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Gain-Scheduling Control of LFT Systems using Parameter-Dependent Lyapunov Functions

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Abstract— **In this paper, we propose a new control design approach for linear fractional transformation (LFT) systems using parameter-dependent Lyapunov functions. Instead of designing a controller with LFT parameter dependency, we consider general parameter-dependent controllers to achieve better closed-loop performance. Using full-block multipliers, new LPV synthesis conditions have been derived in terms of finite number of LMIs. A ship steering examples has been used to demonstrate advantages and benefits of the proposed approach.**

I. INTRODUCTION

Motivated from the gain scheduling control methodology [12], [9], the study of linear parameter-varying (LPV) systems provides a systematic control design framework. LPV control theory is advantageous because it provides stability and performance guarantee over wide range of changing parameters, and has found applications in various industrial problems. A through review of gain-scheduling and LPV control research can be found in [10].

Using the scaled small-gain theorem, an LFT control design technique was developed by Packard [8] and Apkarian and Gahinet [1]. For the plant having linear fractional transformations (LFTs) parameter dependency, the existence of LFT gain-scheduled controller is fully characterized by a finite set of LMIs. This promising approach is applicable whenever the parameters are measurable in real-time. Furthermore, the synthesis problem is a convex one which can be optimized by efficient techniques. Further ramifications have been proposed to count for real gain-scheduling parameters [13]. Recently, a general form of multipliers was introduced in [11], [17] to reduce the conservatism associated with block-diagonal scaling. Nevertheless, it often requires the use of general scheduling functions.

On the other hand, a single or parameter-dependent quadratic Lyapunov functions have also been used in the analysis and control design for parameter-dependent plants [3], [16], [18]. Whereas the analysis test in [3] introduced potential conservatism by measuring performance against arbitrarily fast variations in scheduling parameters, known bounds on the rate of parameter variation were incorporated into the analysis conditions in [16], [18]. Generally speaking, the solution to this type of LPV control analysis and synthesis problems is formulated as a parameter-dependent linear matrix inequalities (LMIs), which is a special type of convex optimization problem with high computational complexity.

In this paper, a new control synthesis approach will be developed for LPV systems with LFT parameter dependency. Different from previous results, our proposed approach is to design a general parameter-dependent controller to achieve better controlled performance. Parallel to the LFT analysis work in [6], [7], quadratic LFT Lyapunov functions and full-block multipliers will be used to convert the infinite dimensional LPV control synthesis conditions into a finite set of LMIs. The advantages of the proposed approach will be demonstrated using a ship steering example. Our work has improved control performance of LFT systems using parameter-dependent Lyapunov functions and derived derived LPV control synthesis conditions in finite number of LMIs.

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II. PRELIMINARIES

We need some preliminary results to derive new LFT control synthesis conditions. The first lemma states a fundamental fact that the multiplication of any two LFTs is also an LFT, and its proof can be found in [19].

Lemma 1: Given two LFTs $(\Delta_1 \star M)$ and $(\Delta_2 \star N)$ with compatible dimensions, then

$$(\Delta_1 \star M)(\Delta_2 \star N) = \begin{bmatrix} \Delta_1 & \\ & \Delta_2 \end{bmatrix} \star W,$$

where

$$W = \left[\begin{array}{cc|c} M_{11} & M_{12}N_{21} & M_{12}N_{22} \\ 0 & N_{11} & N_{12} \\ \hline M_{21} & M_{22}N_{21} & M_{22}N_{22} \end{array}\right].$$

The second lemma provides a basis for the null space of an LFT. It has been shown that the null space is also an LFT [15].

Lemma 2: Given an LFT as

$$L = \Delta \star \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}.$$

Assume $M_{22} \in \mathbf{R}^{m \times r}$ and has rank m, the SVD of $M_{22} = U \begin{bmatrix} \Sigma & 0 \end{bmatrix} V^*$, where V^* is the complex-conjugate transpose of matrix V. Partition V into the first m and last $(r-m)$ columns as $V = \begin{bmatrix} V_1 & V_2 \end{bmatrix}$, then the null space of L is

$$\mathcal{N}(L) = \mathrm{Im}\left(\Delta \star \begin{bmatrix} M_{11} - M_{12}V_1\Sigma^{-1}U^*M_{21} & M_{12}V_2 \\ V_1\Sigma^{-1}U^*M_{21} & V_2 \end{bmatrix}\right).$$

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Proof: Define a transformed version $\bar{L}$ of L by applying V to the input and multiplying the output by U^*.

$$\begin{aligned}\bar{L} &= U^* L \begin{bmatrix} V_1 & V_2 \end{bmatrix} \\ &= \Delta \star \begin{bmatrix} M_{11} & M_{12}V_1 & M_{12}V_2 \\ U^* M_{21} & \Sigma & 0 \end{bmatrix} = \begin{bmatrix} \bar{L}_1(\Delta) & \bar{L}_2(\Delta) \end{bmatrix},\end{aligned}$$

where

$$\bar{L}_1 = \Delta \star \begin{bmatrix} M_{11} & M_{12}V_1 \\ U^*M_{21} & \Sigma \end{bmatrix}, \bar{L}_2 = \Delta \star \begin{bmatrix} M_{11} & M_{12}V_2 \\ U^*M_{21} & 0 \end{bmatrix}.$$

