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Asymptotic and Mean Square Stability Conditions for Hybrid Jump Linear Systems with Performance Supervision

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Abstract— **This paper addresses the asymptotic and mean square stability of a class of linear stochastic hybrid systems. The systems of interest are composed of a high-level supervisor that drives a closed-loop jump linear system using an external Markovian input and a performance measure of the closed-loop system. Two new testable sufficient conditions for the asymptotic and mean square stability of these so-called Hybrid Jump Linear Systems with performance map are introduced. The first condition is also sufficient to determine the asymptotic stability of a deterministic switched system under arbitrary switching. A new testable necessary condition for the asymptotic stability of switched systems is also introduced.**

I. Introduction

This paper addresses the stability analysis of a class of linear stochastic hybrid systems. These systems are composed of low-level, continuous or discrete-time, closed-loop systems and a high-level supervisor, which is generally event-driven. The role of the supervisor is to modify the behavior of the closed-loop system by, for example, engaging different controllers to meet time-dependent performance requirements. Thus, the supervisor produces a switching signal that specifies the closed-loop system's mode of operation.

The particular focus of this paper is on the stability analysis of hybrid jump linear systems [1] where the deterministic dynamics of a system are switched by a stochastic signal that depends on a measure of the state of the system. These systems, similar to the one depicted in Figure 1, are composed of a low-level closed-loop jump linear system and a high-level supervisor which generates the switching signal based on a stochastic input, $\boldsymbol{N}(k)$, and on state information from the closed-loop system, provided by an analog-to-symbol (A/S) map.

Hybrid jump linear systems are natural models for closed-loop control systems deployed in digital computers equipped, for example, with fault recovery mechanisms. In such systems, the supervisor's objective is to correct the effects of faults introduced in the computer system by executing the proper recovery procedure. Generally, the supervisor is implemented as a high level computer program which can be modeled with a finite state machine (FSM). In this paper, two new testable sufficient conditions for asymptotic and mean square stability of these hybrid jump linear systems are introduced, along with a new testable necessary condition for asymptotic stability. The asymptotic stability results are applicable to the whole class of linear stochastic hybrid system and also to the broader class of switched systems.

The rest of the paper is organized as follows. Section II gives a precise definition of a hybrid jump linear system, including the definitions of the A/S map, the performance map and the FSM. In Section III, a recent result in [2] is combined with some classical results from [3] and [4] to provide a testable sufficient condition for the asymptotic stability of switched systems. This section also provides the testable necessary condition for the asymptotic stability of switched systems. Section IV introduces a new testable sufficient condition for the mean square stability of hybrid jump linear systems with performance maps which is *tighter* than the one provided in Section III. Section V illustrates the theory via Monte Carlo simulation examples. Finally, our conclusions are provided in Section VI.

II. Hybrid Jump Linear Systems with Performance Supervision

Consider the block diagram of the class of hybrid jump linear systems in Figure 1. These systems are composed of a finite state machine, an analog-to-symbol (A/S) map, and a jump linear closed-loop system. The FSM is described by two relations: the state evolution equation and the output map. The former is used to compute the next state of the FSM based on the current inputs and state while the latter is used to compute the FSM's output based also on the current inputs and state. Formally, the FSM's inputs constitute a stochastic process defined on the underlying probability space $(\Omega, \mathcal{F}, \Pr)$. The external process, $\boldsymbol{N}(k)$, will be restricted to be a discrete homogeneous Markov chain that takes on symbols from the set $\Sigma_{I_N} = \{\eta_{N1}\eta_{N2}, \ldots, \eta_{Nl_n}\}$. The second input is the output of the A/S map $\psi : \mathbb{R}^n \rightarrow \Sigma_{I_\nu}$ with action $\boldsymbol{x}(k) \mapsto \boldsymbol{\nu}(k) = \psi(\boldsymbol{x}(k))$, where $\Sigma_{I_\nu} = \{\eta_{\nu 1}, \eta_{\nu 2}, \ldots, \eta_{\nu l_\nu}\}$. Clearly, $\boldsymbol{\nu}(k)$ is also a discrete stochastic process with probability measure given by the transformation of the Markovian measure of $\boldsymbol{N}(k)$ corresponding to the dynamics of the FSM and the closed-loop jump linear system.

Let the states of the FSM, $\boldsymbol{z}(k)$, take on values in $\Sigma_S = \{e_1, e_2, \ldots, e_{l_s}\}$, where $e_j = \begin{bmatrix} 0 \cdots 0\ 1\ 0 \cdots 0 \end{bmatrix}'$ with a 1 in the j-th position. The FSM's state evolution equation is given by

$$\boldsymbol{z}(k+1) = S_{(\boldsymbol{N}(k), \boldsymbol{\nu}(k))} \boldsymbol{z}(k), \quad (1)$$

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where each of the $l_n \cdot l_\nu$ matrices $S_{(\eta_N,\eta_\nu)} \in \mathbb{R}^{l_s \times l_s}$ ($(\eta_N,\eta_\nu) \in \Sigma_{I_N} \times \Sigma_{I_\nu}$) is a deterministic matrix with columns containing exactly a single one and $l_s - 1$ zeros. Thus, (1) describes a deterministic FSM with a stochastic input. The FSM's output map is

$$\begin{aligned}&\varpi : \Sigma_{I_N} \times \Sigma_{I_\nu} \times \Sigma_S \to \Sigma_O \qquad (2)\\ &(\boldsymbol{N}(k), \boldsymbol{\nu}(k), \boldsymbol{z}(k)) \mapsto \boldsymbol{\theta}(k) = \varpi(\boldsymbol{N}(k), \boldsymbol{\nu}(k), \boldsymbol{z}(k)),\end{aligned}$$

where $\Sigma_O = \{\xi_1, \xi_2, \ldots, \xi_{l_s}\}$. The evolution of the closed-loop jump linear system is given by

$$\boldsymbol{x}(k+1) = A_{\boldsymbol{\theta}(k)}\boldsymbol{x}(k). \qquad (3)$$

The system (3) can represent, for example, a deterministic plant attached to a bank of control laws, which are switched by $\boldsymbol{\theta}(k)$. The formal definition of a hybrid jump linear system follows.

