled controllers is derived, and conditions for their existence are given. Finally, state-space formulas for these controllers are introduced. These formulas significantly facilitate an implementation and result in a relatively low complexity of the controller's parameter dependence.**

***Index Terms*— Gain-scheduling, LPV systems, ℓ_1-optimal control, Robust control.**

I. INTRODUCTION

Linear parameter-varying (LPV) systems are a powerful system class encompassing linear differential equations with parameter-depending coefficients. Common examples are flight dynamics depending on velocity and height, or robotic systems with varying mass or mass distribution. Furthermore, nonlinear systems linearized around specific trajectories can be cast into the LPV framework. Even general input-affine nonlinear systems

$$\dot{x} = f(x) + g(x)u$$

may be seen as LPV systems. They are rewritten as $\dot{x} = A(x)x + B(x)u$ and treated as the LPV system

$$\dot{x} = A(\rho)x + B(\rho)u,$$

introducing the parameter vector ρ and some conservatism.

Gain-scheduling techniques for LPV systems with measurable parameters have received widespread attention during the 1990s. In particular, schemes for systematic design of gain-scheduled controllers, guaranteeing overall stability and performance, have been developed in contrast to the ad-hoc controller-switching approaches of the 1960s and 70s. An overview is given in [9]. Fruitful developments with respect to gain-scheduling have taken place within the $\mathcal{H}_\infty$ framework (see e.g. [1], [2], [7], [10], and the references therein). However, $\mathcal{H}_\infty$ control lacks certain desirable properties such as specification of time-domain performance criteria and consideration of persistent disturbances or actuator saturation. On the other hand, the ℓ_1 control framework addresses these properties [3], [4]. Until recently, little progress has been made with respect to LPV gain-scheduling in the setup of ℓ_1 control. Except for a state-feedback approach [11], the output-feedback approach proposed in [8] is the only method existing in the ℓ_1 control framework to the best of the authors' knowledge.

In [8], a control structure is introduced, turning the LPV gain-scheduling problem into a classical robust performance problem. Its solution yields an LPV output-feedback controller, which typically has superior performance over robust controllers [8]. The method considers discrete-time LPV systems, in which the parameter dependence is affine, polynomial, or rational, and the rate of parameter variation may be unbounded. The controller synthesis is based on linear programs and achieves suboptimal solutions arbitrarily close to the optimum.

The developments in [8] are restricted to a certain class of plants, and implementation aspects of the controllers are neglected. This paper provides three important extensions to [8]. First, the results are made applicable to more general LPV plants. Second, the explicit construction of the resulting LPV controllers is derived, and conditions on the implementability of the LPV controller are given. Third, state-space formulas of the constructed LPV controllers are introduced. These formulas result in a relatively low complexity of the controller's parameter dependence, and thus an actual implementation is significantly facilitated.

As an additional feature, the proposed control structure and the presented approach to LPV gain-scheduling are likewise applicable to the framework of $\mathcal{H}_\infty$ control. In this context, the same control structure as in [8] has been introduced independently in [12] for $\mathcal{H}_\infty$ control, however under very restrictive assumptions. These assumptions are lifted by the developments of this paper. Although the results of this paper are presented in discrete time, the development remains the same also in continuous time.

The remainder of this article is organized as follows. Section II contains some preliminaries and notation. Section III reviews the problem setup and the control structure for its solution. In Section IV, the construction of LPV controllers is derived under certain plant conditions, whereas Section V gives corresponding state-space formulas. The conclusions are stated in Section VI.

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II. PRELIMINARIES

In this section, some properties and notations are defined. Let $|x|$ denote the absolute value of a real number x. The space of vector-valued right-sided bounded real sequences $x = \{x(k)\}_{k=0}^{\infty}$ with $x(k) = [x_1(k), \ldots, x_n(k)]^T$ is denoted by ℓ_∞^n, with a norm defined by

$$\|x\|_\infty := \max_{1\leq i\leq n} \sup_k |x_i(k)|.$$

The space of matrix-valued right-sided absolutely summable real sequences x with $x(k) = (x_{ij}(k))$ is denoted by $\ell_1^{m\times n}$, equipped with the norm

$$\|x\|_1 := \max_{1\leq i\leq m} \sum_{j=1}^{n} \sum_{k=0}^{\infty} |x_{ij}(k)|.$$

The ℓ_∞-induced norm of an operator $T : \ell_\infty^n \to \ell_\infty^m$ is defined by

$$\|T\|_{\infty-\text{ind}} := \sup_{0\neq w\in\ell_\infty} \frac{\|Tw\|_\infty}{\|w\|_\infty}.$$

For an LTI operator or transfer function, the ℓ_∞-induced norm is the ℓ_1-norm of its impulse response matrix.

