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Parameter-dependent Lyapunov functions for time varying polytopic systems

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Abstract— **This paper provides global asymptotic stability conditions for continuous-time varying polytopic systems, using a parameter dependent Lyapunov function. The time varying parameter uncertainty as well as its time derivative are modelled as belonging to polytopic convex sets and their dependence is made explicit to get less conservative results. A particular case, characterized by the parameter uncertainty satisfying a linear differential equation is analyzed and a simpler version of the aforementioned stability conditions is presented. The results are expressed in terms of linear matrix inequalities being thus numerically solvable with no big difficulty. The theory is illustrated by the determination of the asymptotic stability region of a Mathieu's type equation with an uncertain time varying parameter.**

I. INTRODUCTION

In this paper, asymptotic stability of continuous time varying polytopic linear systems is addressed. For a given set of real matrices $\{A_1, \cdots, A_N\}$ of compatible dimensions, this class of systems obeys the following differential equation

$$\dot{x}(t) = A(\sigma(t))x(t) \ , \ x(0) = x_0 \ , \quad \forall t \geq 0 \qquad (1)$$

where $x(t) \in R^n$ is the state vector, x_0 is the initial condition and $\sigma(t) \in R^N$, defined for all $t \geq 0$ with components $\sigma_i(\cdot)$, $i = 1, \cdots, N$ represents time varying unknown parametric perturbations such that

$$A(\sigma(t)) \in co\{A_1, \cdots, A_N\} \qquad (2)$$

where $co\{\cdot\}$ denotes the convex hull, [8]. The main goal is to determine conditions assuring that the equilibrium solution of (1) is globally asymptotically stable for all $\sigma(t)$ such that (2) holds. One of the first result on this problem for time invariant polytopic systems was presented in [6] and generalized in [7] where new stability conditions based on a parameter dependent Lyapunov function have been introduced. For continuous time varying systems, in the literature to date, there are several papers dealing either with this or with similar problems, [1], [5], [9], [10]. The main difference with the present work is that either they consider affine systems or the proposed Lyapunov function does not explicitly depend on the unknown parameters. Among them the result of [9] seems to be the more general since the proposed quadratic Lyapunov function contains the affine ones as special cases.

In this paper we follow the same route of [7]. The proposed Lyapunov function is parameter dependent and $\dot{\sigma}(t)$ is modelled by means of a norm bounded differential inclusion. The stability conditions apply to any number of extreme matrices $\{A_1, \cdots, A_N\}$ and may be considerably simplified to cope with parameters being given by the solution of linear differential equations with arbitrary initial conditions. This case is of importance whenever it applies since only a set of precise trajectories of interest can be taken into account. As in [7], the present stability conditions for time-varying systems are also expressed in terms of linear matrix inequalities (LMI), being thus solvable by the numeric machinery available in the literature to date. The reader is requested to see [2] where this topic is exhaustively treated.

The notation used throughout is standard. Capital letters denote matrices, small letters denote vectors and small Greek letters denote scalars. For matrices or vectors $(')$ indicates transpose. For symmetric matrices, $X > 0$ (≥ 0) indicates that X is positive definite (nonnegative definite). The set of real numbers are denoted by R. For $h \in R^N$, $\|h\|_\infty = \max\{|h_1|, \cdots, |h_N|\}$. The $\mathcal{L}_2$ squared norm of $x(t) \in R^n$ defined for all $t \geq 0$ equals $\|x(t)\|_2^2 = \int_0^\infty x(t)'x(t)dt$, see [3].