Definition 2.1: The system in Figure 1 described by (1)-(3) is called a Hybrid Jump Linear System (HJLS). Its state evolution is given by

$$\boldsymbol{q}(k+1) = \begin{bmatrix} A_{\boldsymbol{\theta}(k)} & 0 \\ 0 & S_{(\boldsymbol{N}(k),\boldsymbol{\nu}(k))} \end{bmatrix} \boldsymbol{q}(k), \qquad (4)$$

where $\boldsymbol{q}(k) = [\boldsymbol{x}(k)^T, \boldsymbol{z}(k)^T]^T$, and its initial condition, $q(0) = [x_0^T, z_0^T]^T$, belongs to any subset of $\mathbb{R}^n \times \Sigma_S$. Note that $\nu(0)$ is uniquely determined from x_0 and that $\boldsymbol{\theta}(0)$ depends on $\boldsymbol{N}(0)$, x_0, and z_0.

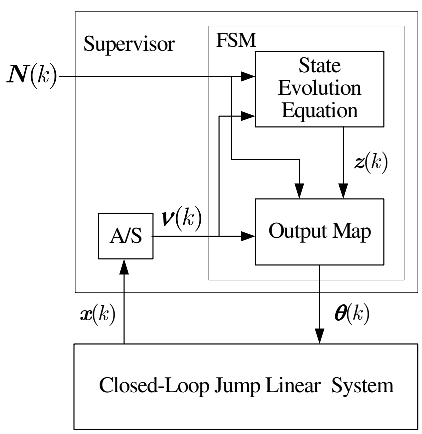

Fig. 1. A stochastic hybrid jump linear system.

The system described in Definition 2.1 is very general in the sense that the FSM representation can model many supervising algorithms. However, this generality significantly complicates the analysis when the supervisor's decisions depend on the supervised system's performance (for an account of hybrid systems without this feedback see [5]). Thus, additional structure is imposed on the A/S map and the FSM shown in Figure 1 as follows. The A/S map provides performance information to the FSM. In particular the performance metric of interest will be the value of the norm of the closed-loop system's state vector, $\boldsymbol{x}(k)$. It is assumed that the supervisor performs its decision process in two steps. In the first step, it uses the information from the input $\boldsymbol{N}(k)$ and its current operating state, $\boldsymbol{z}(k)$, to determine the best subclass of modes to apply to the closed-loop system at the next sample instant. During the second step, the supervisor uses the performance information provided by the A/S map to make the final mode selection from the subclass of modes selected in the first step. Thus, the FSM's state transition matrix depends effectively only on $\boldsymbol{N}(k)$, while the output map depends on both $\boldsymbol{N}(k)$ and $\boldsymbol{\nu}(k)$. Since the output map performs the final mode selection based on performance information, it will be called the performance map.

A precise definition of a HJLS with a performance map follows. The following index set notation will be used $\mathcal{I}_\ell \triangleq \{0, \ldots, \ell - 1\}$. Without loss of generality, let $\Sigma_{I_N} = \mathcal{I}_{l_n}$, $\Sigma_{I_\nu} = \mathcal{I}_{l_\nu}$, and fix positive constants $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_{l_\nu - 1} < \alpha_{l_\nu} = \infty$. The (performance) A/S map is

$$\boldsymbol{\nu}(k) = \psi(\boldsymbol{x}(k)) = \sum_{i=0}^{l_\nu - 1} i \mathbf{1}_{\{\alpha_i \le \|\boldsymbol{x}(k)\| < \alpha_{i+1}\}}, \qquad (5)$$

where $\mathbf{1}_{\{\alpha_i \le \|\boldsymbol{x}(k)\| < \alpha_{i+1}\}} = 1$ whenever $\alpha_i \le \|\boldsymbol{x}(k)\| < \alpha_{i+1}$ and zero otherwise. Clearly, the A/S map quantizes the range of $\|\boldsymbol{x}(k)\|$ into l_ν levels. The FSM's state evolution equation (1) is now simply

$$\boldsymbol{z}(k+1) = S_{\boldsymbol{N}(k)}\boldsymbol{z}(k). \qquad (6)$$

The specific performance map to be analyzed is

$$\begin{aligned}\boldsymbol{\theta}(k) &= \varpi(\boldsymbol{N}(k), \boldsymbol{\nu}(k), z(k)) \\ &= (l_n \cdot l_s)\boldsymbol{\nu}(k) + l_n[0, 1, \ldots, l_s - 1]\boldsymbol{z}(k) + \boldsymbol{N}(k) \\ &= (l_n \cdot l_s)\boldsymbol{\nu}(k) + \boldsymbol{\varphi}(k). \qquad (7)\end{aligned}$$

