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Homogeneous Polynomial Lyapunov Functions for Piecewise Affine Systems

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Abstract— **This paper addresses the construction of piecewise homogeneous polynomial Lyapunov functions (PHPLFs) for piecewise affine (PWA) systems. Sufficient conditions for the existence of PHPLFs of a given degree for both continuous-time and discrete-time PWA systems are obtained in terms of linear matrix inequalities (LMIs). The result contains existing results based on piecewise quadratic Lyapunov functions as special cases and provides a less conservative assertion of the stability of PWA systems, which is supported by numerical examples.**

I. Introduction

Piecewise affine (PWA) systems have been receiving much attention in control community because a large class of nonlinear systems, such as systems with relay, saturation or deadzone, can be modelled as PWA systems. Some approximation of smooth nonlinear systems [14] and fuzzy logic (neural) systems [14], [9] can be modelled as PWA systems as well. Moreover, in [11], Heemels *et. al.* established, under mild assumptions, an equivalence among five classes of hybrid systems, namely, PWA systems, mixed logical dynamical (MLD) systems, linear complementarity (LC) systems, extended linear complementarity (ELC) systems, and max-min-plusscaling (MMPS) systems. Thus PWA systems provide a powerful means of analysis and design for nonlinear systems.

Many results on stability analysis of piecewise linear systems with piecewise quadratic Lyapunov functions (PQLFs) have appeared in recent years. In [14], Johansson gave an inspiring idea on piecewise quadratic Lyapunov functions and relaxation of conservatism for continuous-time PWA systems. Further, discontinuous Lyapunov functions are studied in [16] with the discrete-time counterpart being given in [7], [17]. Other issues such as the well-posedness and the controllability and observability of PWA systems have also been investigated; see, for example, Imura and Schaft [13], [12] and Bemporad *et al.* [1].

Recently, a more general class of Lyapunov functions named homogeneous polynomial Lyapunov functions (HPLFs) was employed for robust stability analysis of uncertain linear systems; see, e.g., [4], [5]. In [4], Chesi *et. al.* constructed HPLFs for continuous-time linear systems with time-varying structured uncertainties and showed that HPLFs is a powerful tool for robustness analysis. In [5], they applied homogeneous polynomial parameter-dependent Lyapunov functions to linear systems with real parameter uncertainties.

In this paper, inspired by HPLFs, we apply piecewise homogeneous polynomial Lyapunov functions (PHPLFs) for PWA systems. We show how PHPLFs can be applied to analyze the stability of continuous-time piecewise linear systems using the power transformation introduced in [4], [3], [18]. A sufficient stability condition is given in terms of the feasibility of a set of LMIs. We also extend the result for discrete-time PWA systems with another kind of transformation. Simulation examples are given to demonstrate the less conservatism of the proposed approach as compared with existing works such as [14], [7].

The rest of this paper is organized as follows. In Section II, we introduce some preliminaries on homogeneous polynomial functions. Two types of transformations for a given matrix to obtain homogeneous polynomial functions are introduced. In Section III we discuss how to construct PHPLFs for continuous-time PWA systems. In Section IV we present the result for discrete-time PWA systems with a different transformation technique. In Section V, we give some numerical examples to illustrate the advantages of the PHPLFs over existing approaches using PQLFs. Some conclusions are drawn in the last section.

For convenience, we introduce the following notations: $A > 0$ ($A < 0$) means that A is positive definite (negative definite). $A \succeq 0$ implies that A is copositive. R denotes the real space, and N the set of all positive integers. We also denote $\begin{pmatrix} n \\ m \end{pmatrix} = \frac{n!}{m!(n-m)!}$.

II. Preliminaries

In this section, we provide some preliminaries on homogeneous polynomial functions. We also introduce two types of transformations related to the homogeneous polynomial functions, and some useful properties of these transformations.

Definition 1: [4], [3], [18]**(Power transformation of degree m)** Consider the vector $x \in R^n$, $x = [x_1, \cdots x_n]^T$. The power transformation of degree m is a nonlinear change of coordinates that forms a new vector $x^{[m]}$ of all integer powered monomials of degree m that can be made from the original x vector,

$$x_l^{[m]} = c_l x_1^{m_{l1}} x_2^{m_{l2}} \cdots x_n^{m_{ln}},\ m_{lj} \in \{1, 2, ..., m\},$$
$$\sum_{j=1}^{n} m_{lj} = m,\ l = 1, ..., d_{n,m},\ d_{n,m} = \begin{pmatrix} n+m-1 \\ m \end{pmatrix} \quad (1)$$

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Usually we take $c_l = 1$.

Definition 2: [4], [3], [18]**(Homogeneous polynomial)** Let $P_{n,m}$ be the set of all polynomials in n variables with degree m, i.e. all terms of $P_{n,m}$ are in the form of (1). If a polynomial $f_{n,m} \in P_{n,m}$ and $f_{n,m}(\lambda x) = \lambda^m f_{n,m}(x)$, then $f_{n,m}$ is a homogeneous polynomial. We define the set of all homogeneous polynomials in n variables with degree m as $H_{n,m}$. The square matricial representation (SMR) of $f_{n,2m} \in H_{n,2m}$ is defined as $f_{n,2m} = x^{[m]^T} C_f x^{[m]}$, where $x^{[m]} \in R^{d_{n,m}}$ is the base vector of homogeneous forms of degree m in x, and $C_f = C_f^T \in R^{d_{n,m}} \times R^{d_{n,m}}$.