II. MODELLING AND PROBLEM FORMULATION

First of all, any system of the form (1)-(2), can be expressed in the more convenient form

$$A(\sigma(t)) = \sum_{i=1}^{N} \sigma_i(t)A_i \ , \quad \sigma(t) \in \Lambda_N \qquad (3)$$

where Λ_N is the simplex

$$\Lambda_N := \left\{ \lambda \in R^N \ : \ \sum_{i=1}^{N} \lambda_i = 1, \ \lambda_i \geq 0 \right\} \qquad (4)$$

Our purpose in this paper, is to state conditions to assure that the equilibrium solution of (1) is globally asymptotically stable, which implies that $\|x(t)\|_2$ is bounded for all $t \geq 0$ and goes to zero as t goes to infinity for all bounded initial condition $x(0) = x_0$ and all $\sigma(t) \in \Lambda_N$. Several authors addressed stability problems for continuous time-varying systems, see for instance [9] and the references therein. The main difference between these works stems from the model used to define the feasible perturbation vector $\sigma(t)$ and its time derivative $\dot{\sigma}(t)$. In this paper, we

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further consider that $\dot\sigma(t)$ exits and it satisfies the differential inclusion $\dot\sigma(t) \in \Omega_M$ for all $t \geq 0$, where Ω_M is the following polyhedral convex set,

$$\Omega_M \in co\{h_1, \cdots, h_M\} \tag{5}$$

where $h_j \in R^N$ for all $j = 1, \cdots, M$ are given vectors. Once again, it is immediately seen that for each $t \geq 0$, there exists $\mu(t) \in R^M$ such that

$$\dot\sigma(t) = \sum_{j=1}^{M} \mu_j(t) h_j \ , \quad \mu(t) \in \Lambda_M \tag{6}$$

When dealing with polytopic continuous time-varying systems, it is important to keep in mind that the vectors used to define Ω_M can not be chosen arbitrarily. In fact, assuming that we want to restrict the magnitude of the time derivative as, for instance, $\|\dot\sigma(t)\|_\infty \leq r$ for all $t \geq 0$, it is essential to take into account that any $\sigma(t) \in \Lambda_N$ satisfies the constraint $c'\sigma(t) = 1$ for $c' = [1 \ \cdots \ 1]$ and all $t \geq 0$. As a consequence, the vectors $h_1, \cdots, h_M$ should be determined in such a way that

$$co\{h_1, \cdots, h_M\} = \{h \in R^N \ : \ \|h\|_\infty \leq r, \ c'h = 0\} \tag{7}$$

Since the set on the right hand side of (7) is convex and polyhedral, it is always possible to determine a finite number of vectors $h_1, \cdots, h_M$ satisfying this condition. In fact, they are precisely the set of all basic solutions of the polyhedral convex set appearing on the right hand side of (7). Notice however that, generally, the minimum number of vectors M needed to keep (7) true is greater than N as we can easily see since for $N = 2$, only two vectors suffice but for $N = 3$ the minimum number of vectors is four.

This model is very general and so it can be useful for modelling polytopic continuous time-varying systems, the only restriction to take into account for modelling purposes is that the set Ω_M must be convex and polyhedral. However, taking M big enough it may approximate, within a desired precision level any convex set, yielding the possibility to consider in (7) other vector norms, as for instance the Euclidean norm. In addition, the particular case characterized by $M = N$ and $\mu(t) = \sigma(t)$ is of great interest because the feasibility of $\sigma(t) \in \Lambda_N$ and $\dot\sigma(t) \in \Omega_N$ for all $t \geq 0$, implies that these signals are coupled together through the linear differential equation $\dot\sigma(t) = H\sigma(t)$ where $H \in R^{N\times N}$. As it will be clear in the sequel, in this case, the stability conditions simplify considerably.

III. STABILITY CONDITION

In this section, before we give the main result of this paper, we need to introduce the following notation. Associated to a set composed by a certain number of matrices $\{U_1, U_2, \cdots\}$ of compatible dimensions, let us define the linear matrix function

$$U(z) := \sum_i z_i U_i \tag{8}$$

for all vectors $z = [z_1 \ z_2 \ \cdots]'$ with $z_i \in R$ being its i-th component. From this definition, it follows that the linear function $P(\lambda)$, defined from the set of positive definite matrices $\{P_1, \cdots, P_N\}$, is positive definite for all $\lambda \in \Lambda_N$. This fact, enables us to define the Lyapunov function

$$v(x(t)) := x(t)'P(\sigma(t))x(t) \tag{9}$$

associated to the system (1) which is a natural generalization, to cope with time varying parameter uncertainty, of the parameter dependent Lyapunov function considered in [7], for the particular case $\dot\sigma(t) = 0$ for all $t \geq 0$. The next theorem provides the stability condition for system (1).