Partition the input of $\bar{L}$ into the first m inputs w_1, and other $r-m$ inputs w_2, then the output z of $\bar{L}$ is

$$z = \begin{bmatrix} \bar{L}_1(\Delta) & \bar{L}_2(\Delta) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

Since Σ is non-singular, then $\bar{L}_1$ is invertible. Therefore

$$\bar{L}_1^{-1} = \Delta \star \begin{bmatrix} M_{11} - M_{12}V_1\Sigma^{-1}U^*M_{21} & M_{12}V_1\Sigma^{-1} \\ -\Sigma^{-1}U^*M_{21} & \Sigma^{-1} \end{bmatrix}.$$

Applying Theorem 3.6 in [15], the null space of the linear map $\bar{L}(w_1, w_2) = \bar{L}_1 w_1 + \bar{L}_2 w_2$ is

$$\mathcal{N}(\bar{L}) = \mathrm{Im}\left(\begin{bmatrix} -\bar{L}_1^{-1}\bar{L}_2 \\ I \end{bmatrix} \right).$$

Note that

$$\begin{aligned}&-\bar{L}_1^{-1}\bar{L}_2 = \begin{bmatrix} \Delta & \\ & \Delta \end{bmatrix} \star \\ &\left[\begin{array}{cc|c} M_{11} - M_{12}V_1\Sigma^{-1}U^*M_{21} & M_{12}V_1\Sigma^{-1}U^*M_{21} & 0 \\ 0 & M_{11} & M_{12}V_2 \\ \hline -\Sigma^{-1}U^*M_{21} & \Sigma^{-1}U^*M_{21} & 0 \end{array}\right] \\ &= \Delta \star \begin{bmatrix} M_{11} - M_{12}V_1\Sigma^{-1}U^*M_{21} & M_{12}V_2 \\ \Sigma^{-1}U^*M_{21} & 0 \end{bmatrix},\end{aligned}$$

then we get the desired result by applying Theorem 3.7 in [15]

$$\begin{aligned}\mathcal{N}(L) &= \mathrm{Im}\left(V \begin{bmatrix} -\bar{L}_1^{-1}\bar{L}_2 \\ I \end{bmatrix} \right) \\ &= \mathrm{Im}\left(\Delta \star \begin{bmatrix} M_{11} - M_{12}V_1\Sigma^{-1}U^*M_{21} & M_{12}V_2 \\ V_1\Sigma^{-1}U^*M_{21} & V_2 \end{bmatrix} \right).\end{aligned}$$

■

The third lemma converts an uncertain matrix inequality to a finite set of inequalities using full-block multipliers [11]. Its proof is omitted here for space reason.

Lemma 3: Given a quadratic matrix inequality

$$G^T(\Theta) M G(\Theta) < 0 \tag{1}$$

with $G(\Theta) = G_{22} + G_{21}\Theta(I - G_{11}\Theta)^{-1}G_{12}$ and $\Theta \in \boldsymbol{\Theta}$. The condition (1) holds if and only if there exists a full-block multiplier Π such that

$$\left[\begin{array}{c} \star \\ \hline \star \end{array}\right]^T \mathrm{diag}\{\Pi, M\} \left[\begin{array}{cc} G_{11} & G_{12} \\ I & 0 \\ \hline G_{21} & G_{22} \end{array}\right] < 0, \tag{2}$$

and for any $\Theta \in \boldsymbol{\Theta}$,

$$[\star]^T \Pi \begin{bmatrix} I \\ \Theta \end{bmatrix} \geq 0. \tag{3}$$

III. MAIN RESULTS

Consider an LFT parameter-dependent plant

$$\begin{bmatrix} \dot{x}(t) \\ q(t) \\ e(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A & B_0 & B_1 & B_2 \\ C_0 & D_{00} & D_{01} & D_{02} \\ C_1 & D_{10} & 0 & D_{12} \\ C_2 & D_{20} & D_{21} & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \\ d(t) \\ u(t) \end{bmatrix}, \tag{4}$$

$$p(t) = \Theta(t) q(t), \tag{5}$$

where $x, \dot{x} \in \mathbf{R}^n$, $d \in \mathbf{R}^{n_d}$, $e \in \mathbf{R}^{n_e}$, $p, q \in \mathbf{R}^{n_p}$, $u \in \mathbf{R}^{n_u}$ and $y \in \mathbf{R}^{n_y}$. The time-varying parameter Θ is assumed to have the following structure