where $\boldsymbol{\varphi}(k) = \tilde{\boldsymbol{z}}(k) + \boldsymbol{N}(k)$ and $\tilde{\boldsymbol{z}}(k) = [0, 1, \ldots, l_s - 1]\boldsymbol{z}(k)$. Thus, if $\boldsymbol{z}(k) = e_j$ then $\tilde{\boldsymbol{z}}(k) = j - 1$. Clearly, $\tilde{\boldsymbol{z}}(k) \in \mathcal{I}_{l_s}$ is isomorphic to $\boldsymbol{z}(k)$. Also, observe that $\boldsymbol{\varphi}(k) = T(\boldsymbol{N}(k), \tilde{\boldsymbol{z}}(k))$, where $T : \Sigma_{I_N} \times \mathcal{I}_{l_s} \to \mathcal{I}_{l_n \cdot l_s - 1}$ with action $(e, d) \mapsto f = l_n \cdot d + e$ is a bijective map. Thus, $\boldsymbol{\varphi}(k)$ is isomorphic to the random process $(\boldsymbol{N}(k), \boldsymbol{z}(k))$, which is known by [5] to be a Markov chain of the same order as the Markov chain $\boldsymbol{N}(k)$. Thus, $\boldsymbol{\varphi}(k)$ is also a Markov chain. Notice, however, that $\boldsymbol{\theta}(k)$ may not be a Markov chain since $\boldsymbol{\nu}(k)$ may not be memoryless. A hybrid jump linear system as described by (3) and (5)-(7) is called a *Hybrid Jump Linear System with Performance Map* with state evolution given by (4).

In [1] it is shown that the set $M = \{\{x_e\} \times \Sigma_S\}$ is an invariant set for (4), where $x_e = 0$. In addition, the stability properties of M are equivalent to the stability properties of the equilibrium point of (3) ($x_e = 0$). So, the stability of a HJLS can be studied by analyzing the stability properties of its associated jump linear system. This is the approach followed in the next two sections. Specifically, in the next section, stability tests that are independent of the switching sequence produced by the performance map (7) are given. A less conservative result which takes into account the specific performance map employed is given in Section IV.

III. Stability Analysis of Discrete-Time Switched Systems

A. Preliminaries

Discrete-time stability results similar to those for continuous-time switched systems are well known (cf. [2], [6]–[8]). It follows from those results that for stability analysis it suffices to consider the following autonomous linear discrete-time switched system

$$x_\sigma(k+1) = A_{\sigma(k)} x_\sigma(k),\ x_\sigma(0) = x_0, \tag{8}$$

where the discrete-time *switching signal*, σ, is a sequence $(\sigma(0), \ldots, \sigma(k), \ldots) \in \mathcal{I}_L^\infty = \mathcal{I}_L \times \mathcal{I}_L \times \cdots$. The set $\mathcal{I}_L^\infty$ of all possible switching sequences is such that each entry in a sequence is used to select $A_{\sigma(k)} \in \mathcal{M} \triangleq \{A_i : A_i \in \mathbb{R}^{n\times n}, i \in \mathcal{I}_L\}$. The effect of the switching signals in $\mathcal{I}_L^\infty$ can also be seen via the set of discrete linear inclusions (DLI), consisting of the set of all possible trajectories $\mathrm{DLI}(\mathcal{M}) \triangleq \{(x_\sigma(0), \ldots, x_\sigma(k), \ldots) : x_\sigma(k) \in \mathbb{R}^n,\ k \in \mathbb{Z}^+\}$ where $x_\sigma(k+1) = A_{\sigma(k)} x_\sigma(k)$ for $A_{\sigma(k)} \in \mathcal{M}$ and $\sigma(k) \in \mathcal{I}_L$. The following stability definition includes the effect of all possible switching sequences.

Definition 3.1: [7] The equilibrium point $x_\sigma = 0$ of the discrete-time switched system (8) is *absolutely asymptotically stable* (AAS) if every trajectory in $\mathrm{DLI}(\mathcal{M})$ satisfies $\lim_{k\to\infty} x_\sigma(k) = 0$.

AAS can be studied by analyzing products of matrices since (8) is AAS if and only if

$$\lim_{k\to\infty} A_{\sigma(k)} \cdots A_{\sigma(0)} = 0,\ A_{\sigma(k)} \in \mathcal{M},\ k \in \mathbb{Z}^+.$$

A particular subclass of arbitrary switched sequences is that formed by sampled-paths of stochastic processes with state space $\mathcal{I}_L$. In particular, let $\tilde{\boldsymbol{\theta}}(k)$ be any second-order stochastic process, with initial distribution $\pi_{\tilde{\theta}}(0)$ and state space $\mathcal{I}_L$. Consider the system

$$\tilde{\boldsymbol{x}}(k+1) = A_{\tilde{\boldsymbol{\theta}}(k)} \tilde{\boldsymbol{x}}(k),\ \ \tilde{\boldsymbol{x}}(0) = \tilde{\boldsymbol{x}}_0, \tag{9}$$

where $A_{\tilde{\boldsymbol{\theta}}(k)} \in \mathcal{M}$. The following definition applies.

Definition 3.2: Let $\boldsymbol{x}_0$ be a second order random variable independent of $\tilde{\boldsymbol{\theta}}(k)$ for all $k \geq 0$ with distribution f_{x_0}. The system (9) is *mean square stable* if for every x_0 and $\pi_{\tilde{\theta}}(0)$, $E\{\|\tilde{\boldsymbol{x}}(k)\|^2\} \to 0$ as $k \to \infty$.

Clearly, if (8) is absolutely asymptotically stable then (9) is mean square stable since for every sample path generated by $\tilde{\boldsymbol{\theta}}(k)$ it follows that $\lim_{k\to\infty} A_{\sigma(k)} \cdots A_{\sigma(0)} = 0$. The switching sequences $\{\sigma(k) : k \in \mathbb{Z}^+\}$ in (8) can be generated, for example, by a high level supervisor as in (3). In this light it is easy to see that the hybrid jump linear systems defined in (4) are a particular type of linear switched systems for which many stability analysis tools are available. The next subsection presents a new testable sufficient condition for the AAS of (8) along with a new testable necessary condition. By extension, both tests apply also to hybrid jump linear systems.