Theorem 1: Assume there exist a set of positive definite matrices $\{P_1, \cdots, P_N\}$ and sets of matrices $\{G_1, \cdots, G_M\}$, $\{V_1, \cdots, V_M\}$ of compatible dimensions satisfying the following linear matrix inequalities

$$\begin{bmatrix} A_i'G_j + G_j'A_i + P(h_j) & P_i + A_i'V_j - G_j' \\ P_i + V_j'A_i - G_j & -V_j - V_j' \end{bmatrix} < 0 \tag{10}$$

for all $i = 1, \cdots, N$ and $j = 1, \cdots, M$. The equilibrium solution $x = 0$ of the system (1) is globally asymptotically stable for all $\sigma(t) \in \Lambda_N$ such that $\dot\sigma(t) \in \Omega_M$.

Proof: For each $i = 1, \cdots, N$ fixed, since by assumption $\dot\sigma(t) \in \Omega_M$, take $\mu(t) \in \Lambda_M$ such that (6) holds and multiply the linear matrix inequality (10) by the nonnegative scalar $\mu_j(t)$. Summing up the results for all $j = 1, \cdots, M$, using the fact that the equality

$$\sum_{j=1}^{M} \mu_j P(h_j) = P\left(\sum_{j=1}^{M} \mu_j h_j\right) = P(\dot\sigma) \tag{11}$$

holds due to the linearity of $P(\cdot)$, it follows that the linear matrix inequality

$$\begin{bmatrix} R_i + P(\dot\sigma) & S_i - G(\mu)' \\ S_i' - G(\mu) & -V(\mu) - V(\mu)' \end{bmatrix} < 0 \tag{12}$$

where

$$\begin{aligned} R_i &= A_i'G(\mu) + G(\mu)'A_i \\ S_i &= P_i + A_i'V(\mu) \end{aligned}$$

is verified for each $i = 1, \cdots, N$. Now, taking $\sigma(t) \in \Lambda_N$, multiplying each inequality (12) by the nonnegative scalar $\sigma_i(t)$ and summing up, one gets

$$\begin{bmatrix} \hat R + P(\dot\sigma) & \hat S - G(\mu)' \\ \hat S' - G(\mu) & -V(\mu) - V(\mu)' \end{bmatrix} < 0 \tag{13}$$

where

$$\begin{aligned} \hat R &= A(\sigma)'G(\mu) + G(\mu)'A(\sigma) \\ \hat S &= P(\sigma) + A(\sigma)'V(\mu) \end{aligned}$$

which multiplied to the left by $[I \ A(\sigma)']$ and to the right by its transpose, yields

$$A(\sigma)'P(\sigma) + P(\sigma)A(\sigma) + P(\dot\sigma) < 0 \tag{14}$$

The theorem is proved since this inequality imposes that the time derivative of the radially unbounded parameter dependent Lyapunov function (9) is negative along all trajectories of the system (1), for all perturbations satisfying $\sigma(t) \in \Lambda_N$ and $\dot{\sigma}(t) \in \Omega_M$, simultaneously. ■

This result deserves some remarks. The first one is related to the fact that $h = 0$ is feasible for (7) which implies that there exists a vector $\mu_0 \in \Lambda_M$ such that

$$\sum_{j=1}^{M} \mu_{0j} h_j = 0 \tag{15}$$

Replacing μ by μ_0 in the proof of Theorem 1, the linear matrix inequalities (12) simplifies to

$$\begin{bmatrix} A_i' G_0 + G_0' A_i & P_i + A_i' V_0 - G_0' \\ P_i + V_0' A_i - G_0 & -V_0 - V_0' \end{bmatrix} < 0 \tag{16}$$