The $\mathcal{Z}$-transform of a one-sided sequence x is defined as $X(z) := \sum_{k=0}^{\infty} x(k)z^{-k}$. A state-space realization of a transfer matrix $G(z)$ is written as

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := C(zI - A)^{-1}B + D = G(z).$$

Where it is not misleading, the dependence on z is sometimes left off for readability and brevity. The symbol I denotes the identity matrix of appropriate dimension, having ones on the diagonal and zeros elsewhere. The left-inverse $M^\dagger$ of an $m \times n$-matrix M ($m \geq n$) with full column rank is defined by $M^\dagger M = I$.

Let $M \in \mathbb{C}^{(m_1+m_2)\times(n_1+n_2)}$ be a matrix partitioned as

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix},$$

and let $\Delta_u \in \mathbb{C}^{n_1\times m_1}$ and $\Delta_l \in \mathbb{C}^{n_2\times m_2}$. Then upper and lower Linear Fractional Transformations (LFTs) are respectively defined as

$$\mathcal{F}_u(M, \Delta_u) := M_{22} + M_{21}\Delta_u(I - M_{11}\Delta_u)^{-1}M_{12}$$
$$\mathcal{F}_l(M, \Delta_l) := M_{11} + M_{12}\Delta_l(I - M_{22}\Delta_l)^{-1}M_{21},$$

assuming that the inverses exist.

III. GAIN-SCHEDULING IN THE ℓ_1 FRAMEWORK

This section introduces the basic concepts and structures of the gain-scheduling technique proposed in [8]. The results in the subsequent sections build on this foundation.

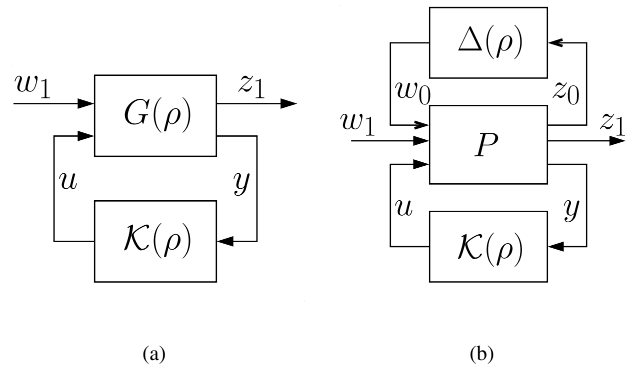

Fig. 1. (a) LPV control problem. (b) Modified problem.

A. LPV Systems and ℓ_1-Optimal Control

Given a discrete-time LPV system

$$\begin{bmatrix} Z_1(z) \\ Y(z) \end{bmatrix} = G(\rho, z) \begin{bmatrix} W_1(z) \\ U(z) \end{bmatrix}$$

with state-space realization

$$G(\rho, z) = \left[\begin{array}{c|cc} \bar{A}(\rho) & \bar{B}_1(\rho) & \bar{B}_2(\rho) \\ \hline \bar{C}_1(\rho) & \bar{D}_{11}(\rho) & \bar{D}_{12}(\rho) \\ \bar{C}_2(\rho) & \bar{D}_{21}(\rho) & \bar{D}_{22}(\rho) \end{array}\right] \tag{1}$$

where $u \in \ell_\infty^{n_u}$ is the control input, $y \in \ell_\infty^{n_y}$ is the measured output, $w_1 \in \ell_\infty^q$ represents exogenous inputs like disturbances or reference commands, and $z_1 \in \ell_\infty^q$ represents outputs for performance specifications (capital letters denote $\mathcal{Z}$-transformed signals). Equal dimensions of w_1 and z_1 can be achieved by introducing "dummy" inputs/outputs with almost no influence on the plant. The matrices depend on the time-varying parameter vector $\rho(k) = [\rho_1(k), \ldots, \rho_p(k)]^T$. The parameters ρ are assumed to be measurable in real-time and to be contained in the set

$$\Pi = \{\rho : \rho_i(k) \in [\rho_{\min,i}, \rho_{\max,i}] \quad \forall k,\ i = 1, \ldots, p\}. \tag{2}$$

In particular, the ρ_i need not be constant, and their rate of variation may be unbounded. It is the goal of the proposed controller design method to solve the following ℓ_1-synthesis problem, see also Fig. 1(a).