B. Stability Analysis

A recent result in [2] provides a sufficient condition for the absolutely asymptotic stability of (8). This result, however, requires the solution of L^2 linear matrix inequalities (LMI) in L unknowns. The following theorem reduces this number to only L. In the sequel, $P > 0$ will denote a real, symmetric, and positive definite matrix $P \in \mathbb{R}^{n\times n}$.

Theorem 3.1: If for a given set of matrices $\{W_i > 0, i \in \mathcal{I}_L\}$ there exists a set of matrices $\{P_i > 0, i \in \mathcal{I}_L\}$ satisfying the condition

$$\sum_{j=0}^{L-1} A_i^T P_j A_i - P_i = -W_i,\ \forall i \in \mathcal{I}_L \tag{10}$$

then (8) is absolutely asymptotically stable.

Proof: Suppose (10) holds and observe for any $x \in \mathbb{R}^n$ that $x^T A_i^T P_i A_i x \geq 0$. Then

$$\begin{aligned} x^T A_i^T P_j A_i x - x^T P_i x &\leq \sum_{j=0}^{L-1} x^T A_i^T P_j A_i x - x^T P_i x \\ &\leq -x^T W_i x, \end{aligned}$$

and therefore $P_i - A_i^T P_j A_i > 0$ for every $i, j \in \mathcal{I}_L$. Since these matrices P_i satisfy the hypothesis in [2, Theorem 2], the system (8) is absolutely asymptotically stable. ■

An advantage of Theorem 3.1 is that there exists a simple test to determine the existence of a solution for (10). The following test has been adapted from the more general results in [4, Proposition 6] and [9, Lemma 1].

Theorem 3.2: Let $\{A_i \in \mathbb{R}^{n\times n}, i \in \mathcal{I}_L\}$ be given and define $\bar{\mathcal{A}}$ as

$$\bar{\mathcal{A}} = (E_L \otimes I_{n^2}) \operatorname{diag}(A_0 \otimes A_0, \ldots, A_{L-1} \otimes A_{L-1}), \tag{11}$$

where E_L is an $L \times L$ matrix with every entry equal to 1, and I_{n^2} is a $n^2 \times n^2$ identity matrix. Then, for every set of matrices $\{W_i > 0, i \in \mathcal{I}_L\}$ there exists a unique set of matrices $\{P_i > 0, i \in \mathcal{I}_L\}$ that satisfy (10) if and only if $\rho(\bar{\mathcal{A}}) < 1$.

As a consequence of this theorem, a spectral radius test determines the absolute asymptotic stability of switched systems.

Theorem 3.3: Consider the system (8) and let $\bar{\mathcal{A}}$ be defined as in (11). If $\rho(\bar{\mathcal{A}}) < 1$ then the system (8) is absolutely asymptotically stable.

Proof: The result follows directly from Theorems 3.1 and 3.2. ■

Observe that the AAS condition $\rho(\bar{\mathcal{A}}) < 1$ in Theorem 3.3 is similar to the sufficient condition for the mean square stability of the system

$$\tilde{\boldsymbol{x}}(k+1) = \sqrt{L} A_{\tilde{\boldsymbol{\theta}}(k)} \tilde{\boldsymbol{x}}(k), \tag{12}$$

where $\tilde{\boldsymbol{\theta}}(k) \in \mathcal{I}_L$ is a Markov chain with transition probabilities $p_{ij} = 1/L$ [4]. Thus, Theorem 3.3 admits the following interpretation: If the mean of the sample paths $\|\tilde{\boldsymbol{x}}(k)\|^2$ of (12) goes asymptotically to zero then so does each possible trajectory of (8). Although the converse argument is not always true, it can be seen that if (8) is absolutely asymptotically stable, then the average of $\|x(k)\|^2$ over the possible trajectories must also approach zero asymptotically. This motivates our next result.

Theorem 3.4: Let $\bar{\mathcal{A}}$ be defined as in (11). A necessary condition for (8) to be absolutely asymptotically stable is $\rho(\bar{\mathcal{A}}) < L$.

The proof of this theorem parallels the necessity proof of [3, Theorem 2.1]. Asymptotic stability of (8) is used to show that for any given set of matrices $\{W_i > 0, i \in \mathcal{I}_L\}$ there exists a unique set of matrices $\{P_i > 0, i \in \mathcal{I}_L\}$ satisfying

$$(1/L)\sum_{j=0}^{L-1} A_i^T P_j A_i - P_i = -W_i. \tag{13}$$

This fact and Theorem 3.2 give the desired result.

Proof: Let the initial condition x_0 be fixed and let the transition matrix for the deterministically switched system (8) be given by

$$\Phi(k+n,k) \triangleq \begin{cases} I_n, & n = 0, \\ A_{\sigma(k+n-1)} \cdots A_{\sigma(k)}, & n \geq 1. \end{cases}$$