for $i = 1, \cdots, N$ where $G_0 = G(\mu_0)$ and $V_0 = V(\mu_0)$. This reproduces the result of [7] assuring robust stability for all $\sigma(t) \in \Lambda_N$ such that $\dot{\sigma}(t) = 0 \in \Omega_M$ for all $t \geq 0$. The second remark is related to the concept of quadratic stability, characterized by the fact that the set of matrices $\{A_1, \cdots, A_N\}$ shares the same Lyapunov function, that is when there exists $P > 0$ such that $A_i' P + P A_i < 0$ for all $i = 1, \cdots, N$. Setting $P_i = P$ for $i = 1, \cdots, N$ and $G_j = P$ for $j = 1, \cdots, M$ and using the equality

$$\begin{aligned} P(h_j) &= \sum_{i=1}^{N} h_{ij} P_i \\ &= c' h_j P \\ &= 0 \ , \ \forall j = 1, \ldots, M \end{aligned} \tag{17}$$

holding true as a consequence of (7), it is immediately verified that (10) is equivalent to

$$V_j + V_j' > 0 \ , \quad A_i' P + P' A_i + A_i' V_j (V_j + V_j')^{-1} V_j' A_i < 0 \tag{18}$$

which is always verified by taking V_j sufficiently small for all $j = 1, \cdots, M$. The equality (17) is of particular importance since for a set of quadratically stable of matrices, the condition provided by Theorem 1 does not depend on a particular choice of r in the set Ω_M. It reproduces, as a particular case, the well known result that under the quadratic stability assumption, robust stability is preserved with arbitrary and possibly unbounded $\dot{\sigma}(t)$.

At this point, let us discuss another particular case of the results obtained so far. It occurs when the set Ω_M is defined by $M = N$ extreme vectors and the convex combination (6) is calculated with $\mu(t) = \sigma(t) \in \Lambda_N$, for all $t \geq 0$. The consequence is that the differential inclusion $\dot{\sigma}(t) \in \Omega_N$ becomes the linear differential equation

$$\dot{\sigma}(t) = H \sigma(t) \tag{19}$$

where $H := [h_1, \ \cdots, \ h_N]$ and, in addition to the constraints appearing in the right hand side of (7) we assume that $h_{ij} \geq 0$ for all $j \neq i = 1, \cdots, N$. In other words, we require that H belongs to a special class of Metzler matrices, defined as

$$\mathcal{M} := \left\{ H \in \ R^{N \times N} \ : \ \sum_{i=1}^{N} h_{ij} = 0, \ h_{ij} \geq 0, \ j \neq i \right\} \tag{20}$$

which assures that for any initial condition $\sigma(0) \in \Lambda_N$ the solution of (19) is always feasible, that is, $\sigma(t) \in \Lambda_N$ for all $t \geq 0$.

Theorem 2: Let $H \in \mathcal{M}$ and assume that there exist a set of positive definite matrices $\{P_1, \cdots, P_N\}$ and matrices G, V of compatible dimensions satisfying the following linear matrix inequalities

$$\begin{bmatrix} A_j' G + G' A_j + P(h_j) & P_j + A_j' V - G' \\ P_j + V' A_j - G & -V - V' \end{bmatrix} < 0 \tag{21}$$

for all $j = 1, \cdots, N$. The equilibrium solution $x = 0$ of the system (1) is globally asymptotically stable for all $\sigma(t)$ solution of the differential equation (19) with initial condition $\sigma(0) \in \Lambda_N$.

Proof: Multiplying the linear matrix inequalities (21) by $\sigma_j \geq 0$, summing up for all $j = 1, \cdots, N$ and taking into account that due to (19) the equality

$$\sum_{j=1}^{N} \sigma_j P(h_j) = P\left(\sum_{j=1}^{N} \sigma_j h_j \right) = P(\dot{\sigma}) \tag{22}$$

holds, once again one gets (14). Hence, the time derivative of the Lyapunov function (9) is negative along all trajectories of the system under consideration. This proves the proposed theorem. ■

From the proofs of Theorem 1 and 2 it is immediate to see that adding a matrix $Q \geq 0$ in the first block of (10) and (21) the asymptotical stability property of (1) is preserved and

$$\int_0^\infty x(t)' Q x(t) dt < x(0)' P(\sigma(0)) x(0) \tag{23}$$

which implies that any trajectory $x(t)$ for all $t \geq 0$ is quadratically integrable and so belongs to the $\mathcal{L}_2$ space, see [3].