$$\begin{aligned}\boldsymbol{\Theta} = \{&\mathrm{diag}\{\theta_1 I_{r_1}, \theta_2 I_{r_2}, \cdots, \theta_s I_{r_s}\} : \ \theta_i \in \mathcal{C}(\mathbf{R}_+, \mathbf{R}), \\ &|\theta_i| \leq 1, \ i = 1, 2, \cdots, s\},\end{aligned}$$

where $\sum_{i=1}^{s} r_i = n_p$. It is assumed that the vector-valued parameter θ and its derivative are measurable in real-time. In addition, its time derivative is bounded and satisfies the constraint $-\bar{\nu}_i \leq \dot{\theta}_i \leq \bar{\nu}_i, i = 1, 2, \cdots, s$. For notational purposes, denote $\mathcal{V} = \{v : \ -\bar{\nu}_i \leq v \leq \bar{\nu}_i, \ i = 1, 2, \cdots, s\}$, i.e., $\mathcal{V}$ is a given convex polytope in $\mathbf{R}^s$ that contains the origin.

By absorbing parameter Θ, the state-space equation of LFT systems can also be written as

$$\begin{bmatrix} \dot{x}(t) \\ e(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A}(\Theta(t)) & \mathcal{B}_1(\Theta(t)) & \mathcal{B}_2(\Theta(t)) \\ \mathcal{C}_1(\Theta(t)) & \mathcal{D}_{11}(\Theta(t)) & \mathcal{D}_{12}(\Theta(t)) \\ \mathcal{C}_2(\Theta(t)) & \mathcal{D}_{21}(\Theta(t)) & \mathcal{D}_{22}(\Theta(t)) \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \\ u(t) \end{bmatrix}, \tag{6}$$

where all state space matrices have LFT parameter dependency on Θ. It is assumed that:

- (A1) The triple $(\mathcal{A}(\Theta), \mathcal{B}_2(\Theta), \mathcal{C}_2(\Theta))$ is parameter-dependent stabilizable and detectable for all $\Theta \in \boldsymbol{\Theta}$,
- (A2) $\begin{bmatrix} \mathcal{C}_2(\Theta) & \mathcal{D}_{21}(\Theta) \end{bmatrix}$ and $\begin{bmatrix} \mathcal{B}_2^T(\Theta) & \mathcal{D}_{12}^T(\Theta) \end{bmatrix}$ have full row rank for all Θ,
- (A3) $\mathcal{D}_{11}(\Theta) = 0$ and $\mathcal{D}_{22}(\Theta) = 0$.

The class of LPV controllers we are interested is in the form of

$$\begin{bmatrix} \dot{x}_k(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} A_k(\Theta(t), \dot{\Theta}(t)) & B_k(\Theta(t)) \\ C_k(\Theta(t)) & D_k(\Theta(t)) \end{bmatrix} \begin{bmatrix} x_k(t) \\ y(t) \end{bmatrix}, \tag{7}$$

where $x_k \in \mathbf{R}^{n_k}$. The dimension of controller states n_k is yet to be determined. Unlike the conventional LFT control techniques, we do not assume LFT parameter dependency in the controller formula.

In references [16], [18], H_∞ synthesis condition of an LPV controller (7) for the LPV system (6) is given by

$$\begin{aligned}&[\star]^T \left[\begin{array}{c} \star \\ \hline \star \end{array}\right]^T \left[\begin{array}{cc|cc} 0 & R(\Theta) & & \\ R(\Theta) & -\dot{R}(\Theta) & \multicolumn{2}{c}{0} \\ \hline & & \gamma^{-1}I & 0 \\ \multicolumn{2}{c|}{0} & 0 & -\gamma I \end{array}\right] \\ &\times \left[\begin{array}{cc} \mathcal{A}^T(\Theta) & \mathcal{C}_1^T(\Theta) \\ I & 0 \\ \hline \mathcal{B}_1^T(\Theta) & \mathcal{D}_{11}^T(\Theta) \\ 0 & I \end{array}\right] \mathcal{N}_R(\Theta) < 0,\end{aligned} \tag{8}$$

$$[\star]^T \left[\frac{\star}{\star}\right]^T \left[\begin{array}{cc|cc} 0 & S(\Theta) & \multicolumn{2}{c}{0} \\ S(\Theta) & \dot{S}(\Theta) & & \\ \hline \multicolumn{2}{c|}{0} & \gamma^{-1}I & 0 \\ & & 0 & -\gamma I \end{array}\right] \times \left[\begin{array}{cc} \mathcal{A}(\Theta) & \mathcal{B}_1(\Theta) \\ I & 0 \\ \hline \mathcal{C}_1(\Theta) & \mathcal{D}_{11}(\Theta) \\ 0 & I \end{array}\right] \mathcal{N}_S(\Theta) < 0, \quad (9)$$

$$\begin{bmatrix} R(\Theta) & I \\ I & S(\Theta) \end{bmatrix} \geq 0, \quad (10)$$

where $R(\Theta), S(\Theta)$ are continuously differentiable functions, and

$$\mathcal{N}_R(\Theta) = \ker \begin{bmatrix} \mathcal{B}_2^T(\Theta) & \mathcal{D}_{12}^T(\Theta) \end{bmatrix}$$
$$\mathcal{N}_S(\Theta) = \ker \begin{bmatrix} \mathcal{C}_2(\Theta) & \mathcal{D}_{21}(\Theta) \end{bmatrix}.$$