Problem 1: Given an LPV system (1), find an LPV output-feedback controller $U(z) = \mathcal{K}(\rho, z)Y(z)$ such that

1) the closed loop is internally stable in the presence of time-varying $\rho \in \Pi$, and
2) the ℓ_∞-gain $\sup_{\rho\in\Pi} \sup_{0\neq w_1\in\ell_\infty} \frac{\|z_1\|_\infty}{\|w_1\|_\infty}$ is minimal. ■

Note that the closed-loop mapping from w_1 to z_1 can be written as

$$T(G(\rho, z), \mathcal{K}(\rho, z)) = \mathcal{F}_l(G(\rho, z), \mathcal{K}(\rho, z)). \tag{3}$$

In [13], [8] it is shown that for several practically useful types of parameter dependence, (1) can be converted into

$$\begin{bmatrix} Z_0(z) \\ Z_1(z) \\ Y(z) \end{bmatrix} = P(z) \begin{bmatrix} W_0(z) \\ W_1(z) \\ U(z) \end{bmatrix} \tag{4}$$

$$W_0(z) = \Delta(\rho) Z_0(z), \tag{5}$$

with state-space realization

$$P(z) = \begin{bmatrix} P_{00}(z) & P_{01}(z) & P_{02}(z) \\ P_{10}(z) & P_{11}(z) & P_{12}(z) \\ P_{20}(z) & P_{21}(z) & P_{22}(z) \end{bmatrix} \tag{6}$$

$$= \left[\begin{array}{c|ccc} A & B_0 & B_1 & B_2 \\ \hline C_0 & D_{00} & D_{01} & D_{02} \\ C_1 & D_{10} & D_{11} & D_{12} \\ C_2 & D_{20} & D_{21} & D_{22} \end{array}\right],$$

where $\Delta : \Pi \to \mathbb{R}^{r \times r}$ is a possibly nonlinear function. The resulting control structure is depicted in Fig. 1(b). In particular, the matrices in (1) may depend on ρ in an affine, polynomial or rational manner. Examples and procedures on how to obtain (4)–(6) from (1) are given in [13], [8], for example. Note that P is an LTI system, all parameter dependence is shifted to the Δ block, and Δ typically is of diagonal structure.

As is standard, in the following it is assumed that (A, B_2) is stabilizable and (A, C_2) is detectable. Furthermore the "uncertainty" block $\Delta(\cdot)$ is viewed as an operator from ℓ_∞^r to ℓ_∞^r and assumed to belong to the set

$$\Theta_\Delta = \big\{\Delta = \mathrm{diag}(\Delta_1, \ldots, \Delta_r) : \Delta_i(\cdot) : \ell_\infty^1 \to \ell_\infty^1 \text{ is causal and } \|\Delta_i\|_{\infty-\mathrm{ind}} \le 1\big\},$$

which can be achieved by appropriate scalings included in P. The proposed setting may be extended in a straightforward manner to include unknown bounded structured uncertainties, which are not subject to measurement of any kind, into the Δ block, allowing for robust gain-scheduling.

B. An LPV Control Structure

To approach Problem 1, a control structure according to Fig. 2(a) is proposed in [8]. The same control structure has been introduced independently in [12], however under very restrictive assumptions. These assumptions are lifted by the developments of this paper.

A discrete-time LTI output-feedback controller

$$U(z) = K(z) \begin{bmatrix} Y(z) \\ \widetilde{Y}(z) \end{bmatrix} \tag{7}$$

with state-space representation

$$K(z) = [K_1(z) \quad K_2(z)] = \left[\begin{array}{c|cc} A_K & B_{K1} & B_{K2} \\ \hline C_K & D_{K1} & D_{K2} \end{array}\right] \tag{8}$$

is considered for control of the LPV system (4). The controller has access to the parameter information by means of the signal $\widetilde{y} = w_0$, and thus changes its behavior along with the plant.

