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Use of Dimensional Analysis to Reduce the Parametric Space for Gain-Scheduling

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***Abstract*—A simple method is presented to reduce the number of scheduling parameters for gain-scheduled controller implementation. The method is based on careful reformulation of the dynamic representation using dimensionless parameters. Parameter re-scaling is necessary to transform from one unit system to another, and this work utilizes unit transformations carefully to act as an implicit method of gain scheduling. The choice of a dimensionless formulation of the parameters simplifies the system representation because the number of dimensionless parameters will always be less than or equal to that in the classical representation. The resulting dimensionless gain-scheduling problem can greatly reduce the complexity of the classical gain-scheduling formulation, potentially even reducing an LPV problem in the classic domain to a zero-parameter or LTI controller design problem in the dimensionless domain. The dimensionless gain-scheduling method is demonstrated using a gantry system with the objective of minimizing the swing of the load.**

***Index Terms*—Gain scheduling, gantry control, pi theorem, dimensionless parameters, nondimensional representation**

I. INTRODUCTION

GAIN scheduling is a popular method to control systems whose linearized model dynamics vary widely over their range of operation. Examples include nonlinear systems and linear parameter-varying (LPV) systems operating over a wide parameter range. There are several methods to implement a gain-scheduling algorithm, including switching control gains as a function of operating conditions (see for e.g. [1]) or interpolating between different linear control laws (see for e.g. [2]). With new mathematical packages, symbolic formulations of dynamics are easier than ever to derive and manipulate, so nonlinear systems are increasingly gain-scheduled directly through the use of LPV controllers or nonlinear control laws [3-6]. Tavakoli [7] and Astrom [8] have utilized dimensional analysis to guide controller designs, specifically in developing dimensionless parameters in the tuning of PID controllers. This work has quite a different focus: to utilize control theory to guide dimensional analysis.

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Fig. 1 shows a generalized representation of a gain scheduled control system. Here the system representation is partitioned between V gain scheduling parameters, $p_1, p_2, ..., p_V$, which affect both the controller and plant. The plant will usually be dependent on a larger set of N parameters inclusive of the varying, gain-scheduled parameters as well as $R = N - V$ constant parameters , i.e. $P(p_1, p_2, ..., p_V, p_{V+1}, ..., p_N)$. The intent of gain-scheduling is therefore simultaneously change the control law as a function of the same time-varying parameters affecting the plant, namely $p_1, p_2, ..., p_V$, as shown in Fig. 1.

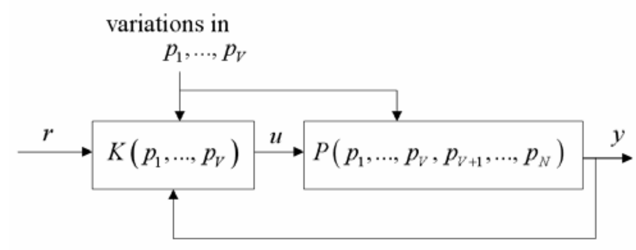

Fig. 1. Controller and plant both scheduled to varying parameters

To formulate the control laws, the controller of $K(p_1, ..., p_V)$ is obtained either from symbolic manipulation or from user-defined heuristic functions. The user defined functions can be obtained from interpolating or extrapolating from point wise controller-synthesis solutions obtained at fixed values of the scheduling variables. User-defined heuristic functions are useful in that they do not require derivation of symbolic controller representations, and in some cases do not even require a symbolic plant representation. This allows gain-scheduling to remain a useful control technique for systems whose dynamic representations are too complex, or for controller synthesis procedures that are too complex to permit closed-form algebraic solutions.