Remark 1: An important property of the SMR is that the matrix C_f is not unique. In fact, we can identify some identities of the form: $x_i^{[m]} x_j^{[m]} - x_k^{[m]} x_h^{[m]} = 0, i, j, k, h \in \{1, 2, ..., d\}$, which is easily translated into a symmetric matrix $C_\ell^{(0)} \in R^{d_{n,m}} \times R^{d_{n,m}}$ with its (i,j) and (j,i) entries being 1 (if $i = j$, then $(i,i) = 2$) and (k,h) and (h,k) entries being -1, or some scalar multiple thereof, and all other entries being zeros. Note that we can have a total of $L_{n,m}$ such identities, i.e. $\ell = 1, 2..., L_{n,m}$, where $L_{n,m} = \frac{1}{2} d_{n,m}(d_{n,m} + 1) - d_{n,2m}$[4]. Thus $f_{n,2m}$ can be represented as $f_{n,2m} = x^{[m]^T}(C_f + \sum_{\ell=1}^{L_{n,m}} \gamma_\ell C_\ell^{(0)}) x^{[m]}$, where $\gamma_\ell \in R$ is scalar.

Firstly, we recall a transformation from matrix A to an extended matrix $A^{[m]}$. This type of transformation connects Ax to homogeneous polynomials of a higher degree.

Lemma 1: [3], [18] Given $A \in R^{r \times n}$, $x \in R^n$ and $m \in N$, then under the power transform, there exists a $A^{[m]} \in R^{d_{r,m} \times d_{n,m}}$ such that $(Ax)^{[m]} = A^{[m]} x^{[m]}, \forall x \in R^n$.

Remark 2: Note that the transformation in Lemma 1 under some constraint is unique, which means a given A is corresponding to an unique $A^{[m]}$, for example, we force the sequence of $x_i^{[m]}$ to satisfy the lexigraphic ordering:

$$\begin{aligned} x^{[m]} = [&x_1^m \;\; x_1^{m-1} x_2 \;\; x_1^{m-2} x_2^2 \;\; x_1^{m-2} x_2 x_3 \;\; \cdots; \\ &x_2^m \;\; x_2^{m-1} x_3 \;\; x_2^{m-2} x_3^2 \;\; \cdots; \;\; \cdots \;; x_n^m]^T \end{aligned}$$

However, the reverse is not true.

Lemma 2: [3], [18] Given $A, B \in R^{n \times n}$, $A^{[m]}$ and $B^{[m]}$ satisfy

$$(AB)^{[m]} = A^{[m]} B^{[m]}, \tag{2}$$

$$(A^q)^{[m]} = (A^{[m]})^q, \tag{3}$$

$$(A^T)^{[m]} = (A^{[m]})^T \tag{4}$$

where q is an integer and A^q is well defined.

Lemma 3: If $Ax \geq 0$, then $A^{[m]} x^{[m]} \geq 0$. If $Ax = 0$, then $A^{[m]} x^{[m]} = 0$.

Remark 3: Note that the transformation $A^{[m]}$ does not possess linear properties, i.e., $[\alpha A + \beta B]^{[m]} \neq \alpha A^{[m]} + \beta B^{[m]}$.

Next, we present the transformation from A to $A_{[m]}$. This transformation is widely used in uncertain linear systems, say, [4], [5]. It expands a differential equation to a homogeneous polynomial of high degree.

Lemma 4: [19], [4] Given $A \in R^{n \times n}$ and $m \in N$, with the relation $\dot{x}(t) = A(t)x(t)$, then there exists a matrix $A_{[m]}(t) \in R^{\binom{n+m-1}{m} \times \binom{n+m-1}{m}}$ such that $\frac{d}{dt}((Ax)^{[m]}) = A_{[m]}(t) x(t)^{[m]}$.

Lemma 5: [19], [4] Letting $\alpha, \beta \in R$ and $A, B \in R^{n \times n}$,

$$[\alpha A + \beta B]_{[m]} = \alpha A_{[m]} + \beta B_{[m]} \tag{5}$$

Lemma 6: [19], [4] If A is Hurwitz, then $A_{[m]}$ is Hurwitz as well.

In the next two sections, we will address how to apply the above transformations and their properties to construct corresponding Lyapunov functions for both continuous-time and discrete-time PWA systems, and thus determine conditions for stability of PWA systems.

III. Continuous-time PWA systems

Consider the following continuous-time PWA system:

$$\dot{x} = A_i x + a_i, \;\; x \in S_i, \;\; i \in \mathcal{I} \tag{6}$$

where $x \in R^n$ is the system state vector, $\{S_i = \{x | E_i x + e_i \geq 0\}\}_{i \in \mathcal{I}} \subseteq \mathcal{R}^n$ denotes a partition of the state space into a set of polyhedral partitions/subspaces/cells [14]. $\mathcal{I}$ is the index set of discrete state i. a_i and e_i are the affine terms. $\bar{S}_i$ is the closure of S_i. Moreover, we assume that if operating regions $\bar{S}_i \cap \bar{S}_j \neq 0$ then there exist F_{ij} and f_{ij} such that $S_{ij} = \bar{S}_i \cap \bar{S}_j = \{x | x = F_{ij} z + f_{ij}, z \in R^{n-1}\}$ [10]. We denote the set of all such index pairs i, j as Ω_c and those index sets with $a_i = e_i = f_i = 0$, i.e., those partitions containing the origin, as $\mathcal{I}_0$. Further, we define $\mathcal{I}_1 = \mathcal{I} \backslash \mathcal{I}_0$. For the sake of compactness, we consider the case that all partitions contain the origin, i.e. $\mathcal{I} = \mathcal{I}_0$. An extension to $\mathcal{I}_1$ will be discussed at the end of this section.