Let $\{W_i > 0, i \in \mathcal{I}_L\}$ be given and define $g(\Phi(k+n,k)) \triangleq \Phi^T(k+n,k)W_{\sigma(k+n)}\Phi(k+n,k)$. Note that for every fixed value of $\sigma(k)$, $g(\Phi(k+n,k))$ can take L^n different values, one for every possible permutation of $\{\sigma(k+1),\ldots,\sigma(k+n)\} \in \mathcal{I}_L$. If $\sigma(k) = i$ and $n \geq 1$, the L^n values of $g(\Phi(k+n,k))$ are obtained by evaluating

$$\begin{aligned} g(\Phi(k+n,k)) = A_i^T \cdot A^T_{\sigma(k+1)=\ell_1} \cdots A^T_{\sigma(k+n-1)=\ell_{n-1}} \\ \cdot W_{\sigma(k+n)=\ell_n} \cdot A_{\sigma(k+n-1)=\ell_{n-1}} \cdots A_{\sigma(k+1)=\ell_1} \cdot A_i, \end{aligned}$$

for each permutation of $\{\ell_1,\ldots,\ell_n\} \in \mathcal{I}_L$. Thus, the mean value of $g(\Phi(k+n,k))$ when $\sigma(k) = i$ is given by
$\hat{E}\{g(\Phi(k+n,k))|\sigma(k) = i\} \triangleq$

$$\begin{cases} W_i, & n = 0 \\ \sum_{\sigma(k+1)} \cdots \sum_{\sigma(k+n)} \frac{g(\Phi(k+n,k+1)A_i)}{L^n}, & n \geq 1. \end{cases}$$

Since $W_{\sigma(k+n)} > 0$, Rayleigh's inequality gives

$$\|g(\Phi(k+n,k))\| \leq \bar{\lambda}\|\Phi(k+n,k)\|^2,$$

where $\bar{\lambda} = \max_i\{\lambda_{\max}(W_i)\}$ and $\lambda_{\max}(W_i)$ denotes the maximum eigenvalue of W_i. Since absolute asymptotic stability of (8) is equivalent to uniform global exponential stability, there exist scalars $\beta > 1$ and $0 < \alpha < 1$ such that $\|\Phi(k+n,k)\|^2 \leq \beta\alpha^n$ (see [10] Theorem 22.7). Thus, it follows that

$$\|\hat{E}\{g(\Phi(k+n,k))|\sigma(k)\}\| \leq \bar{\lambda}\beta\alpha^n.$$

Next, define the matrices $P(n,\sigma(k))$, $n \geq 0$ as

$$P(n,\sigma(k)) \triangleq \sum_{i=k}^{k+n} \hat{E}\{g(\Phi(i,k))|\sigma(k)\},$$

and observe that they are symmetric and positive definite by definition. Furthermore, for any $m > n \geq 0$

$$\begin{aligned} \|P(m,\sigma(k)) - P(n,\sigma(k))\| &\leq \sum_{i=k+n+1}^{k+m} \|\hat{E}\{g(\Phi(i,k))|\sigma(k)\}\| \\ &\leq \sum_{i=n+1}^{m} \bar{\lambda}\beta\alpha^i \\ &\leq \bar{\lambda}\beta\alpha^n/(1-\alpha). \end{aligned}$$

Thus, for any $\epsilon > 0$ there exists a sufficiently large n such that $\bar{\lambda}\beta\alpha^n/(1-\alpha) < \epsilon$, which in turn implies that $P(n,\sigma(k))$ has a unique limit with respect to n. Thus, define $P_i \triangleq \lim_{n\to\infty} P(n,\sigma(k) = i)$ and observe that

$$\begin{aligned} P(n,\sigma(0) = i) &= \sum_{l=0}^{n} \hat{E}\{g(\Phi(l,0))|\sigma(0) = i\} \\ &= W_i + \sum_{l=1}^{n} \hat{E}\{g(\Phi(l,0))|\sigma(0) = i\}. \end{aligned} \tag{14}$$

In addition, since $g(\Phi(k+n,k+1)A_{\sigma(k)}) = A^T_{\sigma(k)}g(\Phi(k+n,k+1))A_{\sigma(k)}$, then

$$\begin{aligned} &\hat{E}\{g(\Phi(l,0))|\sigma(0) = i\} = \\ &\qquad (1/L)A_i^T \sum_{j=0}^{L-1} \hat{E}\{g(\Phi(l,1))|\sigma(1) = j\}A_i. \end{aligned} \tag{15}$$

Substituting (15) in (14) gives

$$P(n,\sigma(0) = i) = W_i + (1/L)\sum_{j=0}^{L-1} A_i^T P(n-1,\sigma(1) = j)A_i.$$

Letting $n \to \infty$ in the expression above implies that the matrices P_i uniquely satisfy (13). Thus, Theorem 3.2 implies that $\rho(\bar{\mathcal{A}}/L) < 1$ or $\rho(\bar{\mathcal{A}}) < L$, completing the proof. ∎

Observe that if a HJLS satisfies the conservative conditions of Theorem 3.3, it is both absolutely asymptotically stable and mean square stable. The next section provides a less conservative sufficient condition for the mean square stability of HJLS with performance maps.

IV. A Sufficient Condition for the Mean Square Stability of a HJLS with a Performance Map

The main result of this section follows.

Theorem 4.1: Consider the following hybrid jump linear system with performance map defined by

$$\begin{aligned} \boldsymbol{x}(k+1) &= A_{\boldsymbol{\theta}(k)}\boldsymbol{x}(k) \\ \boldsymbol{\nu}(k) &= \mathbf{1}_{\{\|\boldsymbol{x}(k)\| \geq \alpha\}},\ \boldsymbol{\theta}(k) = L\boldsymbol{\nu}(k) + \boldsymbol{\varphi}(k), \end{aligned} \tag{17}$$

where $\boldsymbol{x}(k) \in \mathbb{R}^n$, $\alpha > 0$, and $\boldsymbol{\varphi}(k) \in \mathcal{I}_L$ is a Markov chain with transition probability matrix $\Pi_\varphi = [p_{ji}]$. Let $\mathcal{A}$ be defined as

$$\mathcal{A} \triangleq (\Pi_\theta^T \otimes I_{n^2})\,\mathrm{diag}(A_0 \otimes A_0,\ldots,A_{2L-1} \otimes A_{2L-1}), \tag{18}$$

where $\Pi_\theta = \left[\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right] \otimes \Pi_\varphi$. If $\|\mathcal{A}\|_\infty < 1$ then (17) is mean square stable.