The task of controller design and/or controller evaluation grows exponentially more cumbersome with increasing number of scheduling variables. If the number of scheduling variables is V and the number of levels of each variable to be considered is H, then the number of

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different controller designs and/or analyses, N_V, is given by:

$$N_V = H^V \tag{1}$$

For example, if a system is analyzed at two fixed levels of each scheduling variable ($H = 2$), then a three parameter gain-scheduling controller ($V = 3$) will require eight designs and/or analyses. Choosing a more realistic assumption where a nonlinear or LPV system requires analysis at a more levels of each variable – ten levels for instance – then the complexity clearly grows exceedingly quickly: a three variable problem requires 1000 designs, a four variable 10,000 designs, etc. This exponential growth in controller complexity clearly limits the number of scheduling variables that can be easily considered in design or analysis of common gain scheduling algorithms.

Here we present a technique to overcome this issue by attempting to reformulate gain-scheduling problems in a lower number of scheduling variables. The organization of this work is as follows: Section II presents a simple plant – a gantry system – that is used as an illustrative example throughout the remaining sections. Section III presents the underlying approach of gain-scheduling and a simple method of forming dimensionless representations. Section IV further analyzes gain-scheduling in specific but common cases where parameters influencing the system may be varying simultaneously but in a coupled relationship. Section V attempts to quantify the amount of simplification from this method, i.e. the number reduction of scheduling variables. Finally, a summary of the main points is given in section VI.

II. Dynamic Modeling and Numerical Results

Gantry systems are chosen to serve as examples of the gain-scheduling topics presented in this work because they are implemented in many applications ranging from industrial overhead cranes to harbor load handling systems, their dynamics are easily derived from first principles, and their operation requires scheduling with respect to operating conditions. A standard gantry loading cycle involves lifting a load, moving the load to a different position (perhaps while lifting and/or lowering), lowering the load in a new position, then returning the lifting device to a new load to start a new cycle. This process generally requires two or more of the parameters of the system to vary in a given cycle, namely the varying payload on the gantry, m_p, the varying swing-length of the pendulum, L, and the varying mass of the trolley, m_t. Recent work on crane control by Corriga, et. al [9] focusing on implicit gain-scheduling via time scaling can be considered special cases of the current work.

The gantry system is modeled as a pendulum attached to a moving trolley as shown in Fig. 2. The equation of motion of the gantry system, as derived in Franklin, et. al. [10] is simplified by assuming the arm inertia, I, to be very small compared to the rotational inertia of the load, $m_p L^2$:

$$\begin{aligned}&(m_t + m_p)\ddot{x} + b\dot{x} + m_p L\ddot{\theta}\cos(\theta) - m_p L\dot{\theta}^2\sin(\theta) = u\\ &L\ddot{\theta} + g\sin(\theta) = -\ddot{x}\cos(\theta)\end{aligned} \tag{2}$$

For small motions of θ about the equilibrium position $\theta = 0$, it can further be assumed that $\sin(\theta) \approx 0$, $\cos(\theta) \approx 1$ and $\dot{\theta}^2 \approx 0$. Using these assumptions, a linearized form of (2) is given by (3).

$$\begin{aligned}&(m_t + m_p)\ddot{x} + m_p L\ddot{\theta} + b\dot{x} = u\\ &\ddot{x} + L\ddot{\theta} + g\theta = 0\end{aligned} \tag{3}$$

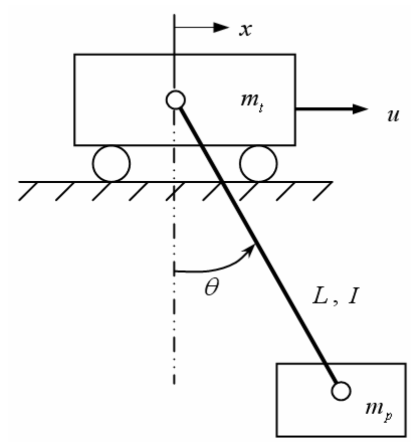

Fig. 2. Schematic of the gantry system

In state-space form, defining $x_1 = x$; $x_2 = \theta$; $x_3 = \dot{x}$ and $x_4 = \dot{\theta}$, the equation of (3) can be written as, $\dot{\mathbf{x}} = A\mathbf{x} + Bu$, where,

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \dfrac{m_p g}{m_t} & -\dfrac{b}{m_t} & 0 \\ 0 & -\dfrac{g}{L}\left(1 + \dfrac{m_p}{m_t}\right) & -\dfrac{b}{m_t L} & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{m_t} \\ -\dfrac{1}{m_t L} \end{bmatrix} \tag{4}$$

In practice, either the nonlinear equations of (2) or the linear equations of (3) could be used for gain-scheduled controller design using the method of nondimensional representation. For simplicity we utilize the linearized model in later sections.