It is interesting to observe that the linear matrix inequalities (10) to be feasible require the set of matrices $\{A_1, \cdots, A_N\}$ be robustly stable. A similar reasoning shows that the necessary condition for the feasibility of (21) is much weaker. In fact, it is well known that for $H \in \mathcal{M}$, the eigenvector associated to the null eigenvalue is non-negative which enables us to say that for this class of matrices there always exists $\lambda_\infty \in \Lambda_N$ such that $H\lambda_\infty = 0$. Since the initial condition $\sigma(0) = \lambda_\infty$ produces $\sigma(t) = \lambda_\infty$ for all $t \geq 0$, then (14) simplifies to

$$A(\lambda_\infty)' P(\lambda_\infty) + P(\lambda_\infty) A(\lambda_\infty) < 0 \tag{24}$$

meaning that the matrix $A(\lambda_\infty)$ must be asymptotically stable. As a final remark, multiplying the inequalities (21) to the right by $[I\ A_j']$ and to the left by its transpose we see that the inequality

$$A_j'P_j + P_jA_j + \sum_{i=1}^{N} h_{ij}P_i < 0 \tag{25}$$

holds for all $j = 1, \cdots, N$. With $H \in \mathcal{M}$, this recovers the condition for stochastic stability of jump linear systems, see [4] for more detail. This fact however, will not be further investigated in the present paper.

Remark 1: Let consider again the system

$$\dot{x} = \left(\sum_{i=1}^{N} \sigma_i(t)A_i\right)x \tag{26}$$

where

$$\dot{\sigma}(t) = H\sigma(t) \tag{27}$$

and let e_k be the $k{-}th$ column of the identity matrix (whose dimension is clear from the context). It turns out that the composite system (26), (27) can be rewritten as

$$\dot{x} = \left(\sum_{i=1}^{N} \sigma_i(0)Z_i(t)\right)x \tag{28}$$

where

$$Z_i(t) = \begin{bmatrix} e_i'e^{H't}B_1 \\ e_i'e^{H't}B_2 \\ \vdots \\ e_i'e^{H't}B_N \end{bmatrix}, \quad B_j = \begin{bmatrix} e_j'A_1 \\ e_j'A_2 \\ \vdots \\ e_j'A_N \end{bmatrix}$$

Hence a necessary condition for robust stability is that the given time-varying matrices $Z_i(t)$ are exponentially asymptotically stable. Notice that when $H = 0$ it follows $Z_i(t) = A_i$. The study of the robust polytopic stability in a purely time-varying context will be the subject of future research.

IV. EXAMPLE

In this section the following second order linear differential equation is considered.

$$\ddot{x}(t) + \xi\dot{x}(t) + (w^2 + \alpha^2 p(t))x(t) = 0 \tag{29}$$

where w, ξ and α are scalars and $p(t)$ is a time varying parameter not exactly specified but such that $|p(t)| \le 1$ and $|\dot{p}(t)| \le \omega$ for all $t \ge 0$. It is interesting to see that this differential equation reduces to the celebrated Mathieu's equation with damping for $p(t) = cos(\omega t)$. In order to rewrite (29) in the form (1), it suffices to set $N = 2$, define the parameter vector

$$\sigma(t) := \begin{bmatrix} 0.5 + 0.5\ p(t) \\ 0.5 - 0.5\ p(t) \end{bmatrix}, \forall\ t \ge 0 \tag{30}$$