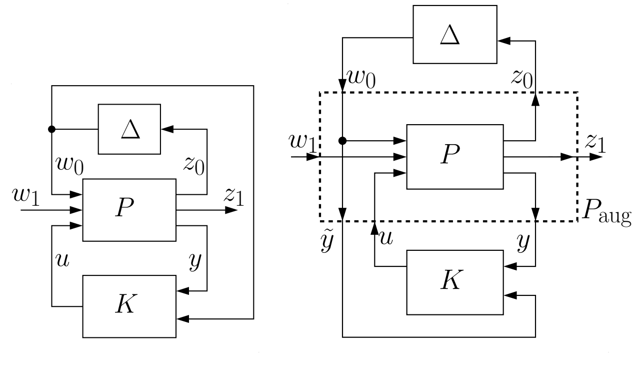

Fig. 2. (a) LPV control structure. (b) Transformed control structure.

The interconnection of Fig. 2(a) is redrawn as in Fig. 2(b). An augmented plant is formed from all parts of the control structure except K and Δ, leading to

$$\begin{bmatrix} Z_0(z) \\ Z_1(z) \\ Y(z) \\ \widetilde{Y}(z) \end{bmatrix} = P_{\text{aug}}(z) \begin{bmatrix} W_0(z) \\ W_1(z) \\ U(z) \end{bmatrix} \tag{9}$$

$$W_0(z) = \Delta(\rho) Z_0(z) \tag{10}$$

and its general state-space realization

$$P_{\text{aug}}(z) = \left[\begin{array}{c|ccc} A & B_0 & B_1 & B_2 \\ \hline C_0 & D_{00} & D_{01} & D_{02} \\ C_1 & D_{10} & D_{11} & D_{12} \\ C_2 & D_{20} & D_{21} & D_{22} \\ 0 & I & 0 & 0 \end{array}\right] \tag{11}$$

with appropriately dimensioned zero matrices. Still, P_{aug} and K are LTI. Note that the closed-loop mapping from w_1 to z_1 can be written as

$$\widetilde{T}(P_{\text{aug}}(z), K(z), \Delta) = \mathcal{F}_u\left(\mathcal{F}_l(P_{\text{aug}}(z), K(z)), \Delta\right). \tag{12}$$

These steps lead from Problem 1 to the following classical robust performance problem in face of structured uncertainty.

Problem 2: Given the system (9), find an LTI output-feedback controller K as in (7) such that

1) the closed loop is robustly internally stable $\forall \Delta \in \Theta_\Delta$, and
2) the closed-loop system achieves minimal ℓ_∞-gain and robust performance:
$\gamma := \inf_{K} \sup_{\Delta \in \Theta_\Delta} \|\widetilde{T}(P_{\text{aug}}, K, \Delta)\|_{\infty-\mathrm{ind}} < 1.$ ■

To solve such a robust performance problem, several steps are necessary. First, determine a Youla parameterization $\Phi(Q) = H - V_1 Q V_2$ of $\mathcal{F}_l(P_{\text{aug}}, K)$ [13], [3]. Second, find Q and L such that $\inf_{L \in \mathcal{L}} \inf_{Q \in \ell_1} \|L^{-1}\Phi(Q)L\|_1 < 1$, where $\mathcal{L} = \{\mathrm{diag}(l_1, \ldots, l_r, l_{r+1} I_q) : l_i > 0\}$ [4]. (Sub)optimal solutions to this infinite-dimensional non-convex optimization problem are possible either by means of L-Q-iterations [4] or via a branch-and-bound procedure [6]. In the end, both approaches are based on computationally attractive linear programs [4], [5], [6].

Note that in classical robust control, a pure LTI controller of the form $U(z) = K_1(z)Y(z)$ is sought to solve such a problem. This is a special case of (7) if the controller dependence on the parameter information w_0 is neglected. Thus, such a classical robust LTI controller clearly is more conservative in general than the proposed gain-scheduled controller. Application examples in [8] show the usefulness of the proposed gain-scheduling technique for LPV systems and its performance enhancement over robust controllers.