Let us introduce the extended system corresponding to (6) defined by

$$\dot{x}^{[m]} = A_{i[m]} x^{[m]}, \;\; x \in S_i, \;\; i \in \mathcal{I} \tag{7}$$

The extended system (7) plays a key role in our stability analysis, which in fact, has been applied to analyze the stability of systems with structured uncertainties [4] and uncertain polytopic systems [5].

Remark 4: The partition properties still hold after the transformation. For $\forall x \in S_i$, we have $E_i x \geq 0$. Thus according to Lemma 1 and Lemma 3, we have $E_i^{[m]} x^{[m]} \geq 0$. Similarly, for $\forall x \in S_{ij}$, we can deduce $x^{[m]} = F_{ij}^{[m]} z^{[m]}$ from $x = F_{ij} z$, $z \in R^{n-1}$.

Based on the Lyapunov stability theory, if we can find a set of HPLFs of degree $2m$, denoted by $v_i^{(2m)}(x)$, such that

$$v_i^{(2m)}(x) > 0, \;\; \forall x \in S_i, \;\; x \neq 0; \tag{8}$$

$$v_i^{(2m)}(x) = v_j^{(2m)}(x), \forall x \in S_{ij}; \tag{9}$$

$$\dot{v}_i^{(2m)}(x) < 0, \;\; \forall x \in S_i, \;\; x \neq 0 \tag{10}$$

Then we can define a PHPLF candidate as

$$v^{(2m)} = \sum_i \tau_i v_i^{(2m)}, \tau_i = \left\{ \begin{array}{ll} 1 & if\ x \in S_i \\ 0 & otherwise \end{array} \right. \tag{11}$$

which implies the asymptotic stability of the corresponding system.

Theorem 3: The continuous-time PWA system (6) is asymptotically stable, if there exists a solution ($P_i = P_i^T, U_i \succeq 0, V_i \succeq 0, \gamma_{il}$) such that

$$P_i - {E_i^{[m]}}^T U_i E_i^{[m]} > 0,\ \forall i \in \mathcal{I}_0 \tag{12}$$

$${F_{ij}^{[m]}}^T (P_i - P_j) F_{ij}^{[m]} = 0,\ \forall i, j \in \Omega_c \tag{13}$$

$$\begin{array}{c} P_i A_{i[m]} + {A_{i[m]}}^T P_i + \sum\limits_{\ell} \gamma_{i\ell} P_{i\ell}^{(0)} + {E_i^{[m]}}^T V_i E_i^{[m]} < 0 \\ \forall i \in \mathcal{I}_0,\ \ell = 1,\ ...,\ L_{n,m} \end{array} \tag{14}$$

where $P_{i\ell}^{(0)}$ is such that ${x^{[m]}}^T P_{i\ell}^{(0)} x^{[m]} = 0$ and $E_i^{[m]}$, $F_{ij}^{[m]}$ and $A_{i[m]}$ are the corresponding transformations of E_i, F_{ij} and A_i, respectively.

Proof: Define $v_i^{(2m)}(x) = {x^{[m]}}^T P_i x^{[m]}$. Now we prove that $v_i^{(2m)}$ is a desired Lyapunov function. For each region S_i, we have $E_i x \geq 0, \forall x \in S_i$. Thus $E_i^{[m]} x^{[m]} \geq 0$. From (12), we have

$$\begin{array}{l} v_i^{(2m)} = {x^{[m]}}^T P_i x^{[m]} \\ > {x^{[m]}}^T {E_i^{[m]}}^T U_i E_i^{[m]} x^{[m]} \geq 0, \quad x \neq 0 \end{array}$$

Since $x = F_{ij} z, \forall x \in S_{ij}$, $x^{[m]} = F_{ij}^{[m]} z^{[m]}$. From (13), we have

$$\begin{array}{rl} & v_i^{(2m)} = {x^{[m]}}^T P_i x^{[m]} = {z^{[m]}}^T {F_{ij}^{[m]}}^T P_i F_{ij}^{[m]} z^{[m]} \\ = & {z^{[m]}}^T {F_{ij}^{[m]}}^T P_j F_{ij}^{[m]} z^{[m]} = {x^{[m]}}^T P_j x^{[m]} = v_j^{(2m)} \end{array}$$

Further, it follows from (14) that

$$\begin{array}{rl} & \frac{d}{dt} v_i^{(2m)} = {x^{[m]}}^T (P_i A_{i[m]} + {A_{i[m]}}^T P_i) x^{[m]} \\ < & -{x^{[m]}}^T (\sum\limits_{\ell} \gamma_{il} P_{il}^{(0)} + {F_i^{[m]}}^T V_i F_i^{[m]}) x^{[m]} \\ = & -{x^{[m]}}^T {F_i^{[m]}}^T V_i F_i^{[m]} x^{[m]} \leq 0,\ x \neq 0 \end{array}$$