Proof: Due to space limitations, part of the proof is only sketched out. Let $Q(k+1) = E\{\boldsymbol{x}(k+1)\boldsymbol{x}(k+1)^T\}$ and define

$$Q_i(k+1) \triangleq E\{\boldsymbol{x}(k+1)\boldsymbol{x}(k+1)^T \mathbf{1}_{\{\boldsymbol{\theta}(k+1)=i\}}\}$$

so that $Q(k+1) = \sum_i Q_i(k+1)$. Then it follows that

$$Q_i(k+1) = \sum_{j=0}^{2L-1} A_j E\{\boldsymbol{x}(k)\boldsymbol{x}(k)^T \mathbf{1}_{\{\boldsymbol{\theta}(k+1)=i,\boldsymbol{\theta}(k)=j\}}\} A_j^T.$$

Now, define $\boldsymbol{q}_x(k) \triangleq \mathrm{vec}(\boldsymbol{x}(k)\boldsymbol{x}(k)^T)$, $\tilde{q}_i(k+1) \triangleq \mathrm{vec}\,(Q_i(k+1)) = E\{\boldsymbol{q}_x(k+1)\mathbf{1}_{\{\boldsymbol{\theta}(k+1)=i\}}\}$ and apply the $\mathrm{vec}(\cdot)$ operator to the equation above to obtain

$$\tilde{q}_i(k+1) = \sum_{j=0}^{2L-1} A_j \otimes A_j E\{\boldsymbol{q}_x(k)\mathbf{1}_{\{\boldsymbol{\theta}(k+1)=i,\boldsymbol{\theta}(k)=j\}}\}. \quad (19)$$

Let $\mathcal{F}_k$ be the sigma algebra generated by the random variables $\{\boldsymbol{\theta}(k), \ldots, \boldsymbol{\theta}(0)\}$ and observe that

$$\begin{aligned} E\{\boldsymbol{q}_x(k)\mathbf{1}_{\{\boldsymbol{\theta}(k+1)=i,\boldsymbol{\theta}(k)=j\}}\} &= \\ &= E\{E\{\boldsymbol{q}_x(k)\mathbf{1}_{\{\boldsymbol{\theta}(k+1)=i,\boldsymbol{\theta}(k)=j\}}|\mathcal{F}_k\}\} \\ &= E\{\boldsymbol{q}_x(k)\Pr\{\boldsymbol{\theta}(k+1)=i|\mathcal{F}_k\}\mathbf{1}_{\{\boldsymbol{\theta}(k)=j\}}\}. \end{aligned} \quad (20)$$

Notice that the value of $\Pr\{\boldsymbol{\theta}(k+1)=i|\mathcal{F}_k\}\mathbf{1}_{\{\boldsymbol{\theta}(k)=j\}}$ depends on one of the following four mutually exclusive conditions. (i) The set $\{\boldsymbol{\theta}(k)=j\}$ is empty. (ii) The set $\{\boldsymbol{\theta}(k)=j\}$ is not empty but there are no non-empty sets of the form $\{\boldsymbol{\theta}(k+1)=i, \boldsymbol{\theta}(k)=j, \boldsymbol{\theta}(k-1)=\ell_{k-1}, \ldots, \boldsymbol{\theta}(0)=\ell_0\}$ where $\ell_0, \ell_1, \ldots, \ell_k, \ell_{k+1} \in \Sigma_O = \mathcal{I}_{2L}$ with $\ell_k \triangleq j, \ell_{k+1} \triangleq i$. (iii) The set $\{\boldsymbol{\theta}(k)=j\}$ and some or all the sets $\{\boldsymbol{\theta}(k+1)=i, \ldots, \boldsymbol{\theta}(0)=\ell_0\}$ are not empty, but $p_{\hat{\jmath}\hat{\imath}} = \Pr\{\boldsymbol{\varphi}(k+1)=\hat{\imath}|\boldsymbol{\varphi}(k)=\hat{\jmath}\} = 0$, where $\hat{\jmath} = j - L\lfloor j/L \rfloor$, $\hat{\imath} = i - L\lfloor i/L \rfloor$, and $\lfloor\cdot\rfloor$ is the floor function. (iv) The set $\{\boldsymbol{\theta}(k)=j\}$, some or all the sets $\{\boldsymbol{\theta}(k+1)=i, \ldots, \boldsymbol{\theta}(0)=\ell_0\}$ are not empty, and $p_{\hat{\jmath}\hat{\imath}} > 0$. In (iii) observe that $\{\boldsymbol{\theta}(k)=j\} = \{\boldsymbol{\varphi}(k)=\hat{\jmath}, \boldsymbol{\nu}(k)=\lfloor j/L \rfloor\}$, so a very simple argument shows that $\{\boldsymbol{\theta}(k+1)=i, \boldsymbol{\theta}(k)=j, \ldots, \boldsymbol{\theta}(0)=\ell_0\} = \{\boldsymbol{\varphi}(k+1)=\hat{\imath}, \boldsymbol{\varphi}(k)=\hat{\jmath}, \ldots, \boldsymbol{\varphi}(0)=\hat{\ell}_0\}$. Thus,

$$\begin{aligned} \Pr\{\boldsymbol{\theta}(k+1)=i|\boldsymbol{\theta}(k)=j, \ldots, \boldsymbol{\theta}(0)=\ell_0\} \\ = \Pr\{\boldsymbol{\varphi}(k+1)=\hat{\imath}|\boldsymbol{\varphi}(k)=\hat{\jmath}, \ldots, \boldsymbol{\varphi}(0)=\hat{\ell}_0\} \\ = p_{\hat{\jmath}\hat{\imath}} = 0. \end{aligned}$$