However, the set of parameter-dependent LMIs (8)-(10) includes infinite number of constraints and is usually difficult to solve. In this paper, we propose to parameterize the matrix functions $R(\Theta)$ and $S(\Theta)$ as quadratic LFT form of scheduling parameter Θ, i.e.,

$$R(\Theta) = T_R^T(\Theta) P T_R(\Theta), \qquad S(\Theta) = T_S^T(\Theta) Q T_S(\Theta),$$

where $T_R(\Theta), T_S(\Theta)$ are pre-specified LFT functions of parameter Θ,

$$T_R(\Theta) = \Theta \star \begin{bmatrix} T_{R11} & T_{R12} \\ T_{R21} & T_{R22} \end{bmatrix}, T_S(\Theta) = \Theta \star \begin{bmatrix} T_{S11} & T_{S12} \\ T_{S21} & T_{S22} \end{bmatrix},$$

and $T_{R22} \in \mathbf{R}^{n_r \times n}, T_{S22} \in \mathbf{R}^{n_s \times n}$ with $n_r, n_s \geq n$. The quadratic LFT type functions are quite general, and include affine functions as a special case. For a two parameters example, if we choose $T_R(\Theta) = \begin{bmatrix} \theta_1 I & \theta_2 I & I \end{bmatrix}^T$ and restrict matrix P as

$$P = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{0} & P_1 \\ & & P_2 \\ \hline P_1^T & P_2^T & P_0 \end{array}\right], \qquad P_0 = P_0^T > 0,$$

then we have $R(\Theta) = P_0 + (P_1 + P_1^T)\theta_1 + (P_2 + P_2^T)\theta_2$ is in affine function form. The quadratic LFT parameterization is also advantageous because the resulting LFT control synthesis condition can be formulated as finite number of LMIs.

Theorem 1: Consider an LFT system (4)-(5) with its parameter and derivative set $\mathbf{\Theta}$ and $\mathcal{V}$, the conditions (8)-(10) are solvable using quadratic LFT functions $R(\Theta), S(\Theta)$ if and only if there exist positive-definite matrices $P \in \mathbf{S}^{(n_r+n)\times(n_r+n)}$ and $Q \in \mathbf{S}^{(n_s+n)\times(n_s+n)}$, and full-block multipliers Π_P, Π_Q and Π, such that

$$\left[\frac{\star}{\star}\right]^T \mathrm{diag}\left\{\Pi_P, \begin{bmatrix} 0 & P & 0 & 0 \\ P & 0 & 0 & 0 \\ 0 & 0 & \gamma^{-1}I & 0 \\ 0 & 0 & 0 & -\gamma I \end{bmatrix}\right\} \times \left[\begin{array}{cc} \hat{G}_{P11} & \hat{G}_{P12} \\ I & 0 \\ \hline \hat{G}_{P21} & \hat{G}_{P22} \end{array}\right] < 0, \quad (11)$$

$$\left[\frac{\star}{\star}\right]^T \mathrm{diag}\left\{\Pi_Q, \begin{bmatrix} 0 & Q & 0 & 0 \\ Q & 0 & 0 & 0 \\ 0 & 0 & \gamma^{-1}I & 0 \\ 0 & 0 & 0 & -\gamma I \end{bmatrix}\right\} \times \left[\begin{array}{cc} \hat{G}_{Q11} & \hat{G}_{Q12} \\ I & 0 \\ \hline \hat{G}_{Q12}^T & \hat{G}_{Q22} \end{array}\right] < 0, \quad (12)$$

$$\left[\frac{\star}{\star}\right]^T \mathrm{diag}\left\{-\Pi, \begin{bmatrix} P & 0 & 0 & 0 \\ 0 & Q & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{bmatrix}\right\} \times \left[\begin{array}{cc} G_{11} & G_{12} \\ I & 0 \\ \hline G_{21} & G_{22} \end{array}\right] \geq 0, \quad (13)$$

and for any $(\Theta, \dot{\Theta}) \in \mathbf{\Theta} \times \mathcal{V}$,

$$[\star]^T \Pi_P \begin{bmatrix} I \\ \hat{\Theta} \end{bmatrix} \geq 0, \quad (14)$$

$$[\star]^T \Pi_Q \begin{bmatrix} I \\ \hat{\Theta} \end{bmatrix} \geq 0, \quad (15)$$