Observe that

$$\frac{d}{dt} v_i^{(2m)} |_{\dot{x}^{[m]} = A_{i[m]} x^{[m]}} = \frac{d}{dt} v_i^{(2m)} |_{\dot{x} = A_i x}$$

Thus we can conclude that the $v_i^{(2m)}$ is a HPLF for system (6) and the system is asymptotically stable. ∎

Remark 5: In [14], Johansson constructed a type of so-called continuity matrices to guarantee the continuity of Lyapunov function. A matrix $\vec{E}_i$ is a continuity matrix for cell S_i, if $\vec{E}_i x = \vec{E}_j x$, for $\forall x \in S_{ij}$. Thus we have: $\vec{E}_i^{[m]} x^{[m]} = \vec{E}_j^{[m]} x^{[m]}$. An alternative approach to replace (13) is to choose $P_i = \vec{E}_i^{[m]T} T \vec{E}_i^{[m]}$, where $T = T^T$, such that for $x \in S_{ij}$,

$$\begin{array}{rl} & v_i^{(2m)} = x^{[m]T} P_i x^{[m]} = {x^{[m]}}^T \vec{E}_i^{[m]T} T \vec{E}_i^{[m]} x^{[m]} \\ = & {x^{[m]}}^T \vec{E}_j^{[m]T} T \vec{E}_j^{[m]} x^{[m]} = {x^{[m]}}^T P_j x^{[m]} = v_j^{(2m)} \end{array}$$

Remark 6: It is obvious that when $m = 1$, the result will reduce to that of Johansson [14] or Hassibi and Boyd [10].

Remark 7: Some SDP solvers such as SDPSOL handle equality constraints such as those given in (13). However, other SDP solvers, say, LMI Toolbox of Matlab, cannot handle equality constraint directly. In this case, we may adopt another approach using discontinuous Lyapunov functions.

In fact, (9) is a strong condition, which guarantees the continuity of the Lyapunov function when the state from one partition enters another one. The requirement that the Lyapunov function be decreasing when the state crosses from the i-th partition to the j-th one implies that:

$$v_i^{(2m)}(x) \geq v_j^{(2m)}(x)\ or\ {x^{[m]}}^T P_i x^{[m]} \geq {x^{[m]}}^T P_j x^{[m]} \tag{15}$$

Note that the set S_{ij} can be represented by $\{x | \vec{F}_{ij} x = 0\}$, where $\vec{F}_{ij}$ is a properly chosen matrix [14]. Thus $\forall x \in S_{ij}$, $\vec{F}_{ij}^{[m]} x^{[m]} = 0$. (15) is equivalent to the following LMI in P_i, P_j, T_{ij} by the Finsler's Lemma [2]:

$$P_i - P_j + \vec{F}_{ij}^{[m]T} T_{ij} + T_{ij}^T \vec{F}_{ij}^{[m]} \geq 0 \tag{16}$$

Thus, (13) of Theorem 3 can be replaced by (16) for $\forall i, j \in \Omega_c$.

Remark 8: For the case that $i \in \mathcal{I}_1$, we can rewrite the system (6) as

$$\dot{\xi} = \bar{A}_i \xi,\ \forall \xi \in \bar{S}_i = \left\{ \xi | \bar{E}_i \xi \geq 0 \right\} \tag{17}$$

where

$$\xi = \left[\begin{array}{c} x \\ 1 \end{array} \right],\ \bar{A}_i = \left[\begin{array}{cc} A_i & a_i \\ 0 & 0 \end{array} \right],\ \bar{E}_i = \left[\begin{array}{cc} E_i & e_i \end{array} \right]$$

$$\bar{F}_{ij} = \left[\begin{array}{cc} F_{ij} & f_{ij} \\ 0 & 1 \end{array} \right],\ \bar{z} = \left[\begin{array}{c} z \\ 1 \end{array} \right]$$

Note that if $\bar{E}_i \xi \geq 0$, then $\bar{E}_i^{[m]} \xi^{[m]} \geq 0$. If $\xi = \bar{F}_{ij} \bar{z}$, then $\xi^{[m]} = \bar{F}_{ij}^{[m]} \bar{z}^{[m]}$. Now we define $\bar{v}_i^{(2m)} = {\xi^{[m]}}^T \bar{P}_i \xi^{[m]}$, which is no longer a HPLF. Fortunately, in (12)-(14), if we replace $A_{i[m]}$ by $\bar{A}_{i[m]}$, $E_i^{[m]}$ by $\bar{E}_i^{[m]}$ and $F_{ij}^{[m]}$ by $\bar{F}_{ij}^{[m]}$ and with the variables properly dimensioned, the inequalities about $\bar{v}_i^{(2m)}$ (replace $v_i^{(2m)}$ with $\bar{v}_i^{(2m)}$) in (8)-(10) still hold. In fact, the last row of $\bar{A}_i$ is zero and the derivative of constant 1 is zero, thus there must exist a $\bar{A}_{i[m]}$, such that $\frac{d}{dt}((\bar{A}\xi)^{[m]}) = \bar{A}_{[m]}(t) \xi^{[m]}$.