In a final step, an LPV controller $\mathcal{K}$ has to be constructed from the auxiliary LTI controller K, since the signal w_0 is generally not available to the controller as it depends on current parameter values and plant states. The next sections give conditions for existence and implementability of such LPV controllers, as well as state-space formulas for facilitating the implementation. Note that [12] on the contrary assumes w_0 to be measurable, which in general is not justified for practical application. Then of course, the controller K could be directly implemented.

IV. CONSTRUCTION AND IMPLEMENTABILITY OF GAIN-SCHEDULED CONTROLLERS

To construct an LPV controller $\mathcal{K}$ from the auxiliary LTI controller K, equations (7)–(11) are taken into account such that the signal w_0 is expressed in terms of u and y. This exposition is not restricted to the ℓ_1-framework, but rather related to the proposed control structure of Fig. 2(a). It can thus be directly applied to a corresponding approach in a framework like $\mathcal{H}_\infty$-optimal control. The general case $P_{01}(z) \not\equiv 0$ is dealt with first, whereas the reduction to the frequently arising case $P_{01}(z) \equiv 0$ is presented afterwards.

A. The case $P_{01}(z) \not\equiv 0$

The following lemma gives a transfer function description of the LPV controller $\mathcal{K}$ and conditions on when this transfer function can be constructed.

Lemma 1: Consider the control structure in Fig. 2(b), a controller K as in (7)–(8), and the augmented plant P_{aug} as in (9)–(11). Let

1) $n_y \geq q$,
2) D_{21} have full column rank,
3) $M_1 := I - \Delta(\rho)(D_{00} - D_{01}D_{21}^{\dagger}D_{20})$ have full rank $\forall\Delta(\rho)$, $\rho \in \Pi$, and
4) $I - D_{K2}M_1^{-1}\Delta(\rho)(D_{02} - D_{01}D_{21}^{\dagger}D_{22})$ have full rank $\forall\Delta(\rho)$, $\rho \in \Pi$.

Then an LPV controller $\mathcal{K}$ related to the structure in Fig. 1(b) exists and is given by

$$\begin{aligned}\mathcal{K}(\rho, z) = &\left(I - K_2\left(I - \Delta(\rho)\widetilde{P}_{00}\right)^{-1}\Delta(\rho)\widetilde{P}_{02}\right)^{-1} \cdot \\ &\left(K_1 + K_2\left(I - \Delta(\rho)\widetilde{P}_{00}\right)^{-1}\Delta(\rho)P_{01}P_{21}^{\dagger}\right),\end{aligned} \tag{13}$$

where $\widetilde{P}_{00} := P_{00} - P_{01}P_{21}^{\dagger}P_{20}$
$\widetilde{P}_{02} := P_{02} - P_{01}P_{21}^{\dagger}P_{22}$.

Proof: From (9) we have

$$W_1 = P_{21}^{\dagger}(Y - P_{20}W_0 - P_{22}U).$$

The left-inverse of P_{21} exists and is proper if $n_y \geq q$ and D_{21} has full column rank [13]. The equations (9)–(11) give

$$W_0 = \Delta Z_0 = \Delta(P_{00}W_0 + P_{01}W_1 + P_{02}U).$$

Inserting the above expression for W_1 and solving for W_0 yields

$$\begin{aligned}W_0 = &\left(I - \Delta(P_{00} - P_{01}P_{21}^{\dagger}P_{20})\right)^{-1} \cdot \\ &\Delta\left(P_{01}P_{21}^{\dagger}Y + (P_{02} - P_{01}P_{21}^{\dagger}P_{22})U\right).\end{aligned}$$

The inverse in this expression exists and is proper if and only if M_1 has full rank $\forall\Delta(\rho)$, $\rho \in \Pi$. This is due to the fact that the inverse of a transfer function exists and is proper if and only if the direct feed-through matrix of its corresponding state-space representation has an inverse [13]. From (7)–(8) we have

$$U = K_1Y + K_2\widetilde{Y} = K_1Y + K_2W_0.$$

Inserting the above expression for W_0 and solving for U results in (13). The outer inverse in (13) exists and is proper if and only if $I - D_{K2}M_1^{-1}\Delta(\rho)(D_{02} - D_{01}D_{21}^{\dagger}D_{22})$ has full rank $\forall\Delta(\rho)$, $\rho \in \Pi$. ■