III. Method of Nondimensional Gain-Scheduling

The proposed control technique for gain-scheduling is based on a reparameterization of the system representation into an equivalent but nondimensional formulation. The following definitions are given to simplify later discussion: Let N be the total number of parameters in a system description. Let R be the number of constant parameters, i.e. parameters that are not being gain-scheduled, and denote each of these parameters by $p_1 ... p_R$. Let V be the number of varying parameters, i.e. parameters that are used to schedule a gain-scheduled controller, and denote each of these parameters by $p_1 ... p_V$. Let M be the number of physical dimensions required to describe all the N parameters in the governing equation.

As an example, the gantry system from before, has $N = 5$ parameters, m_p, m_t, b, L, g. Three of these variables, m_p, L, m_t, are hereafter assumed to be varying significantly and will require gain-scheduling ($V = 3$). The remaining two variables are assumed to be constant ($R = 2$).

The process of transforming to/from nondimensional representation from/to dimensional (classic) representation requires rescaling with respect to M dimensioned parameters, one for each physical dimension. By carefully choosing these scaling parameters, an implicit gain scheduling can be achieved where as many as M of the V parameters are removed from the nondimensional system description. These variables only enter the system representatation for rescaling hence the nondimensional representation and controller for this system will then be scheduled with respect to fewer parameters.

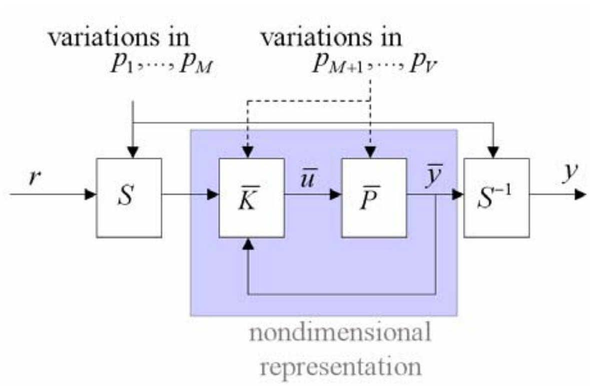

Fig. 3. Scheduling controller and scaling parameters with respect to same varying parameters affecting the model

To determine which of the gain-scheduled variables to use for controller design and which to choose for rescaling the system, the process of dimensional transformation is now generalized and presented. Consider any functional description of dynamic behavior (a plant or controller) dependent on N variables (the same N defined earlier). If the description is a dynamic one, the variables will generally span unit dimensions of length, mass, and distance. Hereafter we assume that the units of each parameter p_i can be written as a vector that is extracted via a dimensional extraction operator, $d = \mathbb{D}(u, p_i)$. To uniquely define this vector, one must specify both the unit space as well as the parameter. For instance, the gravitational constant, $g = 9.81\ \mathrm{m/s^2}$, has dimensional units that depend on the unit space: it can be represented in the unit system, $u = [m \quad kg \quad s]^T$, as a column vector, $d_{g,u} = \mathbb{D}(u, g) = [1 \quad 0 \quad -2]^T$, or in a unit system of $u_2 = [kg \quad N]^T$, as $d_{g,u2} = [-1 \quad 1]^T$.

The dimensional unit system is an arbitrary factor in representing a system, therefore we seek to rescale the system by selecting unit systems that give advantage to the gain-scheduling control problem. Some basis systems, particularly ones producing dimensionless representations, are clearly advantageous for controller design purposes (see Brennan [11]). The transformation from a dimensioned to a unitless system is fairly straightforward and follows a simple basis transformation (see Szirtes [12]). These are formalizations of a procedure first described in the Buckingham-Pi Theorem [12]. First, one notes the units of each variable and writes the vectors representing each variable's unit dependence, $d_i = \mathbb{D}(u, p_i)$, as columns of a matrix. For the gantry example, we select a unit system, $u = [m \quad kg \quad s]^T$. The matrix in (5) summarizes each variable and corresponding unit dimensions as column vectors. This matrix representation will always have M rows and $N + S$ columns. The S term represents the number of additional signals (time, state variables, input signals, etc.) that may also be renormalized in the new dimensional system.