Note that the coefficients of the last entry of ξ in Lemma 1 will not be changed if we take the last entry of ξ as variable. So, for $\bar{E}_i$ and $\bar{F}_{ij}$, there must exist $\bar{E}_i^{[m]}$ and $\bar{F}_{ij}^{[m]}$, such that $(\bar{E}_i \xi)^{[m]} = \bar{E}_i^{[m]} \xi^{[m]}$ and $(\bar{F}_{ij} \xi)^{[m]} = \bar{F}_{ij}^{[m]} \xi^{[m]}$. Thus we have

$$\frac{d}{dt} \bar{v}_i^{(2m)} |_{\dot{\xi}^{[m]} = \bar{A}_{i[m]} \xi^{[m]}} = \frac{d}{dt} \bar{v}_i^{(2m)} |_{\dot{\xi} = \bar{A}_i \xi} = \frac{d}{dt} \bar{v}_i^{(2m)} |_{\dot{x} = A_i x + a_i}$$

We can conclude that $\bar{v}_i^{(2m)}$ is a Lyapunov function segment for the region S_i, $i \in I_1$.

Remark 9: A regular sliding mode occurs when the vector fields in both partitions $\bar{S}_i$ and $\bar{S}_j$ point toward the common boundary S_{ij}. Another sliding mode named higher-order sliding mode arises when the vector fields are tangent to the boundary. However, we do not deal with this issue in this paper. The readers may refer to [14] concerning this issue.

Remark 10: Theorem 3 introduces a PHPLF for the continuous-time PWA system (6). Note that Zachary in [18] showed that non-homogeneous polynomial Lyapunov functions in the form of sum of squares cannot perform better than HPLFs of the same degree for simultaneous stability of multiple linear systems. For PWA systems, we can also show that such non-homogeneous Lyapunov function cannot improve stability analysis. To improve the conservatism of the PHLFs for stability analysis, one can simply increase the degree m.

Remark 11: Some technical improvements can be applied here. For a copositive definite matrix, usually, we simply let all entries of the matrix be positive scalars, which obviously is a strong sufficient condition. In [15], Parrilo suggested a relaxed approach to obtain a copositive definite matrix based on an LMI method. The method can be applied here.

Another technique is suggested by Zachary [18] which applies the set

$$\bar{I}_{n,m} = \{X \geq 0 \in R^{d_{n,m}\times d_{n,m}} : x^{[m]T} X x^{[m]} = \sum_{i=1}^{n} x_i^{2m}\}$$

and the S-procedure to obtain a positive definite $v(x)$. For example, $\bar{I}_{n,1} = I_{n\times n}$, $\bar{I}_{2,2} = diag\{1,0,1\}$. Thus (12) in fact can be replaced by

$$P_i - E_i^{[m]^T} U_i E_i^{[m]} - \epsilon \bar{I}_{n,m} \geq 0 \tag{18}$$

where ϵ is a sufficiently small positive scalar and m is the degree of the corresponding power transformation.

IV. Discrete-time PWA systems

Consider the following discrete-time PWA system:

$$x_{t+1} = A_i x_t + a_i,\ x_t \in S_i,\ i \in \mathcal{I} \tag{19}$$

where $x_t \in R^n$ is the state variable, $\{S_i = \{x_t | E_i x_t + e_i \geq 0\}\}_{i\in\mathcal{I}} \subseteq \mathcal{R}^n$ denotes a set of polyhedral partitions/subspaces of the state space, a_i and e_i are the affine terms. Similar to the continuous-time case, we also assume that if operating regions $\bar{S}_i \cap \bar{S}_j \neq 0$ then there exist F_{ij} and f_{ij} such that $S_{ij} = \bar{S}_i \cap \bar{S}_j = \{x_t | x_t = F_{ij} z_t + f_{ij}, z_t \in R^{n-1}\}$. We denote those index sets with $a_i = f_i = e_i = 0$ as $\mathcal{I}_0$ and $\mathcal{I}_1 = \mathcal{I}\backslash\mathcal{I}_0$. Similar to the previous section, we first consider the set $\mathcal{I}_0$ and give an extension to $\mathcal{I}_1$ at the end of this section. Let Ω_d represent possible index pairs whether in the same region or from one to another based on the measured state space:

$$\Omega_d \triangleq \{i,j | x_t \in S_i, x_{t+1} \in S_j,\ i,j \in \mathcal{I}\}$$

Similar to the continuous-time case, we introduce the extended systems corresponding to (19) defined by

$$x_{t+1}^{[m]} = A_i^{[m]} x_t^{[m]},\ x_t \in S_i,\ i \in \mathcal{I} \tag{20}$$

Note that the partition properties of Remark 4 associated with the extended state also hold. We define a PHPLF similar to (11), and have the following result.

Theorem 4: The discrete-time PWA system (19) is asymptotically stable, if there exists a solution ($P_i = P_i^T, U_i \succeq 0, V_{ij} \succeq 0, \gamma_{i\ell}$) such that

$$P_i - E_i^{[m]^T} U_i E_i^{[m]} > 0,\ \forall i \in \mathcal{I}_0 \tag{21}$$

$$\begin{gathered} {A_i}^{[m]^T} P_j {A_i}^{[m]} - P_i + \sum_{\ell} \gamma_{i\ell} P_{i\ell}^{(0)} + E_i^{[m]^T} V_{ij} E_i^{[m]} < 0 \\ \forall i,j \in \Omega_d,\ i,j \in \mathcal{I}_0,\quad \ell = 1,\ ...,\ L_{n,m} \end{gathered} \tag{22}$$

where $P_{i\ell}^{(0)}$ is such that $x^{[m]^T} P_{i\ell}^{(0)} x^{[m]} = 0$; $E_i^{[m]}$ and ${A_i}^{[m]}$ are the corresponding transformations of E_i and A_i.