$$\left[\frac{\star}{\star}\right]^T \Pi \left[\begin{array}{cc} I & 0 \\ 0 & I \\ \hline \Theta & 0 \\ 0 & \Theta \end{array}\right] \geq 0, \quad (16)$$

where $\hat{\Theta} = \mathrm{diag}\left\{\dot{\Theta}, \Theta, \Theta, \Theta, \Theta\right\}$ and

$$\begin{bmatrix} \hat{G}_{P11} & \hat{G}_{P12} \\ \hat{G}_{P21} & \hat{G}_{P22} \end{bmatrix} = \left[\begin{array}{ccc|c} \hat{R}_{11} & \hat{R}_{12}M_{R21} & \hat{R}_{12}M_{R22}N_{R21} & \hat{R}_{12}M_{R22}N_{R22} \\ 0 & M_{R11} & M_{R12}N_{R21} & M_{R12}N_{R22} \\ 0 & 0 & N_{R11} & N_{R12} \\ \hline \hat{R}_{21} & \hat{R}_{22}M_{R21} & \hat{R}_{22}M_{R22}N_{R21} & \hat{R}_{22}M_{R22}N_{R22} \end{array}\right]$$

with

$$\begin{bmatrix} \hat{R}_{11} & \hat{R}_{12} \\ \hat{R}_{21} & \hat{R}_{22} \end{bmatrix} = \left[\begin{array}{ccc|cccc} 0 & 0 & T_{R11} & 0 & T_{R12} & 0 & 0 \\ T_{R11} & T_{R11} & 0 & T_{R12} & 0 & 0 & 0 \\ 0 & 0 & T_{R11} & 0 & T_{R12} & 0 & 0 \\ \hline -T_{R21} & T_{R21} & 0 & T_{R22} & 0 & 0 & 0 \\ 0 & 0 & T_{R21} & 0 & T_{R22} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \end{array}\right],$$

$$\begin{bmatrix} M_{R11} & M_{R12} \\ M_{R21} & M_{R22} \end{bmatrix} = \left[\begin{array}{c|cc} D_{00}^T & B_0^T & D_{10}^T \\ \hline C_0^T & A^T & C_1^T \\ 0 & I & 0 \\ D_{01}^T & B_1^T & D_{11}^T \\ 0 & 0 & I \end{array}\right],$$

$$\Theta \star \begin{bmatrix} N_{R11} & N_{R12} \\ N_{R21} & N_{R22} \end{bmatrix} = \mathcal{N}_R(\Theta),$$

and

$$\begin{bmatrix} \hat{G}_{Q11} & \hat{G}_{Q12} \\ \hat{G}_{Q21} & \hat{G}_{Q22} \end{bmatrix} = \left[\begin{array}{cccc|c} \hat{S}_{11} & \hat{S}_{12}M_{S21} & \hat{S}_{12}M_{S22}N_{S21} & & \hat{S}_{12}M_{S22}N_{S22} \\ 0 & M_{S11} & M_{S12}N_{S21} & & M_{S12}N_{S22} \\ 0 & 0 & N_{S11} & & N_{S12} \\ \hline \hat{S}_{21} & \hat{S}_{22}M_{S21} & \hat{S}_{22}M_{S22}N_{S21} & & \hat{S}_{22}M_{S22}N_{S22} \end{array}\right]$$

with

$$\begin{bmatrix} \hat{S}_{11} & \hat{S}_{12} \\ \hat{S}_{21} & \hat{S}_{22} \end{bmatrix} = \left[\begin{array}{ccc|cccc} 0 & 0 & T_{S11} & 0 & T_{S12} & 0 & 0 \\ T_{S11} & T_{S11} & 0 & T_{S12} & 0 & 0 & 0 \\ 0 & 0 & T_{S11} & 0 & T_{S12} & 0 & 0 \\ \hline T_{S21} & T_{S21} & 0 & T_{S22} & 0 & 0 & 0 \\ 0 & 0 & T_{S21} & 0 & T_{S22} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \end{array}\right],$$

$$\begin{bmatrix} M_{S11} & M_{S12} \\ M_{S21} & M_{S22} \end{bmatrix} = \left[\begin{array}{c|cc} D_{00} & C_0 & D_{01} \\ \hline B_0 & A & B_1 \\ 0 & I & 0 \\ D_{10} & C_1 & D_{11} \\ 0 & 0 & I \end{array}\right],$$

$$\Theta \star \begin{bmatrix} N_{S11} & N_{S12} \\ N_{S21} & N_{S22} \end{bmatrix} = \mathcal{N}_S(\Theta),$$

and

$$\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \left[\begin{array}{cc|cc} T_{R11} & 0 & T_{R12} & 0 \\ 0 & T_{S11} & 0 & T_{S12} \\ \hline T_{R21} & 0 & T_{R22} & 0 \\ 0 & T_{S21} & 0 & T_{S22} \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{array}\right].$$