	Signals				Parameters				
	t	u	θ	x	b	m_t	m_p	L	g
m	0	1	0	1	0	0	0	1	1
kg	0	1	0	0	1	1	1	0	0
s	1	–2	0	0	–1	0	0	0	–2

(5)

For clarity, the variable labels are shown at top of the matrix and unit dimensions are shown at left of the matrix.

Next, one rearranges the columns of the matrix, selecting M columns such that those selected are linearly independent, i.e. they can together form a new [M x M] matrix of rank M. This rearrangement is always feasible if the unit system is not redundant (see Szirtes [12]). For convenience, we hereafter assume that the matrix in the form of (5) is arranged such that these M columns occur on the right-most columns. This allows the partitioning of the matrix in the form

	physical	variables
dimensions	B_D	A_D
$\pi - groups$	I	C_S

(6)

with A_D square and full rank. One then performs the operation that will represent variables of B_D in the dimensional basis given by the vectors associated with the variables of A_D:

$$C_S = \left(-A_D^{-1}.B_D\right)^T \tag{7}$$

This operation on the gantry variables of (5) forms the lower-right partition below in Equation (8). The lower left partition is always unity to generate dimensionless parameters. The bottom rows of the partitioned matrix in (8) indicate the dimensionless parameters of the system, with the number of dimensionless parameters, Q, given by $Q = N - M$.

	t	m_t	b	x	θ	u	m_p	L	g
m	0	0	0	1	0	1	0	1	1
kg	0	1	1	0	0	1	1	0	0
s	1	0	-1	0	0	-2	0	0	-2
π_1	1	0	0	0	0	0	0	$-1/2$	$1/2$
π_2	0	1	0	0	0	0	-1	0	0
π_3	0	0	1	0	0	0	-1	$1/2$	$-1/2$
π_4	0	0	0	1	0	0	0	-1	0
π_5	0	0	0	0	1	0	0	0	0
π_6	0	0	0	0	0	1	-1	0	-1

(8)

Each bottom row indicates how new dimensionless parameters, hereafter called π-parameters, should be created from powers of each of the column variables. For instance, the fourth row gives $\pi_1 = t^1 \cdot L^{-1/2} \cdot g^{1/2}$. All the gantry π-parameters (including signal normalizations) are shown in (9).

$$\pi_1 = t\sqrt{g/L}, \quad \pi_2 = \frac{m_t}{m_p}, \quad \pi_3 = \frac{b}{m_p}\sqrt{L/g}$$
$$\pi_4 = \frac{x}{L} = \bar{x}, \quad \pi_5 = \theta = \bar{\theta}, \quad \pi_6 = \frac{u}{m_p g} = \bar{u} \tag{9}$$

In the gantry example, the choice of variables m_p, L and g to form a new dimensional basis (sometimes called repeating variables) inherently chooses a new time-scaling of $\sqrt{g/L}$, mass scaling of $1/m_p$, and length scaling of $1/L$.

The resulting system dynamics can always be rewritten in terms of these new π-parameters. For instance, the dimensionless, nonlinear representation of the gantry is obtained from (2):

$$\begin{aligned}(\pi_2+1)\bar{x}'' + \pi_3\bar{x}' + \bar{\theta}''\cos(\bar{\theta}) - \sin(\bar{\theta}) &= \bar{u} \\ \bar{\theta} + \sin(\bar{\theta}) &= -\bar{x}''\cos(\bar{\theta})\end{aligned} \tag{10}$$