Proof: Define $v_i^{(2m)}(t) = x_t^{[m]^T} P_i x_t^{[m]}$. For each region S_i, we have $E_i x_t \geq 0, \forall x_t \in S_i$. Thus $E_i^{[m]} x_t^{[m]} \geq 0$. From (21), we have

$$v_i^{(2m)}(t) = x_t^{[m]^T} P_i x_t^{[m]} > x_t^{[m]^T} E_i^{[m]^T} U_i E_i^{[m]} x_t^{[m]} \geq 0,\ x^{[m]} \neq 0$$

Similar to [7], we assume that the dynamics of the system is governed by the dynamics of the local model of S_i when the state of the system transits from the region S_i to S_j. From (22), we have

$$\begin{aligned} \Delta v(t) &= v^{(2m)}(t+1) - v^{(2m)}(t) = x_t^{[m]^T}({A_i}^{[m]^T} P_j {A_i}^{[m]} - P_i) x_t^{[m]} \\ &< -x_t^{[m]^T}(\sum_{\ell} \gamma_{i\ell} P_{i\ell}^{(0)} + E_i^{[m]^T} V_i E_i^{[m]}) x_t^{[m]} \\ &= -x_t^{[m]^T} E_i^{[m]^T} V_i E_i^{[m]} x_t^{[m]} \leq 0,\ x^{[m]} \neq 0 \end{aligned}$$

for $i,j \in \Omega_d$. Note that

$$\begin{aligned} & \Delta v(t)|_{x_{t+1}=A_i x_t} = x_{t+1}^{[m]T} P_j x_{t+1}^{[m]} - x_t^{[m]T}(t) P_i x_t^{[m]} \\ = \ & (A_i x_t)^{[m]^T} P_j (A_i x_t)^{[m]} - x_t^{[m]^T} P_i x_t^{[m]} \\ = \ & (A_i^{[m]} x_t^{[m]})^T P_j A_i^{[m]} x_t^{[m]} - x_t^{[m]T} P_i x_t^{[m]} \\ = \ & \Delta v(t)|_{x_{t+1}^{[m]} = A_i^{[m]} x_t^{[m]}} \end{aligned} \tag{23}$$

Thus we can conclude that $v_i^{(2m)}$ is the right set of HPLFs for system (19). The proof is completed. ■

Remark 12: It is obvious that when $m = 1$, the result will reduce to that of Feng's [7].

Remark 13: For the case that $i,j \in \mathcal{I}_1$, we can rewrite the system (19) as

$$\dot{\xi}_t = \bar{A}_i \xi_t,\ \forall \xi_t \in \bar{S}_i = \{\xi_t | \bar{E}_i \xi_t \geq 0\} \tag{24}$$

where

$$\xi_t = \begin{bmatrix} x_t \\ 1 \end{bmatrix},\ \bar{A}_i = \begin{bmatrix} A_i & a_i \\ 0 & 1 \end{bmatrix},\ \bar{E}_i = \begin{bmatrix} E_i & e_i \end{bmatrix}$$

We can easily check that in the equalities (21)-(22), when $A_i^{[m]}$ is replaced by $\bar{A}_i^{[m]}$, $E_i^{[m]}$ by $\bar{E}_i^{[m]}$ with the variables properly dimensioned, the properties of $\bar{v}_i^{(2m)}$ still hold for $i,j \in I_1$.

However, different from the continuous-time case, jumps of state from one partition to another are much more complex. In fact, we have to deal with the situation that the state jumps from the partition $i \in I_0$ to the partition $j \in I_1$ or vice versa.

Case 1 ($i \in \mathcal{I}_0,\ j \in \mathcal{I}_1$ and $i, j \in \Omega_d$): We choose $\bar{A}_i = \begin{bmatrix} A_i & 0 \\ 0 & 1 \end{bmatrix}$ and $\bar{F}_i = \begin{bmatrix} F_i & 0 \end{bmatrix}$. Now we have to extend the Lyapunov matrix P_i to a proper dimensioned matrix $\bar{P}_i$. A procedure is described as follows:

Given a variable matrix $Z_i = (z^{(i)}_{i_1,i_2}) \in R^{d_{n+1,m}\times d_{n+1,m}}$. Thus $\bar{W}_i = \xi^{[m]T} Z_i \xi^{[m]} = \sum_{j=1}^{d_{n+1,m}\times d_{n+1,m}} z^{(i)}_{i_1,i_2} x_1^{m_{j1}} \cdots x_n^{m_{jn}}$, where $\sum_{k=1}^{n} m_{jk}$ can take a value from the set $\{0,\ 2m-2,\ 2m-1,\ 2m\}$. Compare $\bar{W}_i$ with $v_i = x^{[m]T} P_i x^{[m]}$ and determine the entries of Z_i such that $z^{(i)}_{i_1,i_2}$ for $x_i^{[2m]}$ are the same as the coefficient of $x_i^{[2m]}$ in v_i, and let the rest entries of Z_i be zeros. Choose such Z_i as $\bar{P}_i$. In this situation, we can easily check that $v_i = x^{[m]T} P_i x^{[m]} = \xi^{[m]T} \bar{P}_i \xi^{[m]} = \bar{v}_i$.