Proof: For quadratic LFT functions $R(\Theta)$ and $S(\Theta)$, the time-derivative of T_S can be expressed as an LFT

$$\dot{T}_S = T_{S21}(I - \Theta T_{S11}^T)^{-1}\dot{\Theta}(I - T_{S11}\Theta)^{-1}T_{S12},$$

and

$$\begin{bmatrix} 0 & S \\ S & \dot{S} \end{bmatrix} = \begin{bmatrix} T_S & \dot{T}_S \\ 0 & T_S \end{bmatrix}^T \begin{bmatrix} 0 & Q \\ Q & 0 \end{bmatrix} \begin{bmatrix} T_S & \dot{T}_S \\ 0 & T_S \end{bmatrix}. \qquad (17)$$

Note that

$$\hat{S}(\Theta, \dot{\Theta}) := \begin{bmatrix} T_S & \dot{T}_S & 0 & 0 \\ 0 & T_S & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} = \begin{bmatrix} \dot{\Theta} & & \\ & \Theta & \\ & & \Theta \end{bmatrix} \star \left[\begin{array}{ccc|cccc} 0 & 0 & T_{S11} & 0 & T_{S12} & 0 & 0 \\ T_{S11} & T_{S11} & 0 & T_{S12} & 0 & 0 & 0 \\ 0 & 0 & T_{S11} & 0 & T_{S12} & 0 & 0 \\ \hline T_{S21} & T_{S21} & 0 & T_{S22} & 0 & 0 & 0 \\ 0 & 0 & T_{S21} & 0 & T_{S22} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \end{array}\right]$$

$$= \begin{bmatrix} \dot{\Theta} & & \\ & \Theta & \\ & & \Theta \end{bmatrix} \star \begin{bmatrix} \hat{S}_{11} & \hat{S}_{12} \\ \hat{S}_{21} & \hat{S}_{22} \end{bmatrix}.$$

One can also write the outer factor matrices in (9) as LFTs

$$M_S(\Theta) := \begin{bmatrix} \mathcal{A}(\Theta) & \mathcal{B}_1(\Theta) \\ I & 0 \\ \mathcal{C}_1(\Theta) & \mathcal{D}_{11}(\Theta) \\ 0 & I \end{bmatrix} = \Theta \star \left[\begin{array}{c|cc} D_{00} & C_0 & D_{01} \\ \hline B_0 & A & B_1 \\ 0 & I & 0 \\ D_{10} & C_1 & D_{11} \\ 0 & 0 & I \end{array}\right],$$

$$= \Theta \star \begin{bmatrix} M_{S11} & M_{S12} \\ M_{S21} & M_{S22} \end{bmatrix}$$

$$\mathcal{N}_S(\Theta) = \Theta \star \begin{bmatrix} N_{S11} & N_{S12} \\ N_{S21} & N_{S22} \end{bmatrix}$$

$$= \ker\left(\Theta \star \left[\begin{array}{c|cc} D_{00} & C_0 & D_{01} \\ \hline D_{20} & C_2 & D_{21} \end{array}\right]\right), \qquad (18)$$

where the null space in (18) is determined using Lemma 2. Therefore,

$$\hat{G}_Q(\Theta, \dot{\Theta}) = \hat{S}(\Theta, \dot{\Theta})M_S(\Theta)N_S(\Theta)$$
$$= \mathrm{diag}\{\dot{\Theta}, \Theta, \Theta, \Theta, \Theta\}\star$$
$$\left[\begin{array}{ccc|c} \hat{S}_{11} & \hat{S}_{12}M_{S21} & \hat{S}_{12}M_{S22}N_{S21} & \hat{S}_{12}M_{S22}N_{S22} \\ 0 & M_{S11} & M_{S12}N_{S21} & M_{S12}N_{S22} \\ 0 & 0 & N_{S11} & N_{S12} \\ \hline \hat{S}_{21} & \hat{S}_{22}M_{S21} & \hat{S}_{22}M_{S22}N_{S21} & \hat{S}_{22}M_{S22}N_{S22} \end{array}\right]$$
$$= \hat{\Theta} \star \begin{bmatrix} \hat{G}_{Q11} & \hat{G}_{Q12} \\ \hat{G}_{Q21} & \hat{G}_{Q22} \end{bmatrix}.$$

Then the condition (9) can be rewritten as

$$\hat{G}_Q^T(\hat{\Theta}) \begin{bmatrix} 0 & Q & 0 & 0 \\ Q & 0 & 0 & 0 \\ 0 & 0 & \gamma^{-1}I & 0 \\ 0 & 0 & 0 & -\gamma I \end{bmatrix} \hat{G}_Q(\hat{\Theta}) < 0,$$

which is equivalent to (12) and (15) by Lemma 3.