where a time normalization is applied to the derivative operator, i.e. $(') \equiv \frac{d}{d\tau} = \sqrt{L/g}\,\frac{d}{dt}$. The linearized form of (10) in state-space form is given by, $\bar{\mathbf{x}}' = \bar{A}\bar{\mathbf{x}} + \bar{B}\bar{u}$, where the states are defined as $\bar{x}_1 = \bar{x}$; $\bar{x}_2 = \bar{\theta}$; $\bar{x}_3 = \bar{x}'$ and $\bar{x}_4 = \bar{\theta}'$, and the dimensionless matrices $\bar{A}$ and $\bar{B}$ given by:

$$\bar{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1/\pi_2 & -\pi_3/\pi_2 & 0 \\ 0 & -(1+1/\pi_2) & \pi_3/\pi_2 & 0 \end{bmatrix} \quad \bar{B} = \begin{bmatrix} 0 \\ 0 \\ 1/\pi_2 \\ -1/\pi_2 \end{bmatrix} \tag{11}$$

Note that the original system of equations (2) and (3) required three gain-scheduled variables. In the dimensionless representation, there are now only two parameters in (10) and (11) that require scheduling for a gain-scheduling algorithm. The other one parameter is absorbed by the unit scaling operation.

By carefully choosing certain gain-scheduling variables for dimensional transformations, these parameters can be mapped to a lower parameter space (i.e., from three to two, in this case) in the non-dimensional system representation.

Indeed, this work suggests that there exists a class of LPV systems that may be transformed to linear time invariant (LTI) system representation with no loss of model information. Specifically, this potentially occurs if $M \geq V$ where V is the number of gain-scheduled variables in the system, and these variables fully span the dimension space of the system description, then a nondimensional representation might be made that has no variable scheduling. This is represented in Fig. 4 below. Section V discusses specific conditions where such gain-scheduling simplification can be established.

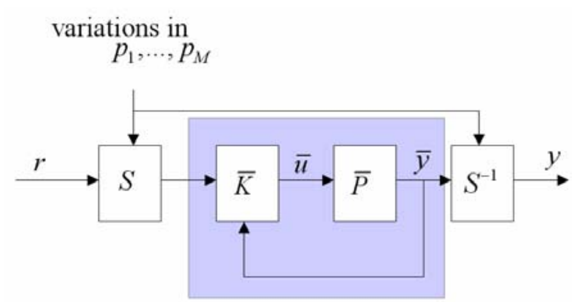

Fig. 4. In certain cases, the scaling parameters alone fully account for varying parameters of the system, resulting in an LTI plant and LTI controller design from what was originally an LPV system.

IV. Nondimensional Gain-Scheduled Controller Design

It may appear questionable that the dynamics and controller design for a dimensioned representation is equivalently performed in a dimensionless representation. To illustrate that they are indeed equivalent, a full state feedback controller is designed using a simple pole placement method [10] in both the dimensional (K) and dimensionless ($\bar{K}$) domain using system representations from (4) and (11). The same physical parameters are used in each design: (i.e., $m_p = 0.196kg$, $m_t = 1.21kg$, $L = 0.311m$, $b = 2.5kg/s$). A diagram of the feedback control structure is shown in Fig. 5.

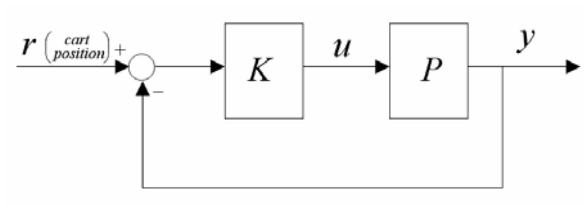

Fig. 5. A simple feedback control structure

The desired pole locations were chosen for this instance to be: $p=[-2.5\pm i, \quad -1.5\pm 2i]$, with resulting gains K and $\overline{K}$ are:

$$\begin{aligned} K &= [1.7137 \quad 3.7978 \quad -0.4955 \quad -2.3871] \\ \overline{K} &= [0.2733 \quad 1.9474 \quad -0.4470 \quad -6.9234] \end{aligned} \tag{12}$$

The equivalence of controller designs in both the dimensional and dimensionless domains are seen by comparison of tracking performance from both closed-loop systems, shown Fig. 6 and 7.