Case 2 ($i \in \mathcal{I}_1,\ j \in \mathcal{I}_0$ and $i, j \in \Omega_d$): Simply let $\bar{A}_j = \begin{bmatrix} A_j & 0 \\ 0 & 1 \end{bmatrix}$ and $\bar{E}_j = \begin{bmatrix} E_j & 0 \end{bmatrix}$, and choose the proper $\bar{P}_j$ from P_j as Case 1.

To complete Theorem 4, the following two conditions shall be added:

$$\bar{P}_i - \bar{E}_i^{[m]T} \bar{U}_i \bar{E}_i^{[m]} > 0,\ \forall i \in \mathcal{I}_0 \tag{25}$$

$$\bar{A}_i^{[m]T} \bar{P}_j A_i{}^{[m]} - \bar{P}_i + \textstyle\sum_l \gamma_{il} \bar{P}^{(0)}_{il} + \bar{E}_i^{[m]T} \bar{V}_{ij} \bar{E}_i^{[m]} < 0, \tag{26}$$

$\forall i, j \in \Omega_d;\ (i, j \in \mathcal{I}_1) \cup (i \in \mathcal{I}_0 \cap j \in \mathcal{I}_1) \cup (i \in \mathcal{I}_1 \cap j \in \mathcal{I}_0)$

where $\bar{P}^{(0)}_{i\ell}$ is such that ${\xi^{[m]}}^T \bar{P}^{(0)}_{i\ell} \xi^{[m]} = 0$; $\bar{E}_i^{[m]}$ and $\bar{A}_{i[m]}$ are the corresponding transformations of $\bar{E}_i$ and $\bar{A}_i$, respectively. Thus, we have already checked all the four possible situations ($i, j \in \mathcal{I}_0$; $i, j \in \mathcal{I}_1$; $i \in \mathcal{I}_0,\ j \in \mathcal{I}_1$; $i \in \mathcal{I}_1,\ j \in \mathcal{I}_0$; for $i, j \in \Omega_d$) that may happen in the state jump for discrete-time systems.

Remark 14: We can also conclude that the non-homogeneous polynomial Lyapunov function in the form of sum of squares cannot improve the stability analysis for discrete-time PWA systems as stated in Remark 10. A similar result about how to alleviate the conservatism as stated in Remark 11 can be applied here.

V. Examples

In this section, we give some examples to illustrate our results. We compare our results with existing works in two aspects: One is the decay rate of the system; the other is the assertion of stability.

Example 1: The following example is from [14]. Here we will compare the decay rate of the continuous-time PWA system (6). The system parameters are given by

$$A_1 = A_3 = \begin{bmatrix} -0.1 & 1 \\ -5 & -0.1 \end{bmatrix},\ A_2 = A_4 = \begin{bmatrix} -0.1 & 5 \\ -1 & -0.1 \end{bmatrix}$$

$$E_1 = -E_3 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix},\ E_2 = -E_4 = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

By setting (14) as

$$P_i A_{i[m]} + {A_{i[m]}}^T P_i - \rho P_i + \sum_\ell \gamma_{i\ell} P^{(0)}_{i\ell} + {E_i^{[m]}}^T V_i E_i^{[m]} < 0$$

where ρ is the corresponding decay ratio. We compare ρ by solving $m = 1$ and $m = 2$, respectively. Note that when $m = 1$, $P^{(0)}_{i\ell} = 0$, thus the method is actually equivalent to the method of Johansson [14] or Hassibi and Boyd [10]. And the optimal $\rho = -0.21$.

When $m = 2$, the corresponding $A_{i[2]}$ and $E_i^{[2]}$ are

$$A_{1[2]} = A_{3[2]} = \begin{bmatrix} -0.2 & 10 & 0 \\ -1 & -0.2 & 5 \\ 0 & -2 & -0.2 \end{bmatrix}$$

$$A_{2[2]} = A_{4[2]} = \begin{bmatrix} -0.2 & 2 & 0 \\ -5 & -0.2 & 1 \\ 0 & -10 & -0.2 \end{bmatrix},$$

$$E_1^{[2]} = E_3^{[2]} = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 0 & -1 \\ 1 & 2 & 1 \end{bmatrix},\ E_2^{[2]} = E_4^{[2]} = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

The optimal $\rho = -0.40$. It is obvious that when $m = 2$, the PHPLF gives a much better result than that of PQLF, or $m = 1$.

If we change the system parameters A_i to

$$A_1 = A_3 = \begin{bmatrix} -0.1 & 1 \\ -5 & 0.1 \end{bmatrix},\ A_2 = A_4 = \begin{bmatrix} -0.1 & 5 \\ -1 & -0.1 \end{bmatrix}$$

When $m = 1$, we cannot get a feasible solution from (12)-(14), i.e., Johansson [14] or Hassibi and Boyd [10]'s methods cannot confirm the stability of the system. Based on the simulation programmed by Johansson [14], we can see that the system is stable as shown in Figure 1.