the extreme matrices

$$A_1 = \begin{bmatrix} 0 & 1 \\ -(w^2+\alpha^2) & -\xi \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ -(w^2-\alpha^2) & -\xi \end{bmatrix} \tag{31}$$

and $r = \omega/2$. Our goal is to determine the region of the plane $\omega \times \alpha$ with $0 \le \omega \le 4$ and $0 \le \alpha \le 1$ such that global asymptotical stability is preserved. For numerical calculations we have considered $w = 1$ and a small damping represented by $\xi = 0.05$. Since (7) provides

$$H = [h_1\ \ h_2] = \begin{bmatrix} -r & r \\ r & -r \end{bmatrix} \in \mathcal{M} \tag{32}$$

then Theorem 1 states that for each pair (ω, α) satisfying the linear matrices inequalities (10), the equilibrium solution of the differential equation (1) is globally asymptotically stable for all $\sigma(t) \in \Lambda_N$ such that $\dot{\sigma}(t) \in \Omega_N$.

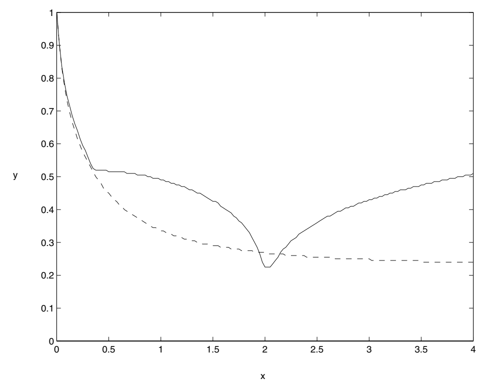

Fig. 1. Stability regions

Clearly, due to the fact that $p(t) = Rmcos(\omega t)$ is a feasible trajectory, the region we have just determined from Theorem 1 is inside the region of global asymptotical stability of the Mathieu's equation.

To apply the result of Theorem 2, we notice that setting $N = 3$, the linear differential equation $\dot{\sigma}(t) = H\sigma(t)$ with

$$H = \begin{bmatrix} q & -3q & q \\ 3q & -q & -q \\ -4q & 4q & 0 \end{bmatrix}, \quad \sigma(0) = \begin{bmatrix} 0.50 \\ 0.25 \\ 0.25 \end{bmatrix} \tag{33}$$

and $q = \omega/4$, has a solution such that $\sigma(t) \in \Lambda_N$ for all $t \ge 0$ and $cos(\omega t) = 4\sigma_1(t) - 1$ which states that the Mathieu's equation can be rewritten as (1) with the extreme matrices given by

$$\begin{aligned} A_1 &= \begin{bmatrix} 0 & 1 \\ -(w^2+3\alpha^2) & -\xi \end{bmatrix} \\ A_2 = A_3 &= \begin{bmatrix} 0 & 1 \\ -(w^2-\alpha^2) & -\xi \end{bmatrix} \end{aligned} \tag{34}$$

and Theorem 2 enables us to conclude that it is globally asymptotically stable for all pairs (ω, α) satisfying the linear matrices inequalities (21).

Figure 1 shows the regions of stability below each curve provided by Theorem 1 (dashed line) and Theorem 2 (solid line) respectively, calculated with an LMI solver that verifies feasibility of each stability condition. It is to be noticed the important improvement when the present result is compared with the one provided by a pure quadratic Lyapunov function independent of the uncertain time varying parameter which assures asymptotical stability only for $0 \leq \alpha \leq 0.22$ and all $\omega \in R$. Moreover, H given in (33) is not a Metzler matrix, but it can be verified that the eigenvector associated to the null eigenvalue is such that $\lambda_\infty = [0.25\; 0.25\; 0.50]' \in \Lambda_N$ and

$$A(\lambda_\infty) = \begin{bmatrix} 0 & 1 \\ -w^2 & -\xi \end{bmatrix} \tag{35}$$

is asymptotically stable. In this case, as we have numerically verified for $\alpha = 1.5$ and $\omega \geq 203$ the linear matrix inequalities (21) are always feasible even though matrices A_2 and A_3 are both unstable. This is an improvement of the stability condition whenever the differential inclusion is replaced by a linear differential equation.