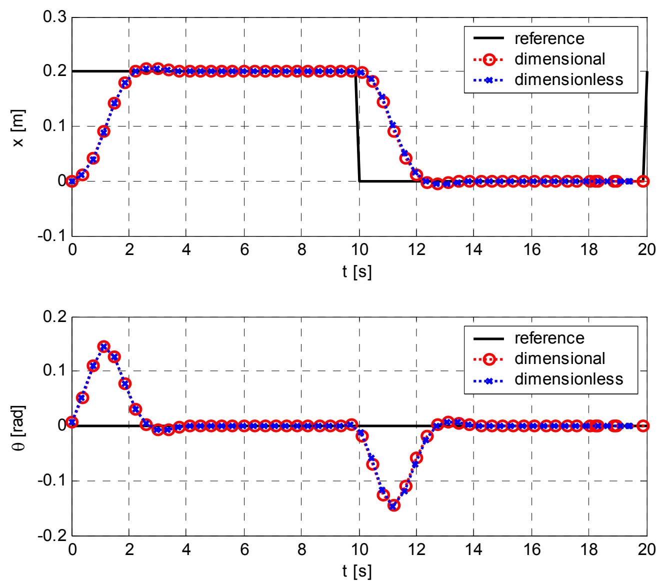

Fig. 6. Tracking response of a square wave, (a) cart position (b) position of the pendulum

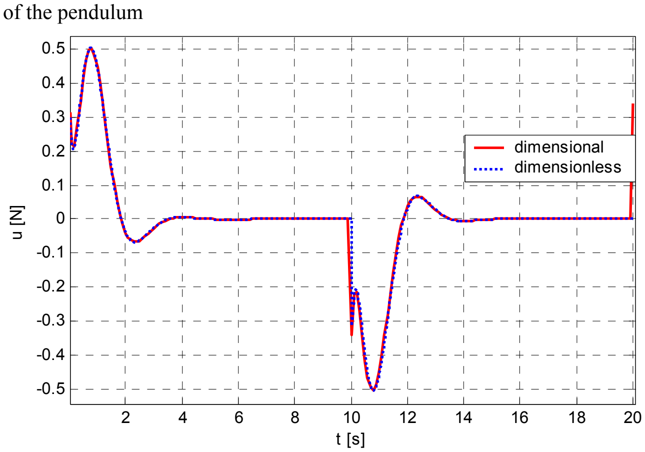

Fig. 7. Control action result of the dimensional and dimensionless cases.

Because the controller gains in the dimensionless case are dependent only on π_2 and π_3, variations in the parameters m_p, L and m_t may not affect the controller as if π_2 and π_3 are constant despite these changes. Indeed, it is common that parameter variations occur in a coupled manner in a physical system, for instance an object with increased mass nearly always has increased rotational inertias. A direct consequence of this coupling is that there is an invariant parameter space. That is, all the plants in this invariant space can be controlled by a single dimensionless controller.

To demonstrate the invariant parameter space of the gantry system and its effect on gain scheduling, the same controllers K and $\overline{K}$ have been used to control the new plant which have different m_p, L and m_t from the previous case but the same π_2 and π_3. More specifically, $m_p = 0.784kg$, $m_t = 4.84kg$, and $L = 4.976m$. The performance of the dimensionless controller and the resulting control action, as shown in Fig. 8 and Fig. 9, respectively, shows that plants with the same dimensionless parameters can really be controlled by a single controller even though significant parameter variation has occurred, as noted by the poor performance of the original, dimensioned controller.