In fact, if we choose $m = 2$, we can show that the system is stable and P_i for (12)-(14) are given by:

$$P_1 = \begin{bmatrix} 39.3319 & -8.47619 & -12597.9 \\ -8.47619 & 25590.7 & -41.6463 \\ -12597.9 & -41.6463 & 982.788 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 981.565 & -36.8693 & -8311.88 \\ -36.8693 & 17019.3 & -13.2532 \\ -8311.88 & -13.2532 & 39.9871 \end{bmatrix}$$

$$P_3 = \begin{bmatrix} 39.3319 & -8.47619 & -11666.2 \\ -8.47619 & 23727.4 & -41.6463 \\ -11666.2 & -41.6463 & 982.788 \end{bmatrix}$$

$$P_4 = \begin{bmatrix} 981.565 & -36.8693 & 510.634 \\ -36.8693 & -625.727 & -13.2532 \\ 510.634 & -13.2532 & 39.9871 \end{bmatrix}$$

Note that in [14], the authors suggested a way to further divide partitions, in order to establish the stability of system. This method is valid here.

Example 2: In this example, we will compare the decay ratio of the discrete-time PWA system (19). The system parameters are given by

$$A_1 = A_3 = \begin{bmatrix} 1 & 0.01 \\ -0.05 & 0.897 \end{bmatrix},\ A_2 = A_4 = \begin{bmatrix} 1 & 0.05 \\ -0.01 & 0.897 \end{bmatrix}$$

$E_i,\ i \in \mathcal{I}$ is the same as these of Example 1.
Since we do not know the system state jump *a priori*, we force $\Omega_d = \mathcal{I} \times \mathcal{I}$. By setting (22) as

$${A_i{}^{[m]}}^T P_j A_i{}^{[m]} - (1+\rho) P_i + \sum_\ell \gamma_{i\ell} P^{(0)}_{i\ell} + {E_i^{[m]}}^T V_{ij} E_i^{[m]} < 0$$

where ρ is the corresponding decay rate. We compare ρ by solving $m = 1$ and $m = 2$, respectively. Note that when

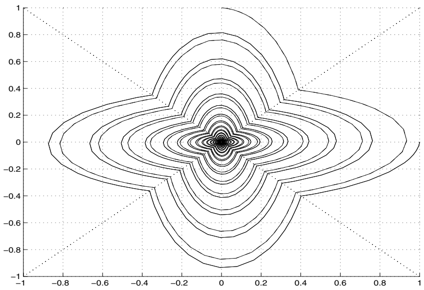

Fig. 1. System state trajectory [0 1]^T and [1 0]^T

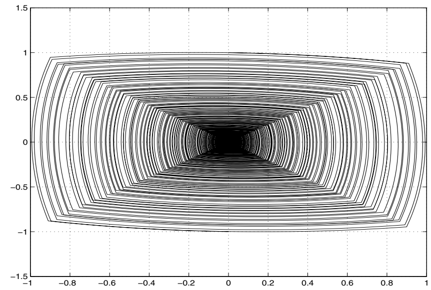

Fig. 2. System state trajectory from [0 1]^T and [0 -1]^T

$m = 1$, $P_{i\ell}^{(0)} = 0$, thus the method is actually equivalent to the method of Feng [7]. And the optimal $\rho = -0.01$.

When $m = 2$, the corresponding $A_i^{[2]}$ are

$$A_1^{[2]} = A_3^{[2]} = \begin{bmatrix} 1 & 0.02 & 0.0001 \\ -0.05 & 0.8965 & 0.00897 \\ 0.0025 & -0.0897 & 0.804609 \end{bmatrix}$$

$$A_2^{[2]} = A_4^{[2]} = \begin{bmatrix} 1 & 0.1 & 0.0025 \\ -0.01 & 0.8965 & 0.04485 \\ 0.0001 & -0.01794 & 0.804609 \end{bmatrix}$$

And $E_i^{[2]}$ are the same as those of Example 1. The optimal $\rho = -0.02$. It is obvious that $m = 2$ gives an much improved result.

In fact, if we change the system parameters to:

$$A_1 = A_3 = \begin{bmatrix} 1 & 0.01 \\ -0.05 & 0.997 \end{bmatrix}, \; A_2 = A_4 = \begin{bmatrix} 1 & 0.05 \\ -0.01 & 0.998 \end{bmatrix}$$

we cannot get a feasible solution when $m = 1$. However, when $m = 2$, Theorem 4 still proves the stability of the system with the corresponding P_i shown as follows.

$$P_1 = \begin{bmatrix} 6.98249 & 0.122781 & 117286 \\ 0.122781 & -234570 & -0.0225782 \\ 117286 & -0.0225782 & 0.249983 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 0.261448 & 0.18896 & 186391 \\ 0.18896 & -372779 & 0.719387 \\ 186391 & 0.719387 & 7.01142 \end{bmatrix}$$

$$P_3 = \begin{bmatrix} 6.98249 & 0.122781 & 137957 \\ 0.122781 & -275911 & -0.0225782 \\ 137957 & -0.0225782 & 0.249983 \end{bmatrix}$$

$$P_4 = \begin{bmatrix} 0.261448 & 0.18896 & 119429 \\ 0.18896 & -238855 & 0.719387 \\ 119429 & 0.719387 & 7.01142 \end{bmatrix}$$

The simulation result in Figure 2 verifies our concluson.

VI. CONCLUSION

In this paper, sufficient conditions for the stability of PWA systems based on piecewise homogeneous polynomial Lyapunov functions (PHPLFs) have been provided. The conditions can be checked by solving a set of LMIs. With respect to the previous work on PQLFs, improved stability analysis results can be obtained by increasing the degree of the PHPLF. Some numerical examples have demonstrated the advantages of the PHPLFs.

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