Remark 2: Associated with (29) one can define the periodic polynomial operator

$$p(\sigma, t) = \sigma^2 + \xi\sigma + \omega^2 + \alpha^2 cos(\omega t)$$

where the symbol σ denotes the derivative operator. The system underlying the Mathieu equation is stable if and only if this polynomial has all characteristic exponents in the open left half plane, see [11]. This happens iff there exists a periodic function $k(t)$ such that

$$\omega^2 + \alpha^2 cos(\omega t) - \xi^2/4 + \dot{k}(t) + k(t)^2 = 0$$

with

$$\left| \frac{1}{T} \int_0^T k(\tau) d\tau \right| < \xi/2$$

The comparison between this exact parametrization and the results obtained above will be the subject of future investigations.

V. CONCLUSIONS AND FUTURE WORKS

A. Conclusions

In this paper we have introduced new stability conditions for continuous time-varying polytopic systems. Assuming that the time derivative of the unknown parameters belong to a polyhedral set, it was possible to describe the time dependence of the parameters taking into account the special structure of the class of dynamic systems under consideration. For illustration purpose, an estimation of the region of global asymptotic stability of a Mathieu's type equation with a time varying uncertain parameter has been determined by means of the proposed parameter dependent Lyapunov function.

B. Future Works

In future work we will investigate on more stringent relations between stabilization of polytopic and switched systems.

VI. ACKNOWLEDGMENTS

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REFERENCES

[1] Almeida, H. L. S., Bhaya, A., Falcão, D. M., and Kaszkurewicz, E. (2001). A team algorithm for robust stability analysis and control design of uncertain time-varying linear systems using piecewise quadratic Lyapunov functions. *Int. Jour. of Robust and Nonlinear Contr.*, vol. 11, pp. 357-371.

[2] S. P. Boyd, L. El Ghaoui, E. Feron and V. Balakrishnan (1994). *Linear Matrix Inequalities in System and Control Theory*, SIAM, Philadelphia.

[3] P. Colaneri, J. C. Geromel and A. Locatelli (1997). *Control Theory and Design - An RH_2 and RH_∞ viewpoint*, Academic Press.

[4] Feng, X., Loparo, K. A., Ji, Y., and Chizeck, H. J. (1992). Stochastic stability properties of jump linear systems. *IEEE Trans. Automat. Contr.*, vol. 37, pp. 38-53.

[5] Gahinet, P., Apkarian, P., and Chilali, M. (1996). Affine parameter dependent Lyapunov functions and real parametric uncertainty. *IEEE Trans. Automat. Contr.*, vol. 41, pp. 436-442.

[6] Geromel, J. C., Peres, P. L. D., and Bernussou, J. (1991). On a convex parameter space method for linear control design of uncertain systems. *SIAM J. Control Optim.*, vol. 29, pp. 381-402.

[7] Geromel, J. C, de Oliveira, M. C, and Hsu, L. (1998). LMI characterization of structural and robust stability. *Linear Algebra and its Applications*, vol. 285, pp. 69-80.

[8] R. Rockafellar (1970). *Convex Analysis*, Princeton Press.

[9] Trofino, A., and de Souza, C. E. (2001). Biquadratic stability of uncertain linear systems. *IEEE Trans. Automat. Contr.*, vol. 46, pp. 1303-1307.

[10] Xie, L., Shishkin, S. and Fu, M. (1997). Piecewise Lypaunov functions for robust stability of linear time-varying systems. *Syst. & Contr. Letters* , vol. 31, pp. 165-171.

[11] P. Colaneri (2004). Periodic Systems: Theoretical aspects Proc. *2nd IFAC Worshop on Periodic Control Systems*, Yokohama (JP), August 30-September 2.

[12] H. Ye, A. N. Michel and L. Hou (1998). Stability theory for hybrid dynamical systems, *IEEE Trans. Automat. Contr.*, vol. 43, pp. 461–474.