Similar to the equation (17), we have

$$\begin{bmatrix} 0 & R \\ R & -\dot{R} \end{bmatrix} = \begin{bmatrix} T_R & -\dot{T}_R \\ 0 & T_R \end{bmatrix}^T \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} T_R & -\dot{T}_R \\ 0 & T_R \end{bmatrix}.$$

Then the same approach can be applied to prove conditions (11) and (14) and coupling conditions (13) and (16). ■

Note that the conditions (14)-(16) are specified for any parameter and its derivative within $\mathbf{\Theta} \times \mathcal{V}$, and consist of infinite number of constraints. However, by introducing additional constrains on the matrices Π_P, Π_Q and Π, these infinite constraint sets can be made finite. For example, if we partition the full-block multiplier Π_Q as

$$\Pi_Q = \begin{bmatrix} \Pi_{Q11} & \Pi_{Q12} \\ \Pi_{Q12}^T & \Pi_{Q22} \end{bmatrix},$$

and assume $\Pi_{Q22} < 0$, then condition (15) will be convex with respect to parameter Θ and its derivative $\dot{\Theta}$. It is therefore adequate to check the matrix inequality at vertices $\hat{\Theta}_i$ of the set $\mathbf{\Theta} \times \mathcal{V}$, that is

$$[\star]^T \begin{bmatrix} \Pi_{Q11} & \Pi_{Q12} \\ \Pi_{Q12}^T & \Pi_{Q22} \end{bmatrix} \begin{bmatrix} I \\ \hat{\Theta}_i \end{bmatrix} \geq 0, \ i = 1, \ldots, 2^{2s} \tag{19}$$

with $\Pi_{Q22} < 0$. Nevertheless, the matrix inequalities (12) and (19) now become sufficient for

$$\hat{G}_Q^T(\hat{\Theta}) \begin{bmatrix} 0 & Q & 0 & 0 \\ Q & 0 & 0 & 0 \\ 0 & 0 & \gamma^{-1}I & 0 \\ 0 & 0 & 0 & \gamma I \end{bmatrix} \hat{G}_Q(\hat{\Theta}) < 0$$

to hold. On the other hand, the condition (15) can be completely removed given $\hat{\Theta}$ as structured uncertainty. Specifically, one can choose $\Pi_{Q11} = -\Pi_{Q22} > 0$, $\Pi_{Q11}\hat{\Theta} = \hat{\Theta}\Pi_{Q11}$, Π_{Q12} skew-symmetric and commutable with $\hat{\Theta}$. With some performance degradation, it will significantly reduce the number of LMIs and optimization variables in the LPV control synthesis condition. Similar arguments are applicable to multipliers Π_P and Π.

The user-defined LFTs $T_R(\Theta)$ and $T_S(\Theta)$ are essentially the basis functions to parameterize Lyapunov functions. Note that the dimensions of T_R, T_S matrices are not necessarily the same as the plant dimension n. Intuitively, this would lead to a search of feasible matrices P, Q in higher dimensional space with increased number of optimization variables. Using $T_R(\Theta), T_S(\Theta)$ as tuning nob, one can improve the performance of controlled LPV systems by incorporating parameter variation information. However, it is not clear at this stage how to select right T_R, T_S matrices.

After solving quadratic LFT matrix functions $R(\Theta), S(\Theta)$, the LPV controller can be constructed through the scheme detailed in [17].

IV. EXAMPLE

In this section, we will design various LPV controllers for a ship control problem. The ship steering problem has been analyzed in [6], and its dynamics can be approximated by the Nomoto model

$$\dot{x}(t) = v(t)(-ax(t) + bv(t)u(t)), \tag{20}$$

$$\dot{\psi}(t) = x(t), \tag{21}$$

where ψ denotes the heading of the ship and x is the angular velocity. u denotes the rudder angle as control input. v is the speed of the ship. It is assumed that $v(t) \geq 0$.

By adding disturbance and measurement, the ship dynamics (20)-(21) can be converted to an LFT parameter-dependent system

$$\begin{bmatrix} \dot{x} \\ \dot{\psi} \\ \hline q_1 \\ q_2 \\ \hline e_1 \\ e_2 \\ \hline y \end{bmatrix} = \left[\begin{array}{cc|cc|cc|c} -a & 0 & h & bh & 1 & 0 & b \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -2a & 0 & h & 2bh & 0 & 0 & 2b \\ 0 & 0 & 0 & h & 0 & 0 & 2 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 0 & 0.01 & 0 \end{array}\right] \begin{bmatrix} x \\ \psi \\ \hline p_1 \\ p_2 \\ \hline d \\ n \\ \hline u \end{bmatrix}, \tag{22}$$

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \delta(t) & \\ & \delta(t) \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \tag{23}$$

where

$$\delta(t) = \frac{v(t) - v_{nom}}{v(t) + v_{nom}} \in [-1, 1].$$

We will design the controllers for the unstable tanker with $a = -0.3, b = 0.8$ and $h = 0.9$. The parameter δ is obviously measurable in real-time, and can be used by LPV controllers. Note that the plant model has its D_{00} term non-zero, therefore its control synthesis problem can not be solved using previous LPV control approaches like [5] and [14].