Note that (20) equals zero according to (i)-(iii) since the random variable $\Pr\{\boldsymbol{\theta}(k+1)=i|\mathcal{F}_k\}\mathbf{1}_{\{\boldsymbol{\theta}(k)=j\}}$ is a constant equal to zero. This is not the case for the last condition. It follows from (iv) that $\Pr\{\boldsymbol{\theta}(k+1)=i|\mathcal{F}_k\}\mathbf{1}_{\{\boldsymbol{\theta}(k)=j\}} \in \{0, p_{\hat{\jmath}\hat{\imath}}\}$ and it can be shown that

$$\begin{aligned} \|E\{\boldsymbol{q}_x(k)\mathbf{1}_{\{\boldsymbol{\theta}(k+1)=i,\boldsymbol{\theta}(k)=j\}}\}\|_\infty \\ \le p_{\hat{\jmath}\hat{\imath}}\|E\{\boldsymbol{q}_x(k)\mathbf{1}_{\{\boldsymbol{\theta}(k)=j\}}\}\|_\infty = p_{\hat{\jmath}\hat{\imath}}\|\tilde{q}_j(k)\|_\infty. \end{aligned}$$

Now, define $\tilde{q}(k) \triangleq [\tilde{q}_0(k)^T, \ldots, \tilde{q}_{2L-1}(k)^T]^T$, $y_i \triangleq [p_{o\hat{\imath}}(A_0 \otimes A_0), \ldots, p_{(L-1)\hat{\imath}}(A_{2L-1} \otimes A_{2L-1})]$, and let $\mathcal{J}(k) \subseteq \mathcal{I}_{2L-1}$ be the subset of indices j for which $\Pr\{\boldsymbol{\theta}(k+1)=i|\mathcal{F}_k\}\mathbf{1}_{\{\boldsymbol{\theta}(k)=j\}} = p_{\hat{\jmath}\hat{\imath}}$ is satisfied under condition (iv). Then it follows from (i)-(iv) that

$$\begin{aligned} &\|\tilde{q}_i(k+1)\|_\infty \\ &\le \|y_i\|_\infty \max_{m\in\mathcal{J}(k)} \left\{ \left\| \frac{E\{\boldsymbol{q}_x(k)\mathbf{1}_{\{\boldsymbol{\theta}(k+1)=i,\boldsymbol{\theta}(k)=m\}}\}}{p_{\hat{m}\hat{\imath}}} \right\|_\infty \right\} \\ &\le \|y_i\|_\infty \max_{m\in\mathcal{J}(k)} \{\|\tilde{q}_m(k)\|_\infty\} \le \|y_i\|_\infty \|\tilde{q}(k)\|_\infty. \end{aligned}$$

Now, $\|\mathcal{A}\|_\infty = \max_i\{\|[p_{0\hat{\imath}}A_0 \otimes A_0, \ldots, p_{(L-1)\hat{\imath}}A_{2L-1} \otimes A_{2L-1}]\|_\infty\} = \max_i\{\|y_i\|_\infty\}$, so taking $\max\{\cdot\}$ on both sides of the equation above gives

$$\begin{aligned} \|\tilde{q}(k+1)\|_\infty &\le \max_{0\le i\le 2L-1}\{\|y_i\|_\infty\}\|\tilde{q}(k)\|_\infty \\ &\le \|\mathcal{A}\|_\infty\|\tilde{q}(k)\|_\infty. \end{aligned}$$

Clearly, if $\|\mathcal{A}\|_\infty < 1$ then $\tilde{q}(k+1) \to 0$ as $k \to \infty$ which in turn implies that $Q(k+1) \to 0$ as $k \to \infty$, and the mean square stability of (3) follows. ■

Finally, observe from its definition that $\mathcal{A} = \left[\begin{smallmatrix}\mathcal{A}_1 & \mathcal{A}_2\\ \mathcal{A}_1 & \mathcal{A}_2\end{smallmatrix}\right]$ where $\mathcal{A}_1 = (\Pi_\varphi^T \otimes I_{n^2})\,\mathrm{diag}(A_0 \otimes A_0, \ldots, A_{L-1} \otimes A_{L-1})$ and $\mathcal{A}_2 = (\Pi_\varphi^T \otimes I_{n^2})\,\mathrm{diag}(A_L \otimes A_L, \ldots, A_{2L-1} \otimes A_{2L-1})$. Thus, a corollary to Theorem 4.1 follows.

Corollary 4.1: For the system considered in Theorem 4.1, if $\left\|\left[\mathcal{A}_1 \quad \mathcal{A}_2\right]\right\|_\infty < 1$ then (17) is mean square stable .