Remark: The conditions of Lemma 1 can be influenced by imposing conditions on the chosen plant description or on the controller matrices. Appropriate modeling can often lead to $D_{00} = 0$ and $D_{01} = 0$, trivially satisfying condition 3. Moreover, imposing $D_{K2} = 0$ may also be helpful, generally resulting in some performance loss. In the most general case, the given controller construction makes it necessary to internally reconstruct the external disturbance w_1 from the measurements y and the control input u. It is not advisable to use the given construction in the presence of unstable zeros of P_{21}. In this case, $P_{21}^{\dagger}$ is an unstable system, and the slightest numerical or modeling errors lead to instability.

B. The case $P_{01}(z) \equiv 0$

The conditions of Lemma 1 relax significantly for the practically important case $P_{01} \equiv 0$.

Lemma 2: Consider the control structure in Fig. 2(b), a controller K as in (7)–(8), and the augmented plant P_{aug} as in (9)–(11). Let

1) $P_{01}(z) \equiv 0$,
2) the control problem be well-posed, that is $M_2 := I - \Delta(\rho)D_{00}$ have full rank $\forall\Delta(\rho)$, $\rho \in \Pi$, and
3) $I - D_{K2}M_2^{-1}\Delta(\rho)D_{02}$ have full rank $\forall\Delta(\rho)$, $\rho \in \Pi$.

Then an LPV controller $\mathcal{K}$ related to the structure in Fig. 1(b) exists and is given by

$$\mathcal{K}(\rho, z) = \left(I - K_2\left(I - \Delta(\rho)P_{00}\right)^{-1}\Delta(\rho)P_{02}\right)^{-1} K_1. \quad (14)$$

Proof: Using $P_{01} \equiv 0$, the equations (9)–(11) give

$$W_0 = \Delta Z_0 = \Delta(P_{00}W_0 + P_{02}U).$$

Solving for W_0 yields

$$W_0 = (I - \Delta P_{00})^{-1}\Delta P_{02}U.$$

The inverse in this expression exists and is proper if and only if M_2 has full rank $\forall\Delta(\rho)$, $\rho \in \Pi$. This is the case if and only if the problem and thus the plant (4)–(6) is well-posed [13]. From (7)–(8) we have

$$U = K_1Y + K_2\widetilde{Y} = K_1Y + K_2W_0.$$

Inserting the above expression for W_0 and solving for U results in (14). The outer inverse in (14) exists and is proper if and only if $I - D_{K2}M_2^{-1}\Delta(\rho)D_{02}$ has full rank $\forall\Delta(\rho)$, $\rho \in \Pi$. ∎

V. STATE-SPACE FORMULAS OF GAIN-SCHEDULED CONTROLLERS

In this section, state-space formulas for the controllers (13) and (14) are derived. These descriptions exhibit a minimal number of states and a relatively low parameter complexity. The ease of implementation compared to the use of the corresponding transfer functions is appealing. Again, the cases $P_{01} \not\equiv 0$ and $P_{01} \equiv 0$ are dealt with separately. A short discussion of controller complexity concludes this section.

The following standard result from linear algebra, the matrix inversion lemma, is useful in the derivation.

Lemma 3: [13, ch. 2.3] Suppose A and D are both nonsingular matrices, then
$(A - BD^{-1}C)^{-1} = A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1}$. ∎

A. The case $P_{01}(z) \not\equiv 0$

First, a modified transfer function of $\mathcal{K}$ is derived to ultimately obtain controller descriptions with a relatively low complexity of the parameter dependence.

Lemma 4: Under the conditions of Lemma 1, the transfer function (13) is equivalent to

$$\mathcal{K}(\rho, z) = \left(I + K_2\left(I - \Delta(\rho)(\widetilde{P}_{00} + \widetilde{P}_{02}K_2)\right)^{-1}\Delta(\rho)\widetilde{P}_{02}\right) \cdot \left(K_1 + K_2\left(I - \Delta(\rho)\widetilde{P}_{00}\right)^{-1}\Delta(\rho)P_{01}P_{21}^{\dagger}\right). \quad (15)$$

Proof: Application of Lemma 3 to the first part of (13) directly leads to (15). ∎

In a second step, state-space descriptions of the transfer functions involved in (15) are taken into account and combined to form a state-space description of (15). This description is then analytically reduced to one with a minimal number of states. In this context, a minimal number of states means that all abundant states arising from state-space operations are canceled. Further state reduction may be achieved by minimal realizations of (6), (8), and the resulting LPV controller, as well as by model reduction techniques. However, due to the controller's parameter dependence, this is not pursued in this general and analytic treatment.