The control design objectives include minimizing the heading error of the ship under disturbances with reasonable control force. They are quantified by rational weighting functions, which apply to the error output and control input channels.

$$W_e = \frac{0.3s + 1.2}{s + 0.04}, \qquad W_u(s) = \frac{s + 0.1}{s + 12500}.$$

For the ship steering problem, five different gain-scheduling control techniques will be considered. The first three cases utilize the proposed control approach in this paper.

1) LFT parameter-dependent Lyapunov function with parameter variation rate $\nu = 0.1, 10, 100$. For this case, we select $T_R(\delta), T_S(\delta)$ as

$$T_R(\delta) = T_S(\delta) = \delta I_4 \star \left[\begin{array}{c|c} 0 & I_4 \\ \hline I_4 & 0 \\ 0 & I_4 \end{array}\right].$$

In fact, this corresponds to quadratic parameter-dependent Lyapunov functions $R(\delta) = R_0 + \delta R_1 + \delta^2 R_2$ and $S(\delta) = S_0 + \delta S_1 + \delta^2 S_2$,

2) LFT parameter-dependent Lyapunov function with parameter variation rate $\nu = 0.1, 10, 100$. $T_R(\delta)$ and $T_S(\delta)$ are chosen as

$$T_R(\delta) = T_S(\delta) = \delta I_4 \star \left[\begin{array}{c|c} I_4 & 2I_4 \\ \hline I_4 & I_4 \\ 0 & I_4 \end{array}\right],$$

3) Constant Lyapunov function with $T_R = T_S = I_4$,
4) LFT control with full-block multipliers [11], [17],
5) LFT control with commutable scaling matrices [1].

TABLE I

OPTIMAL PERFORMANCE FROM DIFFERENT LPV CONTROL.

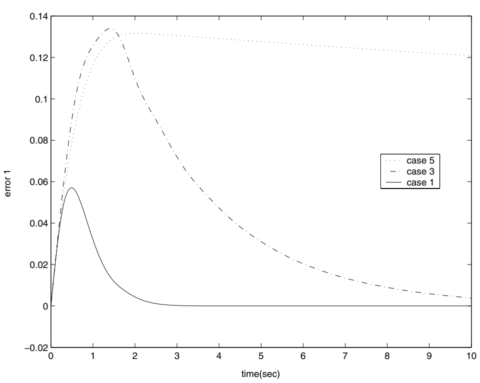
method	ν	$\mathcal{L}_2$ gain
Quadratic LFT Lyapunov function (Case 1)	0.1	0.314
	10	0.784
	100	2.370
Quadratic LFT Lyapunov function (Case 2)	0.1	0.743
	10	2.147
	100	2.386
Constant Lyapunov function	∞	4.183
LFT with full-block multipliers	∞	4.182
LFT with scaling matrices	∞	232.6

From Table I, it is observed that the new approach gets much better performance than the previous LFT ones. As parameter variation rates decrease, the performance bounds are further improved. However, the results clearly depend on the selection of $T_R(\delta)$ and $T_S(\delta)$. Case 3 and Case 4 essentially lead to the same performance level. The small difference between them is due to numerical accuracy. Finally, the performance achieved by conventional LFT control approach [1] is very conservative.

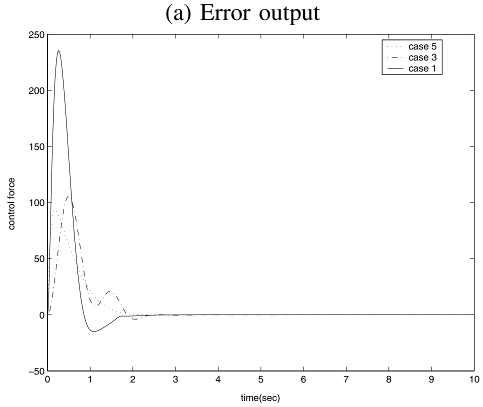
To demonstrate the time-domain performance of these controllers, we choose a particular parameter trajectory as

$$\delta = -\cos(0.3t).$$

For disturbance rejection, an impulse disturbance input is chosen while the measurement noise is set to zero. Fig. 1 shows that the error output using the proposed parameter-dependent controller is much smaller than the others although the control force is relatively large.



(a) Error output



(b) Control force

Fig. 1. Impulse response and control input of different LPV controllers.

V. CONCLUSION

In this paper, we have developed a new approach for gain-scheduling control of LFT systems with bounded parameter variation rates. With the help of quadratic LFT parameter-dependent Lyapunov functions and full-block multipliers, the control synthesis condition is formulated as finite number of LMIs with stringent controlled performance. However, it remains a question how to find the most suitable matrices $T_R(\Theta), T_S(\Theta)$ for the best performance.

The proposed LPV control method was applied on a ship control example. It has been shown that the new approach is very powerful, and always provides better performance than existing LFT control approaches.

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