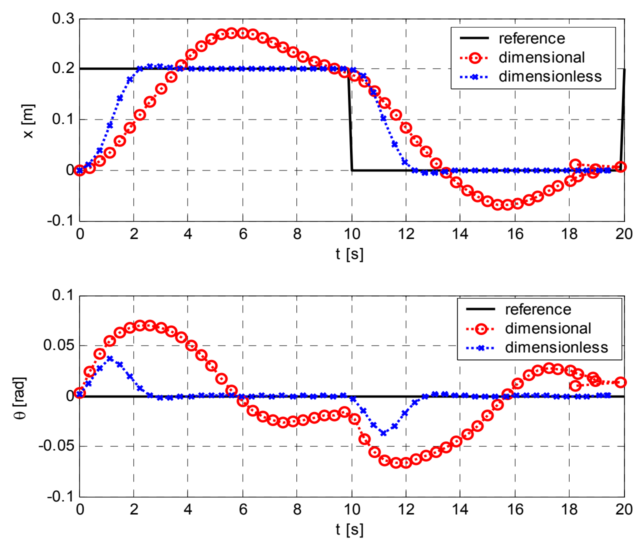

Fig. 8. Use of a single dimensionless controller for plants in the invariant space in relation to the dimensionless parameters.

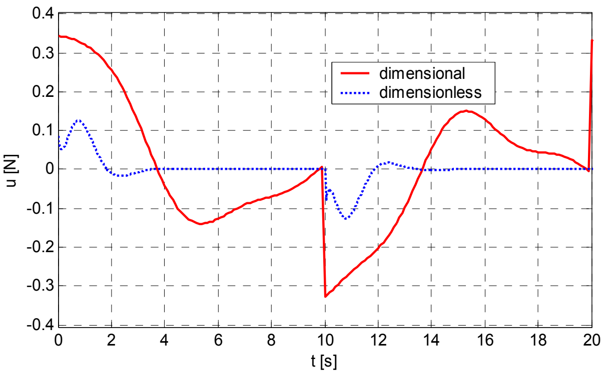

Fig. 9. Control action for plants in the invariant space.

V. CONDITIONS FOR PARAMETRIC SPACE REDUCTION AND GAIN-SCHEDULING

The previous example demonstrated a reduction in the number of necessary gain-scheduled parameters in a gantry problem, but did not specifically address conditions on which a higher space of gain scheduling parameters can be reduced to a lower space. More exactly, the questions remain: when is a nondimensional representation beneficial in terms of gain scheduling, when can the reduction in parameters be quantified, and what parameters should be chosen as repeating parameters? These questions are still being investigated in ongoing work, but answers exist for specific circumstances, and are given below.

In the trivial case where the dynamic representation is already dimensionless ($M = 0$), there can be no reduction in the number of gain-scheduled parameters via direct application of dimensional analysis. Similarly, in the case where all the gain-scheduled variables are dimensionless, i.e.

$$span\left(\mathbb{D}\left(u, v_i\right)\right) = 0 \;\; for \;\; \forall \;\, i = 1:V, \tag{13}$$

then the number of gain-scheduled parameters still cannot be reduced.

In the case where all the problem parameters are being gain scheduled ($V = N$), the problem can be reduced by up to M parameters to a gain-scheduling problem of $N - M$ parameters. A subset of this case is the very special situation where the V parameter gain-scheduling problem is mapped to a constant parameter problem. LPV systems satisfying this case can often be mapped to LTI system representations.

A subset of the above case occurs when the V parameters are the only ones whose dimension vectors span a particular subspace of dimension Q of the M unit dimensions. In this case, the number of gain-scheduled parameters can be reduced up to Q parameters. An example of this case is the previous work by the author on vehicle systems, presented in [11].

Many gain-scheduled problems do not satisfy any of the given cases, and work is nearly complete to define a general method able to predict the reduction in gain-scheduled variables. This method is intended to guide the user how to best select repeating variables for gain-scheduling.

VI. SUMMARY

A method of parametric space reduction has been shown using dimensional analysis. Equivalence between dimensionless and dimensioned controllers was demonstrated using a gantry system. It was shown a by specific example that the nondimensional system had reduced number of gain-scheduled parameters. In the general gain-scheduling problem, this type of reduction in parameters has a profound impact on the performance of gain scheduling methods because any decrease in the dimension of the parametric space implies significant (exponentially smaller) problem simplification and controller representation.

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