Theorem 1: Under the conditions of Lemma 1, a state-space description of the gain-scheduled controller $\mathcal{K}$ with minimal number of states (in the above-mentioned sense) is given by

$$\mathcal{K}(\rho, z) = \left[\begin{array}{c|c} \mathcal{A}(\rho) & \mathcal{B}(\rho) \\ \hline \mathcal{C}(\rho) & \mathcal{D}(\rho) \end{array}\right], \quad (16)$$

where

$$\mathcal{A}(\rho) = \begin{bmatrix} \widetilde{A} + \widehat{B}\widetilde{D}(\rho)\widetilde{C}_0 & (\widetilde{B}_2 + \widehat{B}\widetilde{D}(\rho)\widetilde{D}_{02})C_K \\ B_{K2}\widetilde{D}(\rho)\widetilde{C}_0 & A_K + B_{K2}\widetilde{D}(\rho)\widetilde{D}_{02}C_K \end{bmatrix} \quad (17)$$

$$\mathcal{B}(\rho) = \begin{bmatrix} \widehat{B}_2 + \widehat{B}\widetilde{D}(\rho)\widehat{D}_{02} \\ B_{K1} + B_{K2}\widetilde{D}(\rho)\widehat{D}_{02} \end{bmatrix} \quad (18)$$

$$\mathcal{C}(\rho) = \begin{bmatrix} D_{K2}\widetilde{D}(\rho)\widetilde{C}_0 & (I + D_{K2}\widetilde{D}(\rho)\widetilde{D}_{02})C_K \end{bmatrix} \quad (19)$$

$$\mathcal{D}(\rho) = (I + D_{K2}\widetilde{D}(\rho)\widetilde{D}_{02})(D_{K1} + D_{K2}\widehat{D}(\rho)D_{01}D_{21}^{\dagger}), \quad (20)$$

and

$$\begin{aligned}
\widetilde{D}(\rho) &:= \left(I - \Delta(\rho)(\widetilde{D}_{00} + \widetilde{D}_{02}D_{K2})\right)^{-1}\Delta(\rho) \\
\widehat{D}(\rho) &:= (I - \Delta(\rho)\widetilde{D}_{00})^{-1}\Delta(\rho) \\
\widetilde{A} &:= A - B_1D_{21}^{\dagger}C_2 \\
\widetilde{B}_0 &:= B_0 - B_1D_{21}^{\dagger}D_{20} \\
\widetilde{B}_2 &:= B_2 - B_1D_{21}^{\dagger}D_{22} \\
\widetilde{C}_0 &:= C_0 - D_{01}D_{21}^{\dagger}C_2 \\
\widetilde{D}_{00} &:= D_{00} - D_{01}D_{21}^{\dagger}D_{20} \\
\widetilde{D}_{02} &:= D_{02} - D_{01}D_{21}^{\dagger}D_{22} \\
\widehat{B} &:= \widetilde{B}_0 + \widetilde{B}_2D_{K2} \\
\widehat{B}_2 &:= \widetilde{B}_2D_{K1} + B_1D_{21}^{\dagger} \\
\widehat{D}_{02} &:= \widetilde{D}_{02}D_{K1} + D_{01}D_{21}^{\dagger}.
\end{aligned}$$

Proof: The transfer function (13) of the controller $\mathcal{K}$ follows from Lemma 1. According to Lemma 4, (13) is equivalent to (15). Basic state-space operations [13, ch. 3.6], followed by some tedious algebra and simplification steps, lead to (16)–(20). The number of states is minimal (in the above-mentioned sense), since the number of states is exactly the number of states of A plus the number of states of A_K. ■

B. The case $P_{01}(z) \equiv 0$

Consideration of the case $P_{01}(z) \equiv 0$ yields strongly simplified formulas. These formulas are obtained by canceling terms with P_{01} in (15), as well as terms with D_{01} and/or $D_{21}^{\dagger}$ in (16)–(20).