V. Examples

Three simulation examples are presented to illustrate the theorems in the previous sections and demonstrate their limitations. In these examples, the hybrid jump linear systems represent closed-loop systems implemented on fault recoverable computers. These systems operate as follows: $\boldsymbol{N}(k)$ is a two-state Markov chain, $\boldsymbol{N}(k) \in \{0, 1\}$, representing the absence (0) or presence (1) of computer faults in a system. The objective of the supervisor is to maintain a correct level of performance, which is attained when the norm of the plant's state vector, $\|\boldsymbol{x}(k)\|$, is below a specified level α. Thus, the A/S output is 0 whenever $\|\boldsymbol{x}(k)\| < \alpha$ and 1 otherwise. The supervisor can select one of three operation modes: Nominal, Fault-Recovery, and Performance Recovery modes. The Nominal mode is selected only when there are no faults in the system and the performance is acceptable, i.e., when $\boldsymbol{N}(k) = \boldsymbol{\nu}(k) = 0$. The Fault-Recovery mode is selected whenever there is a fault in the system, regardless of the current performance of the system. This is because the supervisor prioritizes fault correction over performance correction. Finally, the Performance Recovery mode is selected when the performance is unacceptable and there is no fault present in the system, i.e. $\boldsymbol{N}(k) = 0$, $\boldsymbol{\nu}(k) = 1$. The behavior described above can be represented by setting $A_1 = A_3$ in (17).

TABLE I. Stability Conditions Satisfied by the Examples.

	Theorem 3.3	Theorem 3.4	Theorem 4.1
	$\rho(\bar{A}) < 1$	$\rho(\bar{A}/L) < 1$	$\|\mathcal{A}\|_\infty < 1$
Example 1	0.9997 ✓	0.2499 ✓	0.7463 ✓
Example 2	2.2729	0.5682 ✓	0.9947 ✓
Example 3	3.1651	0.7913 ✓	2.0470

The entries with a check mark in Table I indicate which stability conditions are met by the examples. For example, Example 2 satisfies the necessary condition for stability under arbitrary switching but not the sufficient condition clearly showing that the necessary condition is not sufficient to ensure stability under arbitrary switching.

Table II gives the parameters for each example. All the examples use the same initial condition $x_0 = [1 \ \ -1]^T$, the same initial chain distribution $\pi_0 = [1 \ 0]$, and the same performance threshold $\alpha = 1$.

Since it is impossible to show that Example 1 is asymptotically stable for every switching rule the following procedure was followed: 2000 sample paths of $\boldsymbol{N}(k)$ were produced and for each one, the system trajectory was computed. At every time k the square of the norm of the plant's state vector was calculated. Finally, the average of $\|x(k)\|^2$ was computed and plotted in Figure 2. As this figure shows (and was expected), none of the 2000 sample paths diverged to infinity. The mean square stability of Examples 2 and 3 was simulated using also 2000 Monte Carlo runs. Figure 2 also shows the simulation results for those examples.

TABLE II. Hybrid Jump Linear Systems' Parameters.

	Example 1	Example 2	Example 3
A_0	$\begin{bmatrix} 0.75 & 0.15 \\ 0.40 & 0.30 \end{bmatrix}$	$\begin{bmatrix} 0.41 & 0.075 \\ 0.20 & 0.15 \end{bmatrix}$	$\begin{bmatrix} 0.90 & 0.15 \\ 0.40 & 0.30 \end{bmatrix}$
A_1	$\begin{bmatrix} -0.62 & 0.00 \\ 0.00 & 0.57 \end{bmatrix}$	$\begin{bmatrix} -1.01 & 0.00 \\ 0.00 & 0.40 \end{bmatrix}$	$\begin{bmatrix} -1.20 & 0.00 \\ 0.00 & 0.70 \end{bmatrix}$
A_2	$\begin{bmatrix} 0.21 & 0.00 \\ 0.10 & 0.30 \end{bmatrix}$	$\begin{bmatrix} 0.25 & 0.00 \\ 0.05 & 0.30 \end{bmatrix}$	$\begin{bmatrix} 0.50 & 0.00 \\ 0.10 & 0.60 \end{bmatrix}$
Π_N	$\begin{bmatrix} 0.10 & 0.90 \\ 0.05 & 0.95 \end{bmatrix}$	$\begin{bmatrix} 0.70 & 0.30 \\ 0.60 & 0.40 \end{bmatrix}$	$\begin{bmatrix} 0.60 & 0.40 \\ 0.50 & 0.50 \end{bmatrix}$

VI. Conclusions

In this paper two new testable sufficient conditions for absolute asymptotic and mean square stability of discrete-time switched systems were presented. These results can be applied to a large class of linear stochastic hybrid systems as shown by their application to a specific subclass of linear stochastic hybrid systems. These systems consist of a finite state machine driving the mode of operation of a closed-loop system depending on the occurrence of a fault event modeled by a stochastic process and on a state-dependent level of performance of the closed-loop system. A sufficient

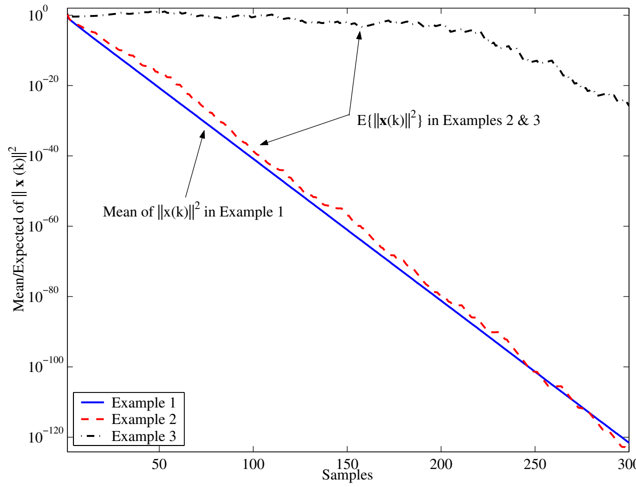

Fig. 2. Mean of the norm squared (Example 1) and second moment of $\boldsymbol{x}(k)$ (Examples 2 and 3). Example 1 satisfies Theorem 3.5 while Example 2 satisfies Theorem 4.1. Example 3 satisfies neither condition.

condition that takes the structure of the hybrid system into account was also developed and shown via an example to be less conservative.

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