Lemma 5: Under the conditions of Lemma 2, the transfer function (14) is equivalent to

$$\mathcal{K}(\rho, z) = \left(I + K_2 \left(I - \Delta(\rho)(P_{00} + P_{02}K_2)\right)^{-1} \Delta(\rho) P_{02}\right) K_1. \tag{21}$$

Proof: Application of Lemma 3 to the first part of (14) directly leads to (21). ■

Theorem 2: Under the conditions of Lemma 2, a state-space description of the gain-scheduled controller $\mathcal{K}$ with minimal number of states (in the above-mentioned sense) is given by

$$\mathcal{K}(\rho, z) = \left[\begin{array}{c|c} \mathcal{A}(\rho) & \mathcal{B}(\rho) \\ \hline \mathcal{C}(\rho) & \mathcal{D}(\rho) \end{array}\right], \tag{22}$$

where

$$\mathcal{A}(\rho) = \begin{bmatrix} A + BD(\rho)C_0 & (B_2 + BD(\rho)D_{02})C_K \\ B_{K2}D(\rho)C_0 & A_K + B_{K2}D(\rho)D_{02}C_K \end{bmatrix} \tag{23}$$

$$\mathcal{B}(\rho) = \begin{bmatrix} (B_2 + BD(\rho)D_{02})D_{K1} \\ B_{K1} + B_{K2}D(\rho)D_{02}D_{K1} \end{bmatrix} \tag{24}$$

$$\mathcal{C}(\rho) = \begin{bmatrix} D_{K2}D(\rho)C_0 & (I + D_{K2}D(\rho)D_{02})C_K \end{bmatrix} \tag{25}$$

$$\mathcal{D}(\rho) = (I + D_{K2}D(\rho)D_{02})D_{K1}, \tag{26}$$

and

$$D(\rho) := \left(I - \Delta(\rho)(D_{00} + D_{02}D_{K2})\right)^{-1} \Delta(\rho)$$
$$B := B_0 + B_2 D_{K2}.$$

Proof: The derivation is analogous to the proof of Theorem 1, or follows directly as a special case of (16)–(20). ■

C. Complexity of the Controller's Parameter Dependence

Concerning the parameter dependence of the controller, a short discussion is in order to give judgement about its complexity and typical structure. This discussion is based on the simpler case $P_{01} \equiv 0$ without loss of generality.

The state-space formulas (23)–(26) depend on the parameter ρ solely by means of the term

$$D(\rho) = \left(I - \Delta(\rho)(D_{00} + D_{02}D_{K2})\right)^{-1} \Delta(\rho). \tag{27}$$

On the other hand, it can be shown that a direct state-space realization of (14) involves terms of the form

$$\begin{aligned} D_{\text{alt}}(\rho) = &(I - \Delta(\rho)D_{00})^{-1}\Delta(\rho)D_{02} \cdot \\ &\left(I - D_{K2}(I - \Delta(\rho)D_{00})^{-1}\Delta(\rho)D_{02}\right)^{-1} \cdot \\ &D_{K2}(I - \Delta(\rho)D_{00})^{-1}\Delta(\rho). \end{aligned} \tag{28}$$

Such a second state-space realization is also a valid one, even with the same number of states. However, the controller's parameter dependence is more complex and involves a larger number of computations. For example, if the functional dependence of $\Delta(\rho)$ on ρ is affine, rational functions in ρ with much higher polynomial order of numerator and denominator arise from (28) than from (27). Thus it is concluded that the achieved reduction in complexity justifies the application of the matrix inversion lemma (Lemma 3) to proceed from (14) to (21). Structurally equal observations are made in the more general case $P_{01} \not\equiv 0$.

VI. CONCLUSIONS

This paper follows up on a novel design method for gain-scheduled output-feedback controllers in the ℓ_1-optimal control framework, introduced in [8]. Three major extensions to the original method are presented: (i) The approach is made applicable to more general linear parameter-varying plants; (ii) a derivation of the explicit controller construction and its conditions is given; (iii) a compact state-space description of the resulting controllers yields convenient implementation possibilities and a low complexity of the parameter dependence. Additionally, restrictive assumptions of the setup in [12] are lifted by the developments of this paper, making the results applicable to other frameworks like $\mathcal{H}_\infty$